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# Stable and unstable solutions to Laplace equations with nonlinear boundary conditions

Junichi Harada (Waseda University)

## 1 Introduction

We consider the Laplace equation with a nonlinear boundary condition.

$$\Delta u=0 \quad \text{in } \mathbb{R}_{+}^{n}, \qquad \partial_{\nu}u=u^{q} \quad \text{on } \partial\mathbb{R}_{+}^{n} \quad (u>0), \tag{1}$$

where $\mathbb{R}_{+}^{n}=\{x\in \mathbb{R}^{n};x_{n}>0\}$ and $\partial_{\nu}=-\partial/\partial x_{n}$. The existence and the nonexistence of positive solutions of (1) depends on a exponent $q>1$. It is known that if $q\in(1, n/(n-2))$, there are no positive solutions of (1) ([6]). On the other hand, there exists a family of positive solutions for $q=n/(n-2)$ ([2], [9]) and for $q>n/(n-2)$ ([3]). These existence and nonexistence theorems for positive solutions of (1) are completely corresponding to those for positive solutions of

$$-\Delta u=u^{p} \quad \text{in } \mathbb{R}^{n}. \tag{2}$$

It is known that (2) has no positive solutions if $p\in(1, (n+2)/(n-2))$ and has a family of positive radially symmetric solutions for $p\geq(n+2)/(n-2)$. Moreover there exists another critical exponent $p_{JL}>p_{S}$ defined by

$$p_{JL}=\begin{cases}\infty & \text{if } n\leq 10,\\ \dfrac{n-2\sqrt{n-1}}{n-4-2\sqrt{n-1}} & \text{if } n\geq 11.\end{cases}$$

This critical exponent $p_{JL}$ is known to be critical for asymptotic expansions and intersection properties of positive radially symmetric solutions of (2) ([8],[12]). A goal of this note is to introduce a new critical exponent corresponding to $p_{JL}$ and to study the properties of positive solutions of (1) for $q>n/(n-2)$.

**Definition 1.1.** *A function $f(x)\in C(\mathbb{R}_{+}^{n})$ is called $x_{n}$-axial symmetric if $f(x)$ can be expressed by $f(x)=f(|x'|, x_{n})$.*

In this note, we often use a polar coordinate:

$$r=|x|, \qquad \tan\theta=|x'|/x_{n}.$$

We introduce a singular solution of (1).

**Lemma 1.1** ([11]). *Let $q > (n-1)/(n-2)$. Then there exists a singular solution $\varphi_\infty(x) = V(\theta)r^{-1/(q-1)}$ of* (1), *where $V(\theta) > 0$ is a unique solution of*

$$\partial_{\theta\theta}V + (n-2)(\cot\theta)\partial_\theta V = \beta V \text{ in } (0,\pi/2), \quad \partial_\nu V = V^q \text{ on } \{\pi/2\},$$

*where $\beta = m_q((n-2) - m_q)$ and $m_q = 1/(q-1)$.*

A new critical exponent is defined as follows.

**Definition 1.2** (JL-critical exponent). *We set*

$$\mu(q) = \inf_{u\in H^1(\mathbb{R}^n_+)} \frac{\left(\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \int_{\partial\mathbb{R}^n_+} (q\varphi_\infty^{q-1})u^2 dx'\right)}{\int_{\partial\mathbb{R}^n_+} |x'|^{-1}u^2 dx'}.$$

*We call $q$ JL-supercritical if $\mu(q) > 0$, JL-critical if $\mu(q) = 0$ and JL-subcritical if $\mu(q) < 0$.*

**Remark 1.1.** *From the trace Hardy inequality* ([4]):

$$\int_{\partial\mathbb{R}^n_+} |x'|^{-1}u^2 dx' \le c_H \int_{\mathbb{R}^n_+} |\nabla u|^2 dx, \tag{3}$$

*$\mu(q)$ is expressed by $\mu(q) = c_H - qV(\pi/2)^{q-1}$. By using this expression, we can show that*

(i) *for $n \ge 3$ there exists $q_0 > n/(n-2)$ such that $\mu(q) < 0$ if $q \in (n/(n-2), q_0)$,*

(ii) *there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ there exists $q_1 > q_0$ such that $\mu(q) > 0$ if*
$q > q_1$.

To state our results, we prepare notations. Let $e_i(\theta)$ be the $i$'th eigenfunction of

$$\begin{cases} -\Delta_S e = \lambda e & \text{in } S^{n-1}_+, \\ \partial_\nu e = qV(\pi/2)^{q-1}e & \text{on } \partial S^{n-1}_+, \end{cases} \tag{4}$$

where $\Delta_S$ is the Laplace Beltrami operator on $S^{n-1}$ and set $B_R = \{x \in \mathbb{R}^n_+; |x| < R\}$. For simplicity of notations, we set

$$m_q = 1/(q-1).$$

**Theorem 1.1** (JL-supercritical, JL-critical). *There exists a family of $x_n$-axial symmetric solutions $\{u_\alpha(x)\}_{\alpha>0}$ satisfying the following properties.*

(a) $u_\alpha(x) = \alpha u_1(\alpha^{(q-1)}x), \quad u_\alpha(0) = \alpha,$

(b) $u_\alpha(x) < \varphi_\infty(x)$, $\lim_{\alpha\to\infty} u_\alpha(x) = \varphi_\infty(x)$,

(c) $u_{\alpha_1}(x) < u_{\alpha_2}(x)$ *if* $\alpha_1 < \alpha_2$,

(d) *there exist* $\gamma_2 > \gamma_1 > m_q$, $c_\alpha \neq 0$ *and* $c'_\alpha \in \mathbb{R}$ *such that*

$$u_\alpha(x) = \begin{cases} V(\theta)r^{-m_q} + c_\alpha e_1(\theta)r^{-\gamma_1} + O(r^{-\gamma_2}) & \text{if JL-supercritical,} \\ V(\theta)r^{-m_q} + (c_\alpha \log r + c'_\alpha)e_1(\theta)r^{-\gamma_1} + O(r^{-\gamma_2}) & \text{if JL-critical.} \end{cases}$$

*Moreover if* $u(x)$, $v(x)$ *are positive* $x_n$*-axial symmetric solutions satisfying* $u(0) = v(0)$ *and* $u(x), v(x) < \varphi_\infty(x)$, *then it holds that* $u(x) \equiv v(x)$.

**Theorem 1.2** (JL-subcritical). *There exists a family of* $x_n$*-axial symmetric solutions* $\{u_\alpha(x)\}_{\alpha>0}$ *satisfying*

$$u_\alpha(x) \leq c_\alpha(1 + |x|)^{-1/(q-1)}. \tag{5}$$

*Moreover one of the following asymptotic expansions holds.*

(i) *there exist* $\gamma_2 > \gamma_1 > m_q$, $A, c_\alpha \neq 0$ *and* $B_\alpha \in \mathbb{R}$ *such that*

$$u_\alpha(x) = V(\theta)r^{-m_q} + c_\alpha e_1(\theta)r^{-\gamma_1}\sin(A\log r + B_\alpha) + O(r^{-\gamma_2}),$$

(ii) *there exist* $\gamma_4 > \gamma_3 > m_q$ *and* $c_\alpha \neq 0$ *such that*

$$u_\alpha(x) = V(\theta)r^{-m_q} + c_\alpha e_2(\theta)r^{-\gamma_3} + O(r^{-\gamma_4}).$$

*Moreover if* $u(x)$, $v(x)$ *are positive* $x_n$*-axial symmetric solutions satisfying* (5) *and* $u(x) \geq v(x)$ *for* $|x| > R$ *with some* $R > 0$, *then it holds that* $u(x) \equiv v(x)$.

**Remark 1.2.** *A solution* $u_\alpha(x)$ $(\alpha > 0)$ *constructed in Theorem* 1.1 *and the singular solution* $\varphi_\infty(x)$ *do not intersect each other for JL-supercritical case and JL-critical case. On the other hand,* $u_\alpha(x)$ $(\alpha > 0)$ *constructed in Theorem* 1.2 *and the singular solution* $\varphi_\infty(x)$ *must intersect each other for JL-subcritcal case. Set*

$$Z_\alpha = \{x \in \mathbb{R}^n_+; u_\alpha(x) = \varphi_\infty(x)\}.$$

*For the case* (i) *in Theorem* 1.2, *it holds that for large* $R > 0$

$$Z_\alpha \setminus B_R \sim \{x \in \mathbb{R}^n_+ : A\log|x| + B_\alpha = k\pi,\ k \in \mathbb{N}\}.$$

*For the case* (ii) *in Theorem* 1.2, *it holds that for large* $R > 0$

$$Z_\alpha \setminus B_R \sim \{x \in \mathbb{R}^n_+ : e_2(\theta) = 0\}.$$

*Unfortunately, we do not know which case* (i) *or* (ii) *actually occurs.*

# 2 Proof

**Step 1-(i) Existence. (JL-supercritical case, JL-critical case)**
For JL-supercritical case and JL-critical case, we construct solutions of (1) satisfying (b) by a different way from [3]. For simplicity, we set

$$D_R = \{x \in \partial\mathbb{R}^n_+; |x| < R\}, \quad S_R = \{x \in \mathbb{R}^n_+; |x| = R\}.$$

First we construct suitable super-solutions.

**Lemma 2.1.** *There exists $\delta_0 > 0$ and a positive $x_n$-axial symmetric function $\bar{u} \in C^2(\overline{B}_{1+\delta_0})$ such that $\bar{u} \equiv \varphi_\infty$ in $B_1$, $\bar{u} < \varphi_\infty$ in $B_{1+\delta_0} \setminus \overline{B}_1$ and*

$$-\Delta\bar{u} \geq 0 \quad \text{in } B_{1+\delta_0}, \quad \partial_\nu\bar{u} = \bar{u}^q \quad \text{on } D_{1+\delta_0}.$$

We fix $\delta \in (0, \delta_0)$. Here we consider approximation problems.

$$\Delta u = 0 \quad \text{in } B_{1+\delta}, \quad \partial_\nu u = u^q \quad \text{on } D_{1+\delta}, \quad u = \bar{u} \quad \text{on } S_{1+\delta}, \tag{6}$$

where $\bar{u}$ is a function given in Lemma 2.1. Here we call $u(x)$ a weak solution of (6) if $u \in \{u \in H^1(B_{1+\delta}); u - \bar{u} = 0 \text{ on } S_{1+\delta}\}$ satisfies

$$\int_{B_{1+\delta}} \nabla u \cdot \nabla\psi = \int_{D_{1+\delta}} u^q\psi$$

for any $\psi \in C_c^\infty(\overline{B}_{1+\delta}\setminus\overline{S}_{1+\delta})$. To construct solutions of (6), we construct a monotone sequence $\{u_i(x)\}_{i\in\mathbb{N}}$. We set $u_0(x) = \bar{u}(x)$ and define $u_{i+1}(x)$ inductively by

$$\Delta u_{i+1} = 0 \quad \text{in } B_{1+\delta}, \quad \partial_\nu u_{i+1} = u_i^q \quad \text{on } D_{1+\delta}, \quad u_{i+1} = \bar{u} \quad \text{on } S_{1+\delta}.$$

Then $u_\delta(x) = \lim_{i\to\infty} u_i(x)$ gives a solution of (6). More precisely, we obtain the following lemma.

**Lemma 2.2.** *Let $q$ be JL-supercritical or JL-critical and $\delta_0 > 0$ be give in Lemma 2.1. Then for $\delta \in (0, \delta_0)$ there exits a positive $x_n$-axial symmetric weak solution $u_\delta \in H^1(B_{1+\delta})$ of (6) such that $u_\delta(x) \leq \bar{u}(x)$.*

Next we show a boundedness of $u_\delta(x)$ near the origin.

**Lemma 2.3.** *Let $q$ be JL-supercritical or JL-critical and $u_\delta(x)$ be a weak solution of (6) constructed in Lemma 2.2. Then it holds that $u_\delta \in L^\infty(B_{1+\delta})$.*

To show Lemma 2.3, we use a technique similar to [10] with local $L^\infty$-estimates. The following local $L^\infty$-estimates are easily derived from the argument of Theorem 8.17 in [5] with Lemma 2.1 in [7].

**Lemma 2.4.** *Let $u \in H^1(B_1)$ be a weak solution of*

$$-\Delta u + b(x)\cdot\nabla u + c(x)u = 0 \text{ in } B_1, \quad \partial_\nu u = K(x')u \text{ on } D_1$$

*with $K \in L^\gamma(D_1)$ for some $\gamma > n-1$. Then there exists $c > 0$ depending on $\|K\|_{L^\gamma(D_1)}$, $\|b\|_{L^\infty(B_1)}$ and $\|c\|_{L^\infty(B_1)}$ such that*

$$\|u\|_{L^\infty(B_{1/2})} \le c\|u\|_{L^2(B_1)}.$$

**Proof of existence of solutions of** (1). Let $u_\delta(x)$ be a function given in Lemma 2.2 and set $M_\delta = \sup_{B_{1+\delta}} u_\delta(x)$. Then Lemma 2.3 implies that $M_\delta < \infty$. First we claim that $\lim_{\delta\to 0} M_\delta = \infty$. On the contrary, suppose that there exist $M_0 > 0$ and a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that $\delta_i \to 0$ and $M_{\delta_i} \le M_0$. Then there exits a subsequence $\{\delta_i\}_{i\in\mathbb{N}}$, which is denoted by the same symbol such that $u_{\delta_i}(x)$ converges to some function $u_\infty(x)$ in $H^1(B_1)$ satisfying $u_\infty(x) \le \varphi_\infty(x) \le \bar{u}(x)$. It is easily verified that the function $u_\infty(x)$ is a positive bounded solution of

$$\Delta u_\infty = 0 \text{ in } B_1, \quad \partial_\nu u_\infty = u_\infty^q \text{ on } D_1, \quad u_\infty = \varphi_\infty \text{ on } S_1.$$

Hence from $u_\infty(x) \le \varphi_\infty(x)$ ($u_\infty \not\equiv \varphi_\infty$), we obtain

$$\int_{B_1} |\nabla(u_\infty - \varphi_\infty)|^2 < qV_B^{q-1}\int_{D_1} r^{-1}(u_\infty - \varphi_\infty)^2.$$

By the trace Hardy inequality (3) and $qV_B^{q-1} \le c_H$ (see Remark 1.1), it holds that $u_\infty \equiv \varphi_\infty$. However this contradicts to $u_\infty \in L^\infty(B_1)$. Hence the claim $\lim_{\delta\to 0} M_\delta = \infty$ is proved. We set

$$\bar{u}_\delta(x) = M_\delta^{-1} u_\delta(M_\delta^{-(q-1)}x).$$

Then $\bar{u}_\delta(x)$ is a solution of $\Delta \bar{u}_\delta = 0$ in $B_{M_\delta^{q-1}}$ and $\partial_\nu \bar{u}_\delta = \bar{u}_\delta^q$ on $D_{M^{q-1}}$. Since

$$M^{-1}\varphi_\infty(M^{-(q-1)}x) = \varphi_\infty(x)$$

for any $M > 0$, it is verified that $\bar{u}_\delta(x) \le \varphi_\infty(x)$ in $B_{M_\delta^{q-1}}$. Hence by $\lim_{|x|\to\infty}\varphi_\infty(x) = 0$, there exists $R > 0$ such that $\max_{B_R} \bar{u}_\delta(x) = 1$ for small $\delta > 0$. Thus taking $\delta \to 0$, we can obtain a positive $x_n$-axial symmetric solution $u(x)$ of (1) satisfying $\max_{x\in\mathbb{R}^n_+} u(x) = 1$ and $u(x) < \varphi_\infty(x)$. Finally we put

$$u_\alpha(x) = \alpha u(\alpha^{q-1}x) \quad (\alpha > 0).$$

Then $u_\alpha(x)$ is a solution of (1) and satisfies $u_\alpha(x) < \varphi_\infty(x)$, which completes the proof. □

**Step 1-(ii) Existence. (JL-subcritical case)** Since the singular solution $\varphi_\infty(x)$ is not stable in a sense of Definition 1.2, arguments in Step 1-(i) can not be applicable. To show the existence of solutions of (1) satisfying (5), we need another technique. Here we omit the detail.

**Step 2 Asymptotic expansion I.**
In this step, we obtain the first asymptotic expansion:

$$\lim_{|x|\to\infty} |x|^{1/(q-1)}u(x) = V(\theta) \quad \text{in } C([0,\pi/2]). \tag{7}$$

Let $u(x)$ be a solution of (1) given in Theorem 1.1 or Theorem 1.2 with $u(0) = 1$. To investigate the asymptotic expansion of $x_n$-axial symmetric solutions of (1), we introduce

$$v(t,\theta) = r^{1/(q-1)}u(r,\theta), \qquad r = e^t \quad (t \in \mathbb{R}).$$

Then $v(t,\theta)$ is a solution of

$$\begin{cases} v_{tt} + \alpha v_t - \beta v + \Delta_S v = 0 & \text{in } \mathbb{R} \times (0,\pi/2), \\ \partial_\theta v = v^q & \text{on } \mathbb{R} \times \{\pi/2\}, \end{cases} \tag{8}$$

where

$$\alpha = (n-2) - 2m_q, \qquad \beta = m_q((n-2) - m_q).$$

It is easily seen that $\alpha, \beta > 0$ if $q > n/(n-2)$. Define the energy function $E(t)$ associated with (8).

$$E(t) = \frac{1}{2}\|\partial_t v(t)\|_2^2 - \frac{\beta}{2}\|v(t)\|_2^2 - \frac{1}{2}\|\partial_\theta v(t)\|_2^2 + \frac{1}{q+1}v(t,\pi/2)^{q+1}.$$

Then it is easily verified that

$$\partial_t E(t) = -\alpha\|v_t(t)\|^2 \le 0.$$

For JL-supercritical case and JL-critical case, since $\varphi_\infty(x) \le |V|_\infty r^{-1/(q-1)}$, from (b) in Theorem 1.1, $v(t,\theta)$ is uniformly bounded on $\mathbb{R} \times (0,\pi/2)$. For JL-subcritical case, from (5), $v(t,\theta)$ is also uniformly bounded on $\mathbb{R} \times (0,\pi/2)$. Hence by a elliptic regularity theory, $v_t(t,\theta), v_\theta(t,\theta)$ are uniformly bounded on $\mathbb{R} \times (0,\pi/2)$. Therefore it follows that

$$\alpha \int_{-\infty}^{\infty}\int_0^{\pi/2} |v_t|^2 d\sigma dt = \lim_{t\to-\infty} E(t) - \lim_{t\to\infty} E(t) < \infty.$$

We set

$$v_\infty(\theta) = \lim_{t\to\infty} v(t,\theta).$$

Since $\lim_{t\to-\infty} E(t) = 0$ and $\partial_t E(t) < 0$, $v_\infty(\theta)$ is a nontrivial nonnegative solution of

$$\Delta_S v = \beta v \ \text{ in } (0,\pi/2), \qquad \partial_\theta v = v^q \ \text{ on } \{\pi/2\}.$$

From Lemma 1.1, it follows that $v_\infty(\theta) \equiv V(\theta)$. Thus (7) is derived.

**Step 3 Asymptotic expansion II.**
Finally we derive more precise asymptotic behavior than Step 2 and obtain (d) in Theorem 1.1 and (i), (ii) in Theorem 1.2. To obtain a higher expansion of $v(t,\theta)$, we study the asymptotic behavior of

$$w(t,\theta) = V(\theta) - v(t,\theta).$$

Then $w(t,\theta)$ is a solution of

$$\begin{cases} w_{tt} + \alpha w_t - \beta w + \Delta_S w = 0 & \text{in } \mathbb{R} \times (0,\pi/2), \\ \partial_\theta w = qV_B^{q-1} w + f(w) & \text{on } \mathbb{R} \times \{\pi/2\}, \end{cases}$$

where $f(w) = \left(V_B^q - (V_B - w)^q - qV_B^{q-1}w\right) = O(w^2)$ and $V_B = V(\pi/2)$. From Step 2, it follows that

$$\lim_{t\to\infty} w(t,\theta) = 0 \quad \text{in } C([0,\pi/2]).$$

By using eigenfunctions of (4), we expand $w(t,\theta)$ by

$$w(t,\theta) = \sum_{i=1}^{\infty} y_i(t) e_i(\theta).$$

The coefficient $y_i(t)$ satisfies

$$y_i'' + \alpha y_i' - (\beta + \lambda_i) y_i = x_i, \tag{9}$$

where

$$x_i(t) = f(w(t,\pi/2)) e_i(\pi/2).$$

The characteristic equation of (9) is given by

$$\gamma_i^2 + \alpha\gamma_i - (\beta + \lambda_i)\gamma_i = 0. \tag{10}$$

Then it is verified that
(A) for the case $i \geq 2$, (10) admits two real roots satisfying $\gamma_i^- < 0$, $\gamma_i^+ > 0$,
(B) for the case $i = 1$ (JL-supercritcal), (10) admits two real roots satisfying $\gamma_1^- < \gamma_1^+ < 0$,
(C) for the case $i = 1$ (JL-critcal), (10) admits one real root satisfying $\gamma_1 < 0$,
(D) for the case $i = 1$ (JL-subcritcal), (10) does not admit real roots.

Then it is verified that for the case (A)

$$\begin{aligned} y_i(t) = y_i(0)e^{\gamma_i^- t} &- \frac{e^{\gamma_i^- t}}{\sqrt{\alpha^2 + 4(\beta+\lambda_i)}} \int_0^t \left(e^{-\gamma_i^- s} - e^{-\gamma_i^+ s}\right) x_i(s)ds \\ &- \frac{e^{\gamma_i^+ t} - e^{\gamma_i^- t}}{\sqrt{\alpha^2 + 4(\beta+\lambda_i)}} \int_t^\infty e^{-\gamma_i^+ s} x_i(s)ds, \end{aligned}$$

for the case (B)

$$y_1(t) = y_1(0)e^{\gamma_1^- t} + \frac{y_1'(0) - \gamma_1^- y_1(0)}{\sqrt{\alpha^2 + 4(\beta + \lambda_1)}} \left(e^{\gamma_1^+ t} - e^{\gamma_1^- t}\right)$$
$$+ \int_0^t \frac{1 - e^{(2\gamma_1^- + \alpha)(t-s)}}{\sqrt{\alpha^2 + 4(\beta + \lambda_1)}} e^{\gamma_1^+ (t-s)} x_1(s) ds,$$

for the case (C)

$$y_1(t) = y_1(0)e^{\gamma_1 t} + (y_1'(0) - \gamma_1 y_1(0))te^{\gamma_1 t} + \int_0^t (t-s)e^{\gamma_1 (t-s)} x_1(s) ds$$

and for the case (D)

$$y_1(t) = \frac{1}{A}\left(\frac{\alpha}{2} y_1(0) + y_1'(0)\right)(\sin At)e^{-(\alpha t)/2} + y_1(0)(\cos At)e^{-(\alpha t)/2}$$
$$+ \frac{1}{A}\int_0^t ((\sin At)(\cos As) - (\sin As)(\cos At))\, e^{-\alpha(t-s)/2} x_1(s) ds,$$

where $A = \sqrt{|\alpha^2 + 4(\beta + \lambda_1)|}/2$. Since

$$\begin{array}{lll} \gamma_1^- < \gamma_1^+ < 0, \quad \gamma_{i+1}^- < \gamma_i^- \ (i \geq 1), \quad \gamma_i^+ > 0 \ (i \geq 2) & \text{if JL-supercritical,} \\ \gamma_1 < 0, \quad \gamma_{i+1}^- < \gamma_i^- \ (i \geq 1), \quad \gamma_i^+ > 0 \ (i \geq 2) & \text{if JL-critical,} \\ \gamma_{i+1}^- < \gamma_i^- < -\alpha/2 \ (i \geq 2), \quad \gamma_i^+ > 0 \ (i \geq 2) & \text{if JL-subcritical,} \end{array}$$

we obtain

$$w(t, \theta) \sim y_1(t)e_1(\theta) = \begin{cases} ce^{\gamma_1^+ t} e_1(\theta) + o\left(e^{\gamma_1^+ t}\right) & \text{if JL-supercritical,} \\ (ct + c')e^{\gamma_1^t} e_1(\theta) + o\left(e^{\gamma_1 t}\right) & \text{if JL-critical,} \\ c\sin(At + B)e^{-\alpha t/2} + o\left(e^{-\alpha t/2}\right) & \text{if JL-subcritical} \end{cases}$$

for some $c, c', B \in \mathbb{R}$. For JL-supercritical and JL-critical case, by using the same technique as [1], we can assure $c \neq 0$. Hence these asymptotic formula give (d) in Theorem 1.1. However for JL-subcritical case, we do not know $c \neq 0$. Hence (i) in Theorem 1.2 holds if $c \neq 0$, on the other hand, (ii) in Theorem 1.2 holds if $c = 0$.

# References


[1] S. Bae, J. Differential Equations **194** (2003) 460-499.

[2] M. Chipot, I. Shafrir, and M. Fila, Adv. Differential Equations **1** (1996) 91-110.

[3] M. Chipot, M. Chlebík, M. Fila, I. Shafrir, J. Math. Anal. Appl. **223** (1998) 429-471.

[4] J. Dávila, L. Dupaigne, M. Montenegro, Commun. Pure Appl. Anal. **7** (2008) 795-817.

[5] D. Gilbarg and N. S. Trudinger, " Elliptic Partial Differential Equations of Second Order, " Springer, Berlin, 1998.

[6] B. Hu, Differential Integral Equations **7** (1994) 301-313.

[7] K. Ishige, T. Kawakami, Calc. Var. Partial Differential Equations **39** (2010) 429-457.

[8] Y. Li, J. Differential Equations **95** (1992) 304-330.

[9] Y. Y. Li and M. Zhu, Duke Math. J. **80** (1995) 383-417.

[10] I. Peral, J. L. Vazquez, Arch. Rational Mech. Anal. **129** (1995) 201-224.

[11] P. Quittner, W. Reichel, Calc. Var. Partial Differential Equations **32** (2008) 429-452.

[12] X.Wang, Trans. Amer. Math. Soc. **337** (1993) 549-590.