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Numerical range of a matrix associated with the graph of a trigonometric polynomial

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Abstract

We present a determinantal representation of a hyperbolic ternary form associated with a trigonometric polynomial. The result is obtained by a joint work with Professor Mao-Ting Chien.

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1. Lax-Fiedler conjecture

Suppose that A is an $n \times n$ complex matrix. The *numerical range* $W(A)$ of A is defined as

$$W(A) = \{\xi^* A\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}. \tag{1.1}$$

In 1918 Toeplitz introduced this set $W(A)$. He characterized $\partial W(A)$ by

$$\max\{\Re(e^{-i\theta}z) : z \in W(A)\} = \max \sigma(H(\theta : A)), \tag{1.2}$$

where

$$H(\theta : A) = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*),$$
$$\sigma(H) = \{\lambda \in \mathbf{R} : \det(\lambda I - H) = 0\}, \tag{1.3}$$

for $H = H^*$. In 1919 Hausdorff proved the simply connectedness of the range $W(A)$. The simply connectedness of the numerical range is also valid for a linear matrix pencil $A\lambda + B$ with $0 \notin W(A)$ ([20]). To compute the eigenvalues of $H(\theta : A)$ we introduce a ternary form

$$F_A(t, x, y) = \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)). \tag{1.4}$$

By the equation

$$\det(tI_n - H(\theta : A)) = F_A(t, -\cos\theta, -\sin\theta),$$

this ternary form determines the eigenvalues of $H(\theta)$ for every angle θ.

In 1951, Kippenhahn [15] showed that

$$W(A) = \mathrm{Conv}(\{X + iY : (X,Y) \in \mathbf{R}^2, Xx + Yy + 1 = 0 \text{ is a tangent of } F(1,x,y) = 0\}.$$

By this result, the boundary of the numerical range $W(A)$ lies on the dual curve of the algebraic curve $F(1,x,y) = 0$ when $W(A)$ is strictly convex.

The form $F_A(t,x,y)$ satisfies (i) $F_A(1,0,0) > 0$ and (ii) For every $(x_0, y_0) \in \mathbf{R}^2$, the equation $F_A(t, x_0, y_0) = 0$ in t has n real solutions couting the multiplicities of the solutions. In 1981, Fiedler [11] conjectured : If $F(t,x,y)$ is a real ternary form of degree n and satisfies (i) $F(1,0,0) = c > 0$ and (ii) For every $(x_0, y_0) \in \mathbf{R}^2$, the equation $F(t, x_0, y_0) = 0$ in t has n real solutions couting the multiplicities of the solutions, then there exists an $n \times n$ complex matrix A with

$$F(t,x,y) = c \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)). \tag{1.5}$$

If a ternary form $F(t,x,y)$ satisfies the above conditions (i) and (ii), then the form is said to be *hyperbolic* with respect to $(1,0,0)$ ([1]). Before Fiedler's formulation, Lax [16] conjectured more strong result in 1958: the above conditions (i), (ii) for F implies the existence of a pair of real symmetric matrices H, K satisfying

$$F(t,x,y) = c \det(tI_n + xH + yK). \tag{1.6}$$

In 2007, Helton and Vinnikov [13] showed thar the Lax conjecture is true (cf. [17]). Hence the Filedler conjecture is true.

We shall consider the determinantal representations of a homogeneous polynomial. Whether a complex homogeneous polynomial $F(x_1, x_2, \ldots, x_m)$ ($m \geq 2$) with Degree n in m indeterminates $x_1, \ldots, x_m$ can be represented as

$$F(x_1, x_2, \ldots, x_n) = \det(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n), \tag{1.7}$$

for some $n \times n$ complex matrices $A_1, A_2, \ldots, A_n$ or not ?

In the case $m = 2$, the form F is expressed as

$$\prod_{j=1}^{n} (\alpha_j x_1 + \beta_j x_2).$$

Hence the diagonal matrices $A_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $A_2 = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ satisfy (1.7). The following results are known.

Theorem [A. C. Dixon, 1901, [9]] For every (non-zero) complex ternary form $F(t,x,y)$ of degree n, there are $n\times n$ complex symmetric matrices A_1, A_2, A_3 satisfying

$$F(t,x,y) = \det(tA_1 + xA_2 + yA_3).$$

Theorem [L. E. Dickson, 1920, [10]] A generic homogeneous polynomials in m variables of degree n has a representation

$$\det(x_1A_1 + x_2A_2 + \ldots + x_mA_m) = 0$$

by $n \times n$ matrices $A_1, A_2, \ldots, A_m$ if and only if

1. $m = 3$ (curves),
2. $m = 4$ and $n = 2, 3$ (surfaces),
3. $m = 4$ and $n = 2$ (threefolds).

Theorem [V. Vinnikov, 1993. [21]] An irreducible real algebraic curve $F(t,x,y) = 0$ has a representation

$$\det(tH_1 + xH_2 + yH_3) = 0, \tag{1.8}$$

by Hermitian matrices H_1, H_2, H_3.

We remark that if H_1 in (1.8) is positive definite, then the real ternary form $\det(tH_1 + xH_2 + yH_3)$ has the property (i) and (ii) mentioned in the above. In such a case, we have the equation

$$\det(tH_1+xH_2+yH_3) = \det(H_1)\det(tI+xH_1^{-1/2}H_2H_1^{-1/2}+yH_1^{-1/2}H_3H_1^{-1/2}).$$

An analogous object of $W(A)$ for a linear operator in an indefinite space satisfies some convexity property (cf. [2], [3], [19]).

We shall consider the joint numerical range of Hermitian matrices. Suupose that $\{H_1, H_2, \ldots, H_m\}$ is an ordered m-ple of $n \times n$ Hermitian matrices. The joint numerical range $W(H_1, H_2, \ldots, H_m)$ is defined as

$$W(H_1, H_2, \ldots, H_m) = \{(\xi^*H_1\xi, \xi^*H_2\xi, \ldots, \xi^*H_m\xi) : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}. \tag{1.9}$$

If $m = 3$, $n \geq 3$, the set $W(H_1, H_2, H_3) \subset \mathbf{R}^3$ is convex. In the case $H_3 = H_1^2 + H_2^2 + i(H_1H_2 - H_2H_1)$, the joint numerical range $W(H_1, H_2, H_3)$ is known as the Davis-Wielandt shell of a matrix $A = H_1 + iH_2$. By using the convexity of the joint numerical range $W(H_1, H_2, (H_1 + iH_2)^*(H_1 + iH_2))$ for $n \geq 3$, we can prove the convexity of the generalized numerical range

$$W_q(A) = \{\eta^* A\xi : \xi, \eta \in \mathbf{C}^n, \xi^*\xi = 1, \eta^*\eta = 1, \eta^*\xi = q\}$$

for an $n \times n$ matrix A and a real number $0 \leq q \leq 1$ (cf. [18], [5], [6]). In the case $q = 1$, the range $W_q(A)$ coincides with the numerical range $W(A)$. The set $W(H_1, H_2, H_3, H_4)$ is not necessarily convex.

Example Let

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_4 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let

$$\Pi = \{(1, x, y, z) : (x, y, z) \in \mathbf{R}^3\}.$$

Then we have

$$W(H_1, H_2, H_3, H_4) = \{(1, x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

Suppose that $\Delta = \text{Conv}(W(H_1, H_2, \ldots, H_m))$ contains $(0, 0, \ldots, 0)$ as an interior point. Then the set

$$\hat{\Delta} = \{(X_1, X_2, \ldots, X_m) \in \mathbf{R}^m, X_1x_1 + X_2x_2 + \ldots + X_mx_m + 1 \geq 0, \text{for}$$
$$(x_1, x_2, \ldots, x_m) \in W(H_1, H_2, \ldots, H_m)\}$$

is a compact convex set. Its boundary point $(X_1, X_2, \ldots, X_m)$ satisfies

$$\det(I_n + X_1H_1 + X_2H_2 + \cdots + X_mH_m) = 0, \quad \det(I_n + t[X_1H_1 + X_2H_2 + \cdots + X_mH_m]) > 0$$

for $0 \leq t < 1$. The coonnected compotent of the set

$$\{(Y_1, Y_2, \ldots, Y_m) \in \mathbf{R}^m : \det(I_m + Y_1H_1 + Y_2H_2 + \ldots + Y_mH_m) \neq 0\}, \quad (1.10)$$

containing $(0, 0, \ldots, 0)$ corresponds to the cross section of the positive cone

$$\{K = (a_{ij}) \in M_n(\mathbf{C}) : K = K^*, \xi^*K\xi > 0 \,\text{for}\xi \in \mathbf{C}^n, \xi \neq 0\}, \qquad (1.11)$$

with the affine plane

$$\{I_n + Y_1H_1 + Y_2H_2 + \ldots + Y_mH_m : (Y_1, Y_2, \ldots, Y_m) \in \mathbf{R}^m\}. \qquad (1.12)$$

Are there an m-ple of Hermitian matrices $H_1, H_2, \ldots, H_m$ and a constant c satisfying

$$F(x_0, x_1, x_2, \ldots, x_m) = c\det(x_0 I_n + x_1 H_1 + x_2 H_2 + \ldots + x_m H_m), \qquad (1.13)$$

if F is a form of degree n hyperbolic with respect to $(1, 0, \ldots, 0)$?

Example 1 Suppose that

$$F(t, x_1, x_2, x_3, x_4) = t^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

Then the form F is hyperbolic with respect to $(1, 0, 0, 0, 0)$. There is no ordered set (H_2, H_2, H_3, H_4) of 2×2 Hermian matrices satifying

$$t^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) = \det(tI_2 + x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4).$$

In fact we asume that there exist such Hermitian matrices H_1, H_2, H_3, H_4. For every point (x_1, x_2, x_3, x_4), we have

$$x_0^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) = (x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2})(x_0 - \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}) = 0$$

and hence $\mathrm{tr}(x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4) = 0$. Thus the Hermitian matrix $x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4$ is expressed as

$$L_1(x_1, x_2, x_3, x_4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + L_2(x_1, x_2, x_3, x_4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + L_3(x_1, x_2, x_3, x_4) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

where $L_j(x_1, x_2, x_3, x_4)$ $(j = 1, 2, 3)$ are linear functionals. We should have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = L_1(x_1, x_2, x_3, x_4)^2 + L_2(x_1, x_2, x_3, x_4)^2 + L_3(x_1, x_2, x_3, x_4)^2.$$

However this equation is impossible since the rank of the quadratic form in the right-hand side is less than or equal to 3 and the rank of the quadratic form in the lect-hand side is 4. Thus the expression as (1.9) is impossible.

Example 2 Suppose that

$$F(t, x_1, x_2, x_3, x_4, x_5) = t^3 - t(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2).$$

Then the form F is hyperbolic with respect to $(1, 0, 0, 0, 0, 0)$. The form $F(t, x_1, x_2, x_3, x_4, 0)$ is realized as

$$\det(\begin{pmatrix} t & x_1 + ix_2 & x_3 + ix_4 \\ x_1 - ix_2 & t & 0 \\ x_3 - ix_4 & 0 & t \end{pmatrix}.$$

Probably the form F itself can not be realized as $\det(tI_3 + x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4 + x_5 H_5)$ by 3×3 Hermitian matrices H_1, H_2, H_3, H_4, H_5. I can not so far prove such a non existence.

2. Henrion's method using Bezoutians

Consider the two polynomials in s, there are coefficients α_j, β_j so that

$$\phi_1(s) = \sum_{j=0}^{m} \alpha_j s^j, \tag{2.1}$$

$$\phi_2(s) = \sum_{j=0}^{m} \beta_j s^j. \tag{2.2}$$

The Bezoutian matrix of (2.1) and (2.2) is the $m \times m$ matrix

$$\mathrm{Bez} = (g_{i,j}),\ 1 \le i, j \le m),$$

where

$$g_{i,j} = \sum_{0 \le k \le \min(i-1, j-1)} (\alpha_{i+j-1-k}\beta_k - \alpha_k \beta_{i+j-1-k}). \tag{2.3}$$

The entries $g_{i,j}$ are characterized as

$$\frac{\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)}{s-t} = \sum_{i,j=1}^{m} g_{i,j} s^{i-1} t^{j-1}.$$

For example, when $m = 4$, the 4×4 Bezoutian matrix

$$\mathrm{Bez} = \{(g_{ij}),\ 1 \le i, j \le 4\} \tag{2.4}$$

is symmetric with entries

$$\begin{aligned}
g_{11} &= \alpha_1\beta_0 - \alpha_0\beta_1,\ \ g_{12} = \alpha_2\beta_0 - \alpha_0\beta_2, \\
g_{13} &= \alpha_3\beta_0 - \alpha_0\beta_3,\ \ g_{14} = \alpha_4\beta_0 - \alpha_0\beta_4, \\
g_{22} &= \alpha_3\beta_0 + \alpha_2\beta_1 - \alpha_1\beta_2 - \alpha_0\beta_3,\ \ g_{23} = \alpha_4\beta_0 + \alpha_3\beta_1 - \alpha_1\beta_3 - \alpha_0\beta_4, \\
g_{24} &= \alpha_4\beta_1 - \alpha_1\beta_4,\ \ g_{33} = \alpha_4\beta_1 + \alpha_3\beta_2 - \alpha_2\beta_3 - \alpha_1\beta_4 \\
g_{34} &= \alpha_4\beta_2 - \alpha_2\beta_4,\ \ g_{44} = \alpha_4\beta_3 - \alpha_3\beta_4
\end{aligned}$$

The two polynomials $\phi_1(s), \phi_2(s)$ have a non-constant common divisor $\psi(s)$ if and only if $\det(\mathrm{Bez}) = 0$.

Henrion [12] provided a more elementary method in the case $F(t, x, y) = 0$ is a rational curve. Henrion started from a parametrized form

$$x = \phi(s), \quad y = \psi(s), \tag{2.1}$$

of the rational curve $F(1, x, y) = 0$ by real rational functions in s.

We express the rational functions $\phi(s), \psi(s)$

$$\phi(s) = \frac{f(s)}{h(s)} \quad \psi(s) = \frac{g(s)}{h(s)}, \tag{2.2}$$

by real polynomials $f(s), g(s), h(s)$

We have

$$L_1(s) = h(s)\,x - f(s) = 0, \tag{2.3}$$

$$L_2(s) = h(s)\,y - g(s) = 0. \tag{2.4}$$

By these equations, he constructed real symmetric matrices H_1, H_2, H_3 satisfying

$$F(t,x,y) = \det(tH_1 + xH_2 + yH_3)$$

by using Bezoutians.

We shall treat the rational curve $F(1,x,y) = 0$ given as the graph of a trigonometric polynomial

$$z(\theta) = c_{-n}\exp(-in\theta)+\ldots+c_0+\cdots+c_n\exp(in\theta) = \sum_{j=-n}^{n} c_j \exp(\sqrt{-1}j\theta), \tag{2.5}$$

$(n = 1, 2, \ldots)$

Then we can obtain a real ternary form $F(t,x,y)$ of degree $2n$ satisfying

$$F(1, \Re(z(\theta)), \Im(z(\theta))) = 0$$

($0 \leq \theta \leq 2\pi$). One method to obtain the non-homogeneous $f(x,y) = F(1,x,y)$ is given as the following. We set $z = x + iy$ and $w = x - iy$ and $u = \exp(i\theta)$. We have

$$M_1(u) = -z\,u^m + c_m u^{2m} + \cdots + c_0 u^m + \cdots + c_{-m} = 0,$$

$$M_2(u) = -w\,u^m + \overline{c_{-m}} u^{2m} + \cdots + \overline{c_0} u^m + \cdots + \overline{c_m} = 0,$$

By using Sylvester determinant, we can eliminate u from these equations and obtain the polynomial $f(x,y)$. However this method does not provide us a method to construct Hermitian matrices H_1, H_2, H_3 satisfying (1.6).

We have another problem. When the form $F(t,x,y)$ assocated with the trigonometric polynomial (2.5) is hyperbolic with respect to $(1,0,0)$? By the condition $F(1,0,0) > 0$, the graph of the trigonometric polynomial does not pass through the origin 0 in the Gausian plane. In an early step, the author supposed the condition

$$|c_n| > \sum_{j=-n}^{n-1} |c_j|$$

for the form $F(t,x,y)$ to be hyperbolic with respect to $(1,0,0)$.

In a letter to the author, Prof. T. Nakazi provided a general condition for the form $F(t,x,y)$ to be hyperbolic under the condition

$$c_n > 0, \qquad (2.6),$$

$$\frac{d\mathrm{Arg}(z(\theta))}{d\theta} > 0$$

$(0 \le \theta \le 2\pi)$.

Nakazi's condition: The equation

$$c_n z^{2n} + \cdots + c_0 z^n + \cdots + c_{-n} = c_n \prod_{j=1}^{2n} (z - \alpha_j), \qquad (2.7)$$

holds for $|\alpha_j| < 1$ $(j = 1, 2, \ldots, 2n)$. His condition is deduced from Rouché's theorem.

Theorem[8] If a trigonometric polynomial

$$z(\theta) = \sum_{j=-n}^{n} c_j \exp(\sqrt{-1} j\theta)$$

satisfies the condition

$$c_n z^{2n} + \cdots + c_0 z^n + \cdots + c_{-n} = c_n \prod_{j=1}^{2n} (z - \alpha_j), \qquad (2.7)$$

for $|\alpha_j| < 1$, then the rational curve obtained as the graph of $z(\theta) = x(\theta) + iy(\theta)$ is realized as

$$\det(H_1 + xH_2 + yH_3) = 0$$

for some $2n \times 2n$ real symmetric matrices H_2, H_3 and a positive definite real symmetric matrix H_1.

To prove the positivity of the Hermitian matrix H_1, Hermite's classical theorem on zeros of a polynomial plays an important role. Let

$$p(z) = \sum_{j=0}^{n} \gamma_j z^j$$

be a polynomial in z with the leading coefficient $\gamma_n \neq 0$. We define two polynomials $\phi_1(z)$ and $\phi_2(z)$ by

$$\phi_1(z) = \sum_{j=0}^{n} \Re(\gamma_j) z^j, \quad \phi_2(z) = \sum_{j=0}^{n} \Im(\gamma_j) z^j.$$

The Bezout matrix of $\phi_2(z)$ and $\phi_1(z)$ is positive definite if and only if the roots of $p(z)$ are contained in the upper half plane $\Im(z) > 0$ (cf. [14], [22]). The graph of a special trigonometric polynomial is treated in [7]. A special rational curve associated with a nilpotent Toeplitz matrix is treated in [4].

Example We give an example to illustrate Hermite's theorem. Let $p(z) = (z-2i)(z-i) = z^2 - 3iz - 2$, $\phi_2(z) = 0 \cdot z^2 - 3z + 0$, $\phi_1(z) = z^2 + 0 \cdot z - 2$. Then the corresponding Bezoutian matrix is given by

$$\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}.$$

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