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Quarter of a Century in the Furuta Inequality

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In 1987, the Furuta Inequality was established in the paper;

T.Furuta, *$A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.

We would like to mention that 2011 is just the year as

"The 25th anniversary of the Furuta Inequality."

1. Road to Furuta Inequality

An operator means a bounded linear operator acting on a Hilbert space. The usual order $A \geq B$ among selfadjoint operators on H is defined by $(Ax, x) \geq (Bx, x)$ for $x \in H$. In particular, A is said to be positive and denoted by $A \geq 0$ if $(Ax, x) \geq 0$ for $x \in H$.

The noncommutativity of operators reflects on the order preservation.

The Löwner-Heinz inequality

$$(LH) \qquad A \geq B \geq 0 \ \Rightarrow \ A^p \geq B^p$$

if and only if $p \in [0, 1]$.

See [24], [21], [25] and [19]. The following is a quite familiar counterexample for which t^2 is not operator monotone;

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that t^p is not order-preserving for $p > 1$ by combining (LH).

The essense of the Löwner-Heinz inequality is the case $p = \frac{1}{2}$:

$$A \geq B \geq 0 \implies A^{\frac{1}{2}} \geq B^{\frac{1}{2}}.$$

It is rephrased as follows: For $A,\ B \geq 0$,

$$AB^2A \leq 1 \implies A^{\frac{1}{2}}BA^{\frac{1}{2}} \leq 1.$$

The assumption $AB^2A \leq 1$ is equivalent to $\|AB\| \leq 1$. Thus, noting the commutativity of the spectral radius, $r(XY) = r(YX)$, we have

$$\|A^{\frac{1}{2}}BA^{\frac{1}{2}}\| = r(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = r(AB) \leq \|AB\| \leq 1.$$

Related to the case $p = \frac{1}{2}$ in the Löwner-Heinz inequality, Chan-Kwong [3] conjectured that

$$(CK) \qquad A \geq B \geq 0 \implies (AB^2A)^{\frac{1}{2}} \leq A^2.$$

Moreover, if it is true, then the following inequality holds;

$$A \geq B \geq 0 \implies (BA^2B)^{\frac{3}{4}} \geq B^3.$$

Here we cite a useful lemma on exponent by Furuta.

Lemma 1. *For $p \in \mathbb{R}$, $(X^*A^2X)^p = X^*A(AXX^*A)^{p-1}AX$ holds for $A > 0$ and invertible X.*

Proof. It is easily checked that

$$Y^*(YY^*)^nY = Y^*Y(Y^*Y)^n \quad n \in \mathbb{N}.$$

This implies that

$$Y^*f(YY^*)Y = Y^*Yf(Y^*Y) \quad \text{for any polynomials } f$$

and so it holds for continuous functions f on a suitable interval. Hence we have the conclusion by applying it to $f(x) = x^{p-1}$ and $Y = AX$. $\square$

Using this trick, Chan-Kwong conjecture is modified in the sense that: If it is true, then

$$A \geq B \geq 0 \implies (AB^2A)^{\frac{3}{4}} \leq A^3.$$

As a matter of fact, we have

$$(AB^2A)^{\frac{3}{4}} = AB(BA^2B)^{-\frac{1}{4}}BA \qquad \text{by Lemma 1}$$

$$= AB((BA^2B)^{-\frac{1}{2}})^{\frac{1}{2}}BA \leq ABB^{-1}BA = ABA \leq A^3.$$

Under such consideration, the Furuta inequality was established in [16] cited in the prologue as follows:

Furuta inequality (FI) *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with

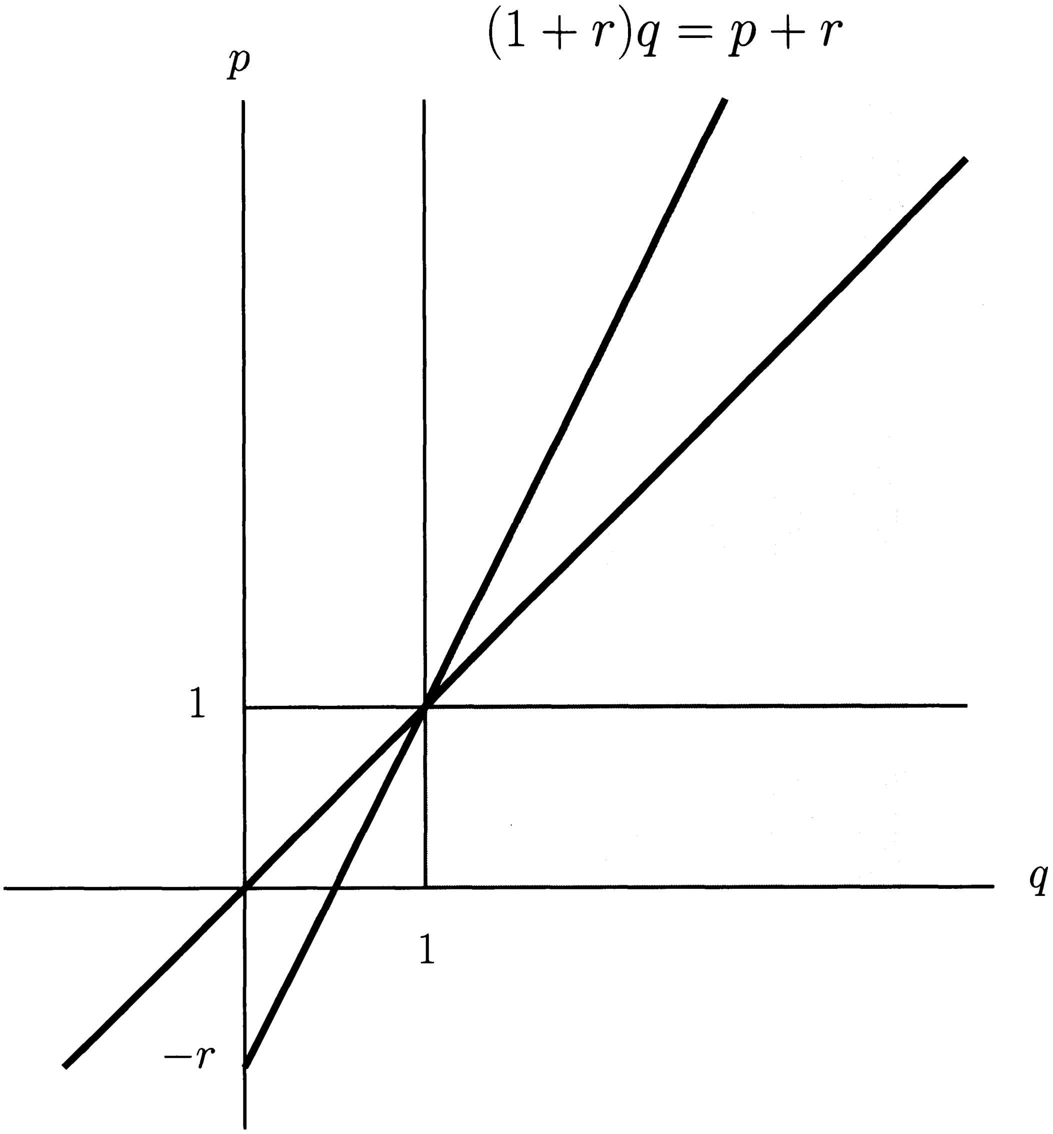
$(*)$ $$(1+r)q \geq p+r.$$

Remark. As a matter of fact, the above modification of (CK) is a critical point for $r = 2$ ($p = 2$ and $q = 4/3$), i.e., $(1+r)q = p+r$ holds. For the Furuta inequality, we refer [16], [17], [4], [22], [26] and [19]. In paticular, the best possibility of the domain determined by $(*)$ is proved by Tanahashi [26].

The figure $(*)$ is understood as the origin of the idea of Furuta inequality. As a matter of fact, Professor Berberian said that the figure determined by $(*)$ is

"Rosetta Stone"

in (FI). The figure $(*)$ is drawed in the next page. By virtue of (LH), it is easily seen that the case where the equality holds in $(*)$, i.e., $(1+r)q = p+r$, is essential in the Furuta inequality. It is reflected in the discussion of Section 3. Precisely it appears as $\frac{1+r}{p+r}$, the index of # in (FI).



2. Chaotic order

We first remark that $\log x$ is operator monotone, i.e., $A \geq B > 0$ implies $\log A \geq \log B$ by (LH) and $\frac{X^p - 1}{p} \to \log X$ for $X > 0$. By this fact, we can introduce the chaotic order as $\log A \geq \log B$ among positive invertible operators, which is weaker than the usual order $A \geq B$. We say it the chaotic order. In this section, we consider Furuta inequality under the chaotic order. We refer [1], [5], [7], [8], and [28] for an elegant proof.

We now recall the Chan-Kwong conjecture (CK):

$$(CK) \qquad A \geq B \geq 0 \implies (AB^2A)^{\frac{1}{2}} \leq A^2.$$

A direct progress of (CK) was done by Ando [1]. In our situation, it is expressed as follows:

Theorem 2. *The following assertions are equivalent for $A,\ B > 0$:*

(i) $A \gg B$, *i.e.*, $\log A \geq \log B$,

(ii) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for* $p \geq 0$.

We added it to 2-variables version in [5] as follows:

Theorem 3. *The following assertions are mutually equivalent for $A,\ B > 0$:*

(i) $A \gg B$, *i.e.*, $\log A \geq \log B$,

(ii) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for* $p \geq 0$,

(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for* $p,\ r \geq 0$.

Proof. We prove the implications: (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): First we note that $(1 + \frac{\log X}{n})^n \longrightarrow X$ for $X > 0$. Since

$$A_n = 1 + \frac{\log A}{n} \geq B_n = 1 + \frac{\log B}{n} > 0$$

for sufficiently large n, Furuta inequality ensures that for given $p,\ r > 0$

$$A_n{}^{1+nr} \geq (A_n{}^{\frac{nr}{2}} B_n{}^{np} A_n{}^{\frac{nr}{2}})^{\frac{1+nr}{n(p+r)}},$$

or equivalently

$$A_n{}^{n(\frac{1}{n}+r)} \geq (A_n{}^{n\frac{r}{2}} B_n{}^{np} A_n{}^{n\frac{r}{2}})^{\frac{1}{n(p+r)}+\frac{r}{p+r}}.$$

Taking $n \to \infty$, we have the desired inequality (iii).

(iii) $\Rightarrow$ (ii) is trivial by setting $r = p$.

(ii) $\Rightarrow$ (i): Note that $\frac{X^p - 1}{p} \to \log X$ for $X > 0$. The assumption (ii) implies that

$$\frac{A^p - 1}{p} \geq \frac{(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} - 1}{p} = \frac{A^{\frac{p}{2}} B^p A^{\frac{p}{2}} - 1}{p((A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} + 1)} = \frac{A^{\frac{p}{2}} (B^p - 1) A^{\frac{p}{2}} + A^p - 1}{p((A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} + 1)}.$$

Taking $p \longrightarrow +0$, we have

$$\log A \geq \frac{\log B + \log A}{2}, \text{ that is, } \log A \geq \log B.$$

So the proof is complete. □

Remark 1. The order preserving operator inequality (i) ⇒ (iii) in above is called chaotic Furuta inequality, simply (CFI). We here note that (iii) ⇒ (i) is directly proved as follows:

Take the logarithm on both side of (iii), that is,

$$r \log A \geq \frac{r}{p+r} \log A^{\frac{r}{2}} B^p A^{\frac{r}{2}}$$

for $p,\ r \geq 0$. Therefore we have

$$\log A \geq \frac{1}{p+r} \log A^{\frac{r}{2}} B^p A^{\frac{r}{2}}.$$

So we put $r = 0$ in above. Namely it implies that

$$\log A \geq \frac{1}{p} \log B^p = \log B.$$

3. Mean theoretic expression

We cite the weighted geometric mean $\#_\alpha$ for $\alpha \in [0,1]$, see [23] for the theory of operator means, and a related binary operation $\natural_s$ for $s \notin [0,1]$:

$$A \#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{and} \quad A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s A^{\frac{1}{2}}.$$

We cite a useful lemma which we will use frequently in the below.

Lemma 4. *For $X,\ Y > 0$ and $a,\ b \in [0,1]$,*

(i) *monotonicity:*

$$X \leq X_1 \quad \textit{and} \quad Y \leq Y_1 \implies X \#_a Y \leq X_1 \#_a Y_1,$$

(ii) *transformer equality:*

$$T^*XT \#_a T^*YT = T^*(X \#_a Y)T \quad \textit{for invertible } T,$$

(iii) *transposition:* $X \#_a Y = Y \#_{1-a} X$,

(iv) *multiplicativity:* $X \#_{ab} Y = X \#_a (X \#_b Y)$.

Proof. First of all, (iii) follows from Lemma 1.2, and (iv) does from a direct computation.

To prove (i), we may assume that $X, Y > 0$. If $Y \le Y_1$, then $X \#_a Y \le X \#_a Y_1$ is assured by (LH) (and the formula of $\#_a$). We prove (ii). We put $Z = X^{\frac{1}{2}}T = U|Z|$, the polar decomposition of Z, where U is unitary. Then it follows that

$$\begin{aligned}
T^*XT \#_a T^*YT &= Z^*Z \#_a T^*YT \\
&= |Z|(|Z|^{-1}T^*YT|Z|^{-1})^a|Z| \\
&= |Z|(|Z|^{-1}Z^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})Z|Z|^{-1})^a|Z| \\
&= |Z|(U^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})U)^a|Z| \\
&= |Z|U^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^aU|Z| \\
&= Z^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^aZ \\
&= T^*X^{\frac{1}{2}}(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^aX^{\frac{1}{2}}T \\
&= T^*(X \#_a Y)T.
\end{aligned}$$

□

In this context, the Furuta inequality has the following expression:

The Furuta inequality (FI). If $A \ge B > 0,\ t \in [0,1]$, then

$$A^{-r}\#_{\frac{1+r}{p+r}} B^p \le A \quad \text{for } r \ge 0 \text{ and } p \ge 1.$$

We recall Theorem 3 (iii); a chaotic version of (FI)

$$\log A \ge \log B \iff A^r \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \quad \text{for} \quad p,\ r \ge 0.$$

It leads us a weaker form than Theorem 3. (The assumption is stronger, but conclusion is the same as Theorem 3 (iii).)

The chaotic Furuta inequality (CFI)

$$A \ge B > 0 \implies A^{-r} \#_{\frac{r}{p+r}} B^p \le I \quad \text{for} \quad p \ge 0 \text{ and } r \ge 0.$$

Closely related to (FI), we here note a satellite of it due to Kamei [22]:

Satellite of Furuta inequality (SF).

$$A \geq B > 0 \Rightarrow A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \ (\leq A) \quad \text{for } r \geq 0,\ p \geq 1.$$

The meaning of (SF) looks like SF in the following sense: If $A \gg B$, then

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \quad \text{for } r \geq 0 \text{ and } p \geq 1.$$

That is, (SF) holds under the chaotic order. As a matter of fact, since $B^p \#_{\frac{p}{p+r}} A^{-r} = A^{-r} \#_{\frac{r}{p+r}} B^p \leq 1$, we have

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p = B^p \#_{\frac{p-1}{p+r}} A^{-r} = B^p \#_{\frac{p-1}{p}} (B^p \#_{\frac{p}{p+r}} A^{-r}) \leq B^p \#_{\frac{p-1}{p}} 1 = 1 \#_{\frac{1}{p}} B^p = B.$$

On the other hand, Ando-Hiai [2] established a log-majorization inequality, whose principal part is the following;

Ando-Hiai inequality (AH).

$$(AH) \qquad X \#_\alpha Y \leq 1 \ \Rightarrow \ X^r \#_\alpha Y^r \leq 1 \quad \text{for } r \geq 1,$$

Theorem 5. (FI), (CFI) *and* (AH) *are mutually equivalent:*

Proof. Suppose that (CFI) holds. To prove (FI), we assume $A \geq B > 0$. Then

$$\begin{aligned} A^{-r} \#_{\frac{1+r}{p+r}} B^p &= B^p \#_{\frac{p-1}{p+r}} A^{-r} = B^p \#_{\frac{p-1}{p}} (B^p \#_{\frac{p}{p+r}} A^{-r}) \\ &= B^p \#_{\frac{p-1}{p}} (A^{-r} \#_{\frac{r}{p+r}} B^p) \leq B^p \#_{\frac{p-1}{p}} I = B \leq A, \end{aligned}$$

which means that (FI) is shown.

Next we suppose that (FI) holds. Then we prove (AH), so that we assume $A \#_\alpha B \leq I$ and $r \geq 0$. Then, putting $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ and $p = \frac{1}{\alpha} > 1$, we have

$$B_1 = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha = C^{\frac{1}{p}} \leq A^{-1} = A_1.$$

Applying (FI) to $A_1 \geq B_1$, it follows that

$$A_1^{-r} \#_{\frac{r}{p+r}} B_1^p \leq I \quad \text{for } r \geq 0.$$

Moreover it follows that for $p \geq 1$,

$$A_1^{-r} \#_{\frac{1+r}{p+r}} B_1^p = B_1^p \#_{\frac{p-1}{p+r}} A_1^{-r} = B_1^p \#_{\frac{p-1}{p}} (B_1^p \#_{\frac{p}{p+r}} A_1^{-r})$$

$$= B_1^p \#_{\frac{p-1}{p}} (A_1^{-r} \#_{\frac{r}{p+r}} B_1^p) \leq B_1^p \#_{\frac{p-1}{p}} I = B_1 \leq A_1.$$

Summing up the above discussion, for each $p > 1$,

$$A \#_{\frac{1}{p}} B \leq I \ \Rightarrow \ A^r \#_{\frac{1+r}{p+r}} A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-1}, \text{ or } A^{r+1} \#_{\frac{1+r}{p+r}} B \leq I \text{ for } r \geq 0.$$

Note that, putting $q = \frac{p+r}{p-1} \geq 1$,

$$B \#_{\frac{1}{q}} A^{r+1} = B \#_{\frac{p-1}{p+r}} A^{r+1} = A^{r+1} \#_{\frac{1+r}{p+r}} B \leq I$$

holds. Hence, applying the above

$$A \#_{\frac{1}{p}} B \leq I \ \Rightarrow \ A^{r+1} \#_{\frac{1+r}{p+r}} B \leq I$$

for $q = \frac{p+r}{p-1} \geq 1$ and $B \#_{\frac{1}{q}} A^{r+1} \leq 1$, it implies that

$$I \geq B^{r+1} \#_{\frac{1+r}{q+r}} A^{r+1}.$$

Since $1 - \frac{1+r}{q+r} = \frac{1}{p}$,

$$I \geq B^{r+1} \#_{\frac{1+r}{q+r}} A^{r+1} = A^{r+1} \#_{\frac{1}{p}} B^{r+1}.$$

Namely we obtain (AH).

Finally we prove (AH) $\Rightarrow$ (CFI). So we assume that $A \geq B > 0$ and $p, r > 1$ because it holds for $0 \leq p, r \leq 1$ by (LH). For given $p, r > 1$, we put $\alpha = \frac{r}{p+r}$ and $r_1 = \frac{r}{p}$. Then we have

$$A^{-r_1} \#_{\frac{r_1}{1+r_1}} B \leq A^{-r_1} \#_{\frac{r_1}{1+r_1}} A = I.$$

We here apply (AH) to this and so we have

$$I \geq A^{-r_1 p} \#_{\frac{r_1}{1+r_1}} B^p = I \geq A^{-r_1 p} \#_{\frac{r_1 p}{p+r_1 p}} B^p = A^{-r} \#_{\frac{r}{p+r}} B^p,$$

as desired. $\square$

4. Generalization of Ando-Hiai inequality

Recall the Ando-Hiai inequality:

If $A \#_\alpha B \le I$ *for* $A, B > 0$, *then* $A^r \#_\alpha B^r \le I$ *for* $r \ge 1$.

Based on an idea of Furuta inequality, we propose two variables version of Ando-Hiai inequality, see [6], [11], [12], [13] and [14]:

Theorem 6 (Generalized Ando-Hiai inequality (GAH)). *For* $A,\ B > 0$ *and* $\alpha \in [0,1]$, *if* $A \#_\alpha B \le I$, *then*

$$A^r \#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^s \le I \quad for \quad r, s \ge 1.$$

It is obvious that the case $r = s$ in Theorem 6 is just Ando-Hiai inequality.

Now we consider two one-sided versions of Theorem 6:

Proposition 7. *For* $A,\ B > 0$ *and* $\alpha \in [0,\ 1]$, *if* $A \#_\alpha B \le I$, *then*

$$A^r \#_{\frac{\alpha r}{\alpha r+1-\alpha}} B \le I \quad for \quad r \ge 1.$$

Proposition 8. *For* $A,\ B > 0$ *and* $\alpha \in [0,\ 1]$, *if* $A \#_\alpha B \le I$, *then*

$$A \#_{\frac{\alpha}{\alpha+(1-\alpha)s}} B^s \le I \quad for \quad s \ge 1.$$

We investigate relations among them and Theorem 6.

Theorem 9. (1) *Propositions 7 and 8 are equivalent.*

(2) *Theorem 6 follows from Propositions 7 and 8.*

Proof. (1) We first note the transposition formula $X \#_\alpha Y = Y \#_\beta X$ for $\beta = 1 - \alpha$. Therefore Proposition 7 (for β) is rephrased as follows:

$$B \#_\beta A \le I \quad \Rightarrow \quad B^s \#_{\frac{\beta s}{\beta s+\alpha}} A \le I \quad \text{for} \quad s \ge 1.$$

Using the transposition formula again, it coincides with Proposition 8 because

$$1 - \frac{\beta s}{\beta s + \alpha} = \frac{\alpha}{\beta s + \alpha} = \frac{\alpha}{(1-\alpha)s + \alpha}.$$

(2) Suppose that $A \#_\alpha B \leq I$ and $r, s \geq 1$ are given. Then it follows from Proposition 7 that $A^r \#_{\alpha_1} B \leq I$ for $\alpha_1 = \frac{\alpha r}{\alpha r+1-\alpha}$. We next apply Proposition 8 to it, so that we have

$$1 \geq A^r \#_{\frac{\alpha_1}{\alpha_1+(1-\alpha_1)s}} B^s = A^r \#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^s,$$

as desired. □

We now point out that Proposition 7 is an equivalent expression of Furuta inequality of Ando-Hiai type:

Theorem 10. *Proposition 7 is equivalent to the Furuta inequality.*

Proof. For a given $p \geq 1$, we put $\alpha = \frac{1}{p}$. Then $A \geq B (\geq 0)$ if and only if

$$A^{-1} \#_\alpha B_1 \leq 1, \text{ for } B_1 = A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}. \tag{1}$$

If $A \geq B > 0$, then (2.1) holds for $A,\ B > 0$, so that Proposition 7 implies that for any $r \geq 0$

$$1 \geq A^{-(r+1)} \#_{\frac{\frac{r+1}{p}}{(1-\frac{1}{p})+\frac{r+1}{p}}} B_1 = A^{-(r+1)} \#_{\frac{1+r}{p+r}} B_1 = A^{-(r+1)} \#_{\frac{1+r}{p+r}} A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}.$$

Hence we have (FI);

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq A.$$

Conversely suppose that (FI) is assumed. If $A^{-1} \#_\alpha B_1 \leq 1$, then $A \geq (A^{\frac{1}{2}} B_1 A^{\frac{1}{2}})^\alpha = B$, where $p = \frac{1}{\alpha}$. So (FI) implies that for $r_1 = r - 1 \geq 0$

$$A \geq A^{-r_1} \#_{\frac{1+r_1}{p+r_1}} B^p = A^{-(r-1)} \#_{\frac{r}{p+r-1}} A^{\frac{1}{2}} B_1 A^{\frac{1}{2}}.$$

Since $\frac{r}{p+r-1} = \frac{\alpha r}{1+\alpha r-\alpha}$, we have Proposition 7. □

As in the discussion as above, Theorem 6 can be proved by showing Proposition 7. Finally we cite its proof. Since it is equivalent to the Furuta inequality, we have an alternative proof of it. It is done by the usual induction, whose technical point is a multiplicative property of the index $\frac{\alpha r}{(1-\alpha)+\alpha r}$ of $\#$ as appeared below.

Proof of Proposition 7. For convenience, we show that if $A^{-1} \#_\alpha B \le I$, then

$$A^{-r} \#_{\frac{\alpha r}{(1-\alpha)+\alpha r}} B \le I \quad \text{for} \quad r \ge 1. \tag{2.2}$$

Now the assumption says that

$$C^\alpha = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^\alpha \le A.$$

For any $\epsilon \in (0,1]$, we have $C^{\alpha\epsilon} \le A^\epsilon$ by (LH) and so

$$\begin{aligned} A^{-(1+\epsilon)} \#_{\frac{\alpha(1+\epsilon)}{(1-\alpha)+\alpha(1+\epsilon)}} B &= A^{-\frac{1}{2}}(A^{-\epsilon} \#_{\frac{\alpha(1+\epsilon)}{1+\alpha\epsilon}} A^{\frac{1}{2}} B A^{\frac{1}{2}}) A^{-\frac{1}{2}} \\ &\le A^{-\frac{1}{2}}(C^{-\alpha\epsilon} \#_{\frac{\alpha(1+\epsilon)}{1+\alpha\epsilon}} C) A^{-\frac{1}{2}} \\ &= A^{-\frac{1}{2}} C^\alpha A^{-\frac{1}{2}} = A^{-1} \#_\alpha B \le I. \end{aligned}$$

Hence the conclusion (2.2) is proved for $1 \le r \le 2$. So we next assume that (2.2) holds for $1 \le r \le 2^n$. Then the discussion of the first half ensures that

$$(A^{-r})^{r_1} \#_{\frac{\alpha_1 r_1}{(1-\alpha_1)+\alpha_1 r_1}} B \le I$$

holds for $1 \le r_1 \le 2$, where $\alpha_1 = \frac{\alpha r}{(1-\alpha)+\alpha r}$.

Thus the multiplicative property of the index

$$\frac{\alpha_1 r_1}{(1-\alpha_1)+\alpha_1 r_1} = \frac{\alpha r r_1}{(1-\alpha)+\alpha r r_1}$$

shows that (2.2) holds for all $r \ge 1$. □

We here consider an expression of (AH)-type for satellite of (FI): Suppose that $A^{-1} \#_\alpha B \le I$ and put $\alpha = \frac{1}{p}$. It is equivalent to $C = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{p}} \le A$. So (SF) says that

$$A^{-r} \#_{\frac{1+r}{p+r}} C^p \le C,$$

Multiplying $A^{-\frac{1}{2}}$ on both sides,

$$A^{-(r+1)} \#_{\frac{1+r}{p+r}} B \le A^{-\frac{1}{2}} C A^{-\frac{1}{2}} = A^{-1} \#_{\frac{1}{p}} B.$$

Namely (SF) has an (AH)-type representation as follows:

Theorem 11. *Let A and B be positive invertible operators. Then*

$$A \#_\alpha B \leq I \;\Rightarrow\; A^r \#_{\frac{\alpha r}{\alpha r+1-\alpha}} B \leq A \#_\alpha B \; (\leq I) \quad \textit{for} \;\; r \geq 1.$$

5. Grand Furuta Inequality

To compare with (AH) and (FI), we arrange (AH) as a Furuta type operator inequality. First of all, the assumption of (AH) $A \#_\alpha B \leq I$ is equivalent to that

$$B_1 = C^\alpha = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha \leq A^{-1} = A_1.$$

Similarly, the conclusion $A^r \#_\alpha B^r \leq I$ is equivalent to that

$$A^{-r} \geq [A^{-\frac{r}{2}} B^r A^{-\frac{r}{2}}]^\alpha = [A^{-\frac{r}{2}} (A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}})^r A^{-\frac{r}{2}}]^\alpha.$$

Replacing $p = \alpha^{-1}$, (AH) is reformulated that

$$A_1 \geq B_1 > 0 \implies A_1^r \geq (A_1^{\frac{r}{2}} (A_1^{-\frac{1}{2}} B_1^p A_1^{-\frac{1}{2}})^r A_1^{\frac{r}{2}})^{\frac{1}{p}} \qquad (\dagger)$$

for $r \geq 1$ and $p \geq 1$.

Moreover, to make a simultaneous extension of both (FI) and (AH), Furuta added variables as in the case of (FI). Actually he paid his attention to $A^{-\frac{1}{2}}$ in ($\dagger$), presicely, he replaced it to $A^{-\frac{t}{2}}$ $(t \in [0,1])$. Consequently he established so-called grand Furuta inequality, simply (GFI). It is sometimes said to be generalized Furuta inequality. We refer [18], [19], [9], [10], [15], [27], [29], [30], and [20] for a generalization.

Theorem 12 (Grand Furuta inequality (GFI)). *If $A \geq B > 0$ and $t \in [0,1]$, then*

$$[A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

It is easily seen that

$$\text{(GFI) for } t = 1, \; r = s \iff \text{(AH)}$$

$$\text{(GFI) for } t = 0, \; (s = 1) \iff \text{(FI)}.$$

Next we point out that (GFI) for $t = 1$ includes both Ando-Hiai and Furuta inequalities. Since Ando-Hiai inequality is just (GFI; $t = 1$) for $r = s$, it suffices to check that

Furuta inequality is contained in (GFI; $t=1$). As a matter of fact, it is just (GFI; $t=1$) for $s=1$.

Theorem 13. *Furuta inequality* (FI) *is equivalent to* (GFI) *for* $t=s=1$.

Proof. We write down (GFI; $t=1$) for $s=1$: If $A \geq B > 0$, then

$$[A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})A^{\frac{r}{2}}]^{\frac{r}{p-1+r}} \leq A^r$$

for $p, r \geq 1$, or equivalently,

$$A^{-(r-1)} \#_{\frac{r}{p-1+r}} B^p \leq A$$

for $p, r \geq 1$. Replacing $r-1$ by r_1, (GFI; $t=1$) for $s=1$ is rephrased as follows: If $A \geq B > 0$, then

$$A^{-r_1} \#_{\frac{1+r_1}{p+r_1}} B^p \leq A$$

for $p \geq 1$ and $r_1 \geq 0$, which is nothing but Furuta inequality. □

Furthermore Theorem 6, generalized Ando-Hiai inequality, is understood as the case $t=1$ in (GFI):

Theorem 14. (GFI; $t=1$) *is equivalent to* (GAH).

Proof. (GFI; $t=1$) is written as

$$A \geq B > 0 \;\Rightarrow\; [A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}} \leq A^r$$

for $p, r, s \geq 1$. We here put

$$\alpha = \frac{1}{p}, \quad B_1 = A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}}.$$

Then we have

$$A \geq B > 0 \;\Leftrightarrow\; A^{-1} \#_{\frac{1}{p}} A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}} \leq 1 \;\Leftrightarrow\; A^{-1} \#_\alpha B_1 \leq 1$$

and for each $p, r, s \geq 1$

$$\begin{aligned} &[A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}} \leq A^r \\ \Longleftrightarrow\; &A^{-r} \#_{\frac{r}{(p-1)s+r}} (A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^s \leq 1 \\ \Longleftrightarrow\; &A^{-r} \#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B_1^s \leq 1. \end{aligned}$$

This shows the statement of Theorem 6 (GAH). □

6. The Löwner-Heinz property

In this section, we discuss the Löwner-Heinz property on (GFI). A family $\{F(t); t \in [0,1]\}$ of (operator) inequalities has the Löwner-Heinz property if $F(1)$ implies $F(t)$ for $t \in [0,1]$.

Theorem 15. *The family* (GFI; $t \in [0,1]$) *has the Löwner-Heinz property, i.e.,* (GFI; $t = 1$) *implies* (GFI; $t \in [0,1]$).

To prove this, we recall the following lemmas:

Lemma 16. *If $A \geq B > 0$ and $t \in [0,1]$, then*

$$A^t \natural_s B^p \leq B^{(p-t)s+t}$$

holds for $p \geq 1$ and $1 \leq s \leq 2$.

Proof. Since $A^{-t} \leq B^{-t}$ by (LH), we have

$$A^t \natural_s B^p = B^p(B^{-p} \#_{s-1} A^{-t})B^p \leq B^p(B^{-p} \#_{s-1} B^{-t})B^p = B^{(p-t)s+t}.$$

More generally, we know the following fact:

Lemma 17. *If $A \geq B > 0$ and $t \in [0,1]$, then*

$$(A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leq B \leq A$$

holds for $p, s \geq 1$.

Proof. We fix $p \geq 1$ and $t \in [0,1]$. By the privious Lemma and (LH), if $s \in [1,2]$, then

$$(\dagger) \qquad A \geq B > 0 \ \Rightarrow \ B_1 = (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leq B \leq A.$$

So assume ($\dagger$) for some $s \geq 1$, and prove that

$$B_2 = (A^t \natural_{2s} B^p)^{\frac{1}{2(p-t)s+t}} \leq B_1 \leq B.$$

Actually we apply ($\dagger$) to $B_1 \leq A$. Then we have

$$(A^t \natural_2 B_1^{p_1})^{\frac{1}{(p_1-t)2+t}} \leq B_1 \leq B, \text{ where } p_1 = (p-t)s+t;$$

$$(A^t \natural_2 B_1^{p_1})^{\frac{1}{(p_1-t)2+t}} = [A^t \natural_2 (A^t \natural_s B^p)]^{\frac{1}{(p-t)2s+t}} = (A^t \natural_{2s} B^p)^{\frac{1}{(p-t)2s+t}} = B_2.$$

Proof of Theorem 18. Suppose that (GFI; $t=1$) holds, i.e., if $A \geq B > 0$, then

$$A^{-r+1} \#_{\frac{r}{(p-1)s+r}} (A \natural_s B^p) \leq A$$

holds for all $p, r, s \geq 1$.

For given $0 < t < 1$, $r \geq t$, $p \geq 1$ and $A \geq B > 0$, we put

$C = (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}}$, $p_1 = (p-t)s+t$, $r_1 = r-t+1$

and $s_1 = 1$. Then it follows from the preceding lemma that

$C \leq A$; $p_1 \geq 1$, $r_1 \geq 1$ by $r \geq t$.

Hence, $A \geq C > 0$ and (GFI; $t=1$) imply that

$$A^{-r_1+1} \#_{\frac{r_1}{(p_1-1)s_1+r_1}} (A \natural_{s_1} C^{p_1}) \leq A$$

holds. Since $\frac{r_1}{(p_1-1)s_1+r_1} = \frac{r-t+1}{(p-t)s+r}$ and $C^{p_1} = A^t \natural_s B^p$, we have

$$A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A,$$

as desired.

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