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Global dynamics beyond the ground state energy for nonlinear wave equations

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This is an introduction for the recent joint work with Wilhelm Schlag (University of Chicago) and Joachim Krieger (EPFL). We study global behavior of solutions for the nonlinear Klein-Gordon equation (NLKG)

$$\ddot{u} - \Delta u + u = |u|^{p-1}u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \quad 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}, \tag{1}$$

the nonlinear Schrödinger equation (NLS)

$$i\dot{v} - \Delta v = |v|^{p-1}v, \quad v(t,x) : \mathbb{R}^{1+d} \to \mathbb{C}, \quad 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}, \tag{2}$$

and the nonlinear critical wave equation (NLW)

$$\ddot{w} - \Delta w = |w|^{4/(d-2)}w, \quad w(t,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \quad d \geq 3, \tag{3}$$

which have unstable ground states, i.e., the nontrivial stationary solution

$$u(t,x) = Q(x), \quad w(t,x) = W(x), \tag{4}$$

or the standing wave

$$v(t,x) = e^{-it\omega}Q(x) \quad (\omega > 0), \tag{5}$$

with the least energy and positive profiles. Here the conserved energy is defined respectively by

$$\begin{aligned} &E(u) = \int_{\mathbb{R}^d} \frac{|\dot{u}|^2 + |\nabla u|^2 + |u|^2}{2} - \frac{|u|^{p+1}}{p+1} dx, \\ &E(v) = \int_{\mathbb{R}^d} \frac{|\nabla v|^2 + \omega |v|^2}{2} - \frac{|v|^{p+1}}{p+1} dx, \quad E(w) = \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{2} - \frac{|w|^{p+1}}{p+1} dx. \end{aligned} \tag{6}$$

Hence the energy space is naturally defined by $(u, \dot{u}) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $v \in H^1(\mathbb{R}^d)$, or $(w, \dot{w}) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, in which we investigate the global dynamics.

The existence as well as the instability of the ground state is well known for the above equations. $Q(x)$ decays exponentially as $|x| \to \infty$, while $W(x)$ decays algebraically. The Lorentz or Galilei invariance of the equation generates a class of solitons with various traveling velocity from any stationary or standing wave solution.

Besides those solitons, it is known that these equations have so-called scattering solutions, which are asymptotic to some solutions of the free equation (the equation without the nonlinearity) as $t \to \infty$ in the energy space, and also blow-up solutions, for which the energy norm diverges or the energy concentrates in finite time (the latter happens only for the critical equation w). More precisely, it is characterized by a finite maximal existence time of the unique local solution which is strongly continuous in the energy space.

The goal of this study is to classify and predict global dynamics for all solutions or initial data in the energy space, revealing the topological relation of solutions with different evolutions. This is for now too ambitious, since there are too many possible scenarios and the global dynamics can be very complex. However, restricting the energy at most slightly above the ground state, we can completely classify the global dynamics, which shows in particular that even if the scattering, blowup and the solitons are mixed together, the dynamics can stay reasonably ordered, thanks to the global dispersion of the equations.

More precisely, our main results state that all initial data in the energy space with energy at most slightly above the ground state are split into 9 sets, according to the global behavior of the solution, which include scattering, solitons and blowup, as well as transitions among them from $t < 0$ to $t > 0$. The classification is given in terms of center-stable and center-unstable manifolds of the ground state, constructed globally under the energy restriction, as the hypersurfaces separating scattering and blowup solutions for $t > 0$ and $t < 0$, respectively. For simplicity, the full statement is given below in the model case: radial, cubic and 3D NLKG [3]. The other cases [5, 2, 4, 1, 6] can be understood as extension of it.

Theorem 1. *Let $d = p = 3$ for (NLKG). For each $\sigma = \pm$, let $\mathcal{S}_\sigma, \mathcal{B}_\sigma, \mathcal{T}_\sigma$ be the maximal subsets of $\mathcal{H} := H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ such that for each initial data $(u(0), \dot{u}(0)) \in \mathcal{H}$, the corresponding solution u of NLKG satisfies*

$$\begin{aligned}
(u(0), \dot{u}(0)) \in \mathcal{S}_\sigma &\implies u \text{ scatters as } \sigma t \to \infty,\\
(u(0), \dot{u}(0)) \in \mathcal{B}_\sigma &\implies u \text{ blows up in } \sigma t > 0, \\
(u(0), \dot{u}(0)) \in \mathcal{T}_\sigma &\implies u \mp Q \text{ scatters as } \sigma t \to \infty.
\end{aligned} \tag{7}$$

For each subset $X \subset \mathcal{H}$ and $\varepsilon > 0$, let X^ε be the restriction

$$X^\varepsilon = \{\varphi \in X \mid E(\varphi) < E(Q) + \varepsilon^2\}. \tag{8}$$

Then there exists $\varepsilon > 0$ such that the following holds. We have

$$\mathcal{H}^\varepsilon = \mathcal{S}_+^\varepsilon \cup \mathcal{B}_+^\varepsilon \cup \mathcal{T}_+^\varepsilon = \mathcal{S}_-^\varepsilon \cup \mathcal{B}_-^\varepsilon \cup \mathcal{T}_-^\varepsilon \quad (\textit{disjoint union}), \tag{9}$$

and each of the 9 intersections

$$\begin{aligned}
&\mathcal{S}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon, \quad \mathcal{S}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon, \quad \mathcal{S}_+^\varepsilon \cap \mathcal{T}_-^\varepsilon,\\
&\mathcal{B}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon, \quad \mathcal{B}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon, \quad \mathcal{B}_+^\varepsilon \cap \mathcal{T}_-^\varepsilon,\\
&\mathcal{T}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon, \quad \mathcal{T}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon, \quad \mathcal{T}_+^\varepsilon \cap \mathcal{T}_-^\varepsilon
\end{aligned} \tag{10}$$

contains infinitely many orbits. $\mathcal{S}_\pm^\varepsilon$ and $\mathcal{B}_\pm^\varepsilon$ are unbounded connected open sets, $\mathcal{T}_\pm^\varepsilon$ are unbounded smooth manifold of codimension 1 with two connected components, and $\mathcal{T}_+^\varepsilon \cap \mathcal{T}_-^\varepsilon$ is a smooth manifold of codimension 2 with two components within $O(\varepsilon)$ distance from $\pm Q$.

The key ingredient in the proof is what we call the one-pass theorem, which precludes orbits departing from and returning to a small neighborhood of the ground state. Once it is established, the analysis is essentially decomposed into a scattering region, a blowup region and the small neighborhood of the ground state. In each of them, some modification of the preceding works suffices.

Theorem 2. *Let $d = p = 3$ and $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Then there are small $\varepsilon > 0$, $O(\varepsilon)$-neighborhood U of $\pm Q$, a continuous function $\mathfrak{S} : \mathcal{H}^\varepsilon \setminus U \to \{\pm 1\}$ such that* *the following hold. For any solution u of NLKG in $\mathcal{H}^\varepsilon$, let $I(u) \subset \mathbb{R}$ be the maximal existence interval, and*

$$\begin{aligned} I_{\mathcal{S}}(u) &= \{t \in I(u) \mid \mathfrak{S}((u(t),\dot{u}(t)) = +1\}, \\ I_{\mathcal{B}}(u) &= \{t \in I(u) \mid \mathfrak{S}((u(t),\dot{u}(t)) = -1\}, \\ I_{\mathcal{T}}(u) &= \{t \in I(u) \mid (u(t),\dot{u}(t)) \in U\}. \end{aligned} \tag{11}$$

Then $I_{\mathcal{T}}(u)$ is connected, so $I_{\mathcal{S}}(u)$ and $I_{\mathcal{B}}(u)$ together have at most two connected components. $I_{\mathcal{S}}(u)$ consists of unbounded intervals, while $I_{\mathcal{B}}(u)$ consists of bounded intervals. For each $X = \mathcal{S}, \mathcal{B}, \mathcal{T}$, we have $(u(0), \dot{u}(0)) \in X_+$ iff $\exists T \in I(u)$ such that $(T, \infty) \cap I(u) \subset I_X(u)$. For those $(\varphi, \psi) \in \mathcal{H}$ close to $(\pm Q, 0)$, we have

$$\mathfrak{S}((\varphi,\psi)) = \operatorname{sign} \langle Q \mp \varphi | \rho \rangle, \tag{12}$$

where ρ is the ground state of the linearized operator $-\Delta + 1 - 3Q^2$, and for (φ, ψ) away from $(\pm Q, 0)$, we have

$$\mathfrak{S}((\varphi,\psi)) = \operatorname{sign} \int_{\mathbb{R}^3} [|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4] dx. \tag{13}$$

Moreover, $\mathfrak{S}$ is uniquely determined by the above two formulas.

The one-pass theorem is proved by combining the hyperbolic dynamics of the linearized equation near but slightly away from the ground state, and the variational analysis far from the ground state, together with a virial identity localized in space-time for an almost homoclinic orbit.

Despite the lengthy statements, the proofs are not technically complicated at all, which are based on numerous classical and recent ideas. The interested reader is invited to look first into the above simplest setting, either in the paper [3] or in the book [7], where one needs no estimate or computation which lasts for several lines.

References

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