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Lower bound of L^2 decay of the Navier-Stokes flow in the half space R_+^n

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1 Introduction

In this paper, we consider an asymptotic behavior in L^2 of weak solutions of the Navier-Stokes equations in the half-space $\mathbb{R}_+^n$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}_+^n \times (0,\infty) \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^n \times (0,\infty) \\ u = 0 & \text{on } \partial\mathbb{R}_+^n \times (0,\infty) \\ u(0) = a & \text{in } \mathbb{R}_+^n, \end{cases} \tag{N-S}$$

where $n \geq 3$, $\mathbb{R}_+^n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n ; x_n > 0\}$ denotes the upper half-space. Here, $u = u(x,t) = (u^1(x,t), \ldots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and pressure of the fluid at point $(x,t) \in \mathbb{R}_+^n \times (0,\infty)$, respectively, while $a = a(x) = (a^1(x), \ldots, a^n(x))$ is the given initial velocity.

In his celebrated paper [7], Leray proposed the problem whether or not weak solutions of (N-S) tend to zero in L^2 as the time goes to infinity. Masuda [8] first gave a partial answer to Leray's problem and clarified that the energy inequality of strong type plays an important role in L^2 decay of weak solutions. Here, we mean by the energy inequality of strong type:

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2\, d\tau \leq \|u(s)\|_2^2 \tag{1.1}$$

for almost all $s \geq 0$, including $s = 0$, and for all $t \geq s$. Leray called a weak solution with (1.1) a *turbulent solution.* Later on, exact algebraic decay rate of energy decay was shown by Schonbek [12], Kajikiya-Miyakawa [5] and Wiegner [21]. For example, by the asymptotic expansion of heat kernel, Fujigaki-Miyakawa showed that there exists a turbulent solution of (N-S) such that

$$\|u(t)\|_2 = O(t^{-\frac{n+2}{4}}) \quad \text{as} \quad t \to \infty, \tag{1.2}$$

if initial data $a \in L^2_\sigma(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n}(1+|x|)|a(x)|dx < 0$. They also found the necessary and sufficient condition that decay rate $t^{-(n+2)/4}$ is optimal. Furthermore, it is well known that the decay rate as in (1.2) is one of the nonlinear Duhamel term. Indeed, Kajikiya-Miyakawa [5] and Borchers-Miyakawa [1] proved that the decay rate of the difference between the nonlinear Navier-Stokes flow and the linear Stokes flow in L^2 for the case of $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Proposition 1.1 ([5],[1]). *Let $1 \le r < 2$ and $a \in L^2_\sigma$. If $1 \le r < 2n/(n+2)$, then every weak solution $u(t)$ of (N-S) with (1.1) satisfies*

$$\|u(t) - e^{-tA}a\|_2 = O(t^{-\frac{n+2}{4}}) \quad \textit{as} \quad t \to \infty, \tag{1.3}$$

where e^{tA} is the Stokes semigroup and A is the Stokes operator.

Form this proposition, it is easy to see that if we require the slower decay for the nonlinear Navier-Stokes flow, the linear Stokes flow is dominant and should be investigated. Here we note that our aim this article is to establish the lower bound of the energy decay for the Navier-Stokes flow in the half space, i.e.,

$$\|u(t)\|_2 \ge Ct^{-\alpha}, \quad t \gg 1, \tag{1.4}$$

where $n/4 \le \alpha < (n+2)/4$.

Our original motivation and background is based on the energy concentration phenomenon in the frequency space in order to investigate the asymptotic profile of the Naiver Stokes flow in the whole space $\mathbb{R}^n$. For this purpose, we consider the following asymptotic behavior:

$$\lim_{t\to\infty} \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} = 1 \tag{1.5}$$

where $\{E_\lambda\}_{\lambda\ge 0}$ is a family of projection operators on $L^2_\sigma(\mathbb{R}^n)$ associated with the spectral decomposition of the Stokes operator A. Furthermore, for the case of the whole space $\mathbb{R}^n$, $E_\lambda u$ can be regarded as a low frequency component of u in the frequency space. Indeed, we introduce a cut-off function χ_R:

$$\chi_R(\xi) := \begin{cases} 1 & |\xi| \le R \\ 0 & |\xi| > R. \end{cases}$$

Then by the Fourier transform, we see that

$$\widehat{E_\lambda u}(\xi) = \chi_{\sqrt{\lambda}}(\xi)\widehat{u}(\xi).$$

Hence, (1.5) meas that the partial energy of the lower frequency component of $u(t)$ up to $\sqrt{\lambda}$ becomes dominant over the whole energy of $u(t)$ as $t \to \infty$. So it is an interesting problem to clarified that whether concentration phenomenon (1.5) does occur or not, that what initial data causes (1.5) and that what λ energy concentrates.

To prove (1.5), we found out the following inequality:

$$1 - \frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2} \le \frac{1}{\lambda} \frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2}, \tag{1.6}$$

for all $t > 0$ and all $\lambda > 0$. Here, it is well-known that $\|\nabla u(t)\|_2$ decays with the same rate as one of L^p-L^q estimate of the Stokes semigroup e^{-tA}, if initial data $a \in L^r \cap L^2_\sigma$ with some $1 \le r < 2$. Hence, in order to prove the convergence of the L.H.S. in (1.6), it suffices to derive the lower bound of the decay of $\|u(t)|_2$ and to compare with each rate. However, the fastest decay of $\|\nabla u(t)\|_2$ is $t^{-(n+2)/4}$ via L^p-L^q estimate for $r = 1$. This is why we need such a slow decay (1.4).

In this direction, the author established precise behavior of solutions of the lower bound in $L^2(\mathbb{R}^n)$. We note that to derive such a slow decay, the analysis on the linear Stokes flow is essential. By the Fourier splitting method, the behavior at $t = \infty$ of the Stokes flow is controlled by the lower frequency component of initial data. Indeed, introducing a class $K^m_{\alpha,\delta}(\mathbb{R}^n)$ for initial data, defined by

$$K^m_{\alpha,\delta}(\mathbb{R}^n) := \{\phi \in L^2(\mathbb{R}^n)\,;\, |\hat{\phi}(\xi)| \ge \alpha|\xi|^m, \quad |\xi| \le \delta\}, \quad m \ge 0,\, \alpha, \delta > 0, \tag{1.7}$$

he [10] proved that if $a \in K^m_{\alpha,\delta}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with $1 < r < 2$, then the weak solution $u(t)$ of (N-S) satisfies

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \ge C(1+t)^{-\frac{n+2m}{4}}, \tag{1.8}$$

for $n = 2, 3, 4$. We note that the set $K^m_{\alpha,\delta}(\mathbb{R}^n)$ has a different character of the initial profile from that of [13, 14] and [9], and that in particular, our characterization covers the results of [12, 13, 14], when $0 \le m < 1$.

From this observation, to derive energy concentration (1.5), the slow decay of $\|u(t)\|_2$ is essential. Here, we notice that the method to derive the lower bound depends heavily on the Fourier transform. Hence it is interesting problem to establish the lower bound of the energy decay in other domains where the Fourier transform does not work well.

Next we consider the Navier-Stokes flow in the half-space $\mathbb{R}^n_+$. In the half-space, there are many results for the upper bound of the temporal decay of the Stokes flow and the Navier-Stokes flow. See, for instance, Borchers and Miyakawa [1], Fujigaki and Miyakawa [3, 4]. However, up to now, it seems that there are few results for the *lower bound* of the energy decay. In such a situation, [3, 4] obtained the same lower bound as (1.8) under some condition on initial data. Especially, in [4], it was clarified that the strong solution $u(t)$ of (N-S) satisfies $\|u(t)\|_2 \geq Ct^{-n/4}$ if and only if the Stokes flow $v(t)$ satisfies $\|v(t)\|_2 \geq Ct^{-n/4}$. As is mentioned in [4], it seems to be an interesting problem to characterize a class of the initial data which exhibits a lower bound of the Stokes flow in the half-space $\mathbb{R}^n_+$. In the present article, focusing on the profile of initial data, we investigate the lower bound such as (1.8) for weak solutions of (N-S) which satisfy the energy inequality of strong type (1.1) in the half-space $\mathbb{R}^n_+$. Our rate as in (1.8) improves the rate given by [3] like (**??**). Furthermore, we give a positive answer to the question of [4] for the slow decay of the Stokes flow by the concrete characterization of the initial data in $\mathbb{R}^n_+$ which is similar to (1.7).

To study on the asymptotic behavior of the Navier-Stokes flow in the half-space, we first consider the Stokes flow and establish the estimate from below in terms of the explicit solution formula given by Ukai [20]. In the whole space $\mathbb{R}^n$, a number of decay properties of lower bounds relies heavily on the Fourier transform. However, in order to overcome such difficulty, we split the variables of the initial data a with the following form:

$$a(x) = a'(x')\eta(x_n),$$

where $x = (x', x_n) \in \mathbb{R}^n$ and $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Moreover, under some restriction on a' and η, we notice that the property of a' is dominant to the slow decay of the Stokes flow. By this form, the problem is reduced to that on the lower dimensional whole space $\mathbb{R}^{n-1}$. Conversely, we see that the 2-dimensional initial data can be embedded in the 3-dimensional half-space $\mathbb{R}^3_+$ and also the whole space $\mathbb{R}^3$, where the slow decay properties are preserved. In the same manner, for every $n \in \mathbb{N}$, we find out a hierarchy structure between $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ for the decay of the lower bounds of solutions with respect to the initial data. On the other hand, instead of $K^m_{\alpha,\delta}(\mathbb{R}^n)$ as in (1.7), we introduce a more general profile on the lower frequency part on initial data such as

$$T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n) := \{\phi \in L^2(\mathbb{R}^n);\, |\hat{\phi}(\xi)| \geq \alpha|\xi_n|^m, \quad |\xi_n| \leq \gamma,\, |\xi'| \leq \delta\}, \quad (1.9)$$

for $m \geq 0$, $\alpha, \gamma, \delta > 0$, where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ and $\xi' := (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. It should be noted that the class $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ can be characterized in terms of the estimate from below of the low frequency $\xi = (\xi', \xi_n)$ in the ξ_n-direction. It turns out that such a profile of initial data only in one direction to ξ_n dominates the asymptotic behavior in time from below of the Stokes flow. We also note that by making use of $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$, we can improve the previous result in [10] for the whole space $\mathbb{R}^n$. By the virtue of Ukai's solution formula of the Stokes flow, the profile of initial data can be directly applicable to the exact exponent of the decay in (1.8). If we take $m = 0$ in (1.8) and (1.9), then we obtain such a lower bound as:

$$\|u(t)\|_2 \geq Ct^{-\frac{n}{4}} \quad t \gg 1. \tag{1.10}$$

In addition, if $|\widehat{a'}(\xi')| \leq M$ for near $\xi' = 0$, it is easy to see that

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{n}{4}}. \tag{1.11}$$

Therefore, (1.11) gives the optimal decay rate of the energy of the Navier-Stokes flow in the half-space $\mathbb{R}^n_+$ for such a initial data. Indeed, we construct an initial data which causes both (1.10) and (1.11), as an example in $T^0_{\alpha,\gamma,\delta}(\mathbb{R}^n)$.

2 Results

We consider the following assumption on initial data:

Assumption. *(A1)* $a(x) = (a^1(x), \ldots, a^{n-1}(x), 0) =: (a'(x), 0)$

(A2) $a'(x) = a''(x')\eta(x_n)$

(A3) $\eta(-x_n) = -\eta(x_n)$ *and* $|\widehat{\eta}(\xi_n)| \geq C$ *near* $\xi_n = 0$

(A4) $a'' \in T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$*, i.e.,* $|\widehat{a}''(\xi')| \geq C|\xi_{n-1}|^m$ *near* $\xi' = 0$

Now our results read:

Theorem 2.1. *Let $n \geq 3$, and let r and m satisfy either (i) or (ii):*

(i) $1 < r \leq 2n/(n+2)$, $0 \leq m < 1$,

(ii) $2n/(n+2) < r < 2n/(n+1)$, $0 \leq m < 2n/r - n - 1$.

If $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$ satisfies the assumptions (A1), (A2), (A3) and (A4) for some α, γ, $\delta > 0$, then there exist $T > 1$ and a constant $C > 0$ such that every weak solution $u(t)$ of (N-S) with (1.1) fulfills the estimate,

$$\|u(t)\|_2 \geq Ct^{-\frac{n+2m}{4}} \tag{2.1}$$

for all $t \geq T$.

Remark 2.1. (i) We note that (2.1) improves the result in [4] when $0 \le m < 1$.

(ii) The estimate (2.1) inspires us that the optimal decay rate for such an initial data seems to be $n/4$. Indeed, by taking $m = 0$ in (2.1), we obtain

$$Ct^{-\frac{n}{4}} \le \|u(t)\|_2 \le C_r(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}, \quad t > T, \tag{2.2}$$

for $a \in L^r(\mathbb{R}^n_+) \cap L^2(\mathbb{R}^n_+)$, $1 < r < 2$. Letting $r \to 1$ in (2.2) formally, we may expect an exact estimate both from below and above such that

$$Ct^{-\frac{n}{4}} \le \|u(t)\|_2 \le C(1+t)^{-\frac{n}{4}}, \quad t \ge T.$$

However, up to now, we do not establish any uniform estimate with respect to $1 < r < 2$ on the constant C_r in (2.2).

(iii) In addition to the case $m = 0$, if $|\widehat{a''}(\xi')| \le M$ for near $\xi' = 0$ and $|\widehat{\eta^*}(\xi_n)| \le M$ for near $\xi_n = 0$ then we obtain the optimal decay rate $n/4$ for such an initial data, since it holds that

$$Ct^{-\frac{n}{4}} \le \|u(t)\|_2 \le C(1+t)^{-\frac{n}{4}}, \quad t \ge T.$$

3 Stokes flow in the half-space $\mathbb{R}^n_+$

To prove our main theorem, it is essential to investigate the energy decay of the linear Stokes flow in the half-space. For this purpose, we first introduce some specific properties of solutions, $v = (v', v^n)$, $v' = (v^1, \ldots, v^{n-1})$, of the Stokes equations:

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \nabla p = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\ \text{div } v = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\ v = 0 \quad \text{on } \partial\mathbb{R}^n_+ \times (0,\infty) \\ v(0) = a \quad \text{in } \mathbb{R}^n_+. \end{cases} \tag{S}$$

Ukai [20] gave a explicit solution formula for (S). To state Ukai's formula we prepare some notations. Let $R = (R', R_n)$ with $R' = (R_1, \ldots, R_{n-1})$ and $S = (S_1, \ldots, S_{n-1})$ denote the Riesz transform over $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Each R_j (resp. S_j) is a bounded linear operator on $L^r(\mathbb{R}^n)$ (resp. $L^r(\mathbb{R}^{n-1})$), $1 < r < \infty$. For a function $f(x', x_n)$, we understand that S_j acts as a convolution with respect to the variables x', so S_j is regarded as a bounded operator on both $L^r(\mathbb{R}^n)$ and $L^r(\mathbb{R}^n_+)$, $1 < r < \infty$. Let $B = B_r = -\Delta$ be the Laplacian on $\mathbb{R}^n_+$ with domain $D(B) := W^{2,r}(\mathbb{R}^n_+) \cap W^{1,r}_0(\mathbb{R}^n_+)$. It is well known that $-B$ generates a bounded analytic semigroup $\{e^{-tB}\}_{t\geq 0}$ on $L^r(\mathbb{R}^n_+)$, $1<r<\infty$. More precisely, we have

$$e^{-tB}f = e^{t\Delta}f^*\big|_{\mathbb{R}^n_+}, \quad \text{for } f\in L^r(\mathbb{R}^n_+), \quad 1<r<\infty,$$

where $e^{t\Delta}$ is the usual heat operator on $\mathbb{R}^n$ and f^* denotes the odd extension with respect to variable x_n, i.e.,

$$f^*(x',x_n) := \begin{cases} f(x',x_n), & x_n>0,\\ -f(x',-x_n), & x_n<0.\end{cases}$$

The solution formula of Ukai [20] is now read:

Proposition 3.1 (Ukai [20]). *For $a\in L^r_\sigma(\mathbb{R}^n_+)$, $1<r<\infty$, the solution $v=(v',v^n)$ of (S) is expressed as*

$$v^n(t) = Ue^{-tB}[a^n + S\cdot a'], \quad v'(t) = e^{-tB}[a' - Sa^n] + Sv^n$$

where U is the bounded operator on $L^r(\mathbb{R}^n_+)$, indeed, $Uf = R'\cdot S(R'\cdot S - R_n)ef|_{\mathbb{R}^n_+}$, which is also expressed with the Fourier transform on $\mathbb{R}^{n-1}$ as

$$\widehat{Uf}(\xi',x_n) = |\xi'|\int_0^{x_n} e^{-|\xi'|(x_n-y)}\hat{f}(\xi',y)\,dy.$$

Here, ef denotes the zero extension of f from $\mathbb{R}^n_+$ over $\mathbb{R}^n$:

$$ef(x',x_n) = \begin{cases} f(x',x_n) & x_n>0\\ 0 & x_n<0.\end{cases} \tag{3.1}$$

Remark 3.1. In this paper, we use the Fourier transform with the following form:

$$\hat{f}(\xi) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)\,dx, \quad i:=\sqrt{-1}.$$

Furthermore, we note that the symbols of Riesz's operator R_j and S_j are

$$\sigma(R_j) = -i\xi_j/|\xi|, \quad j=1,\dots,n,$$
$$\sigma(S_j) = -i\xi_j/|\xi'|, \quad j=1,\dots,n-1,$$

which have opposite signs of ones in [20, 1].

With Ukai's solution formula for the linear Stokes flow in the half-space, we can directly calculate the Stokes flow if the initial data is given. Indeed, we have the following Theorem for the lower bound of the energy decay for the Stokes flow in the half-space:

Theorem 3.1 (The half-space). *Let $n \geq 3$ and put $v(t) = e^{-tA}a$. If $a \in L^2_\sigma(\mathbb{R}^n_+)$ satisfies assumptions (A1), (A2), (A3) and (A4) then the Stokes flow $v(t)$ satisfies*

$$\|v(t)\|_2 \geq Ct^{-\frac{n+2m}{4}} \quad \text{for } t \geq 1 \tag{3.2}$$

where $C = C(n, m, \alpha, \gamma, \delta) > 0$.

Since we focus on the initial data with splitting variables $a(x) = a''(x')\eta(x_n)$, the following lemma for the Stokes flow in the whole space plays an important role for $a''(x')$

Lemma 3.1 (The whole space). *Let $n \geq 2$ and put $v(t) = e^{-tA}a$ with the Stokes semigroup e^{-tA} on $L^2_\sigma(\mathbb{R}^n)$. If $a \in L^2_\sigma(\mathbb{R}^n) \cap T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ for some $m \geq 0$ and $\alpha, \gamma, \delta > 0$, then $v(t)$ satisfies*

$$\|v(t)\|_2 \geq Ct^{-\frac{n+2m}{4}} \quad \text{for } t \geq 1, \tag{3.3}$$

where $C = C(n, m, \alpha, \gamma, \delta) > 0$.

Proof. By Plancherel's theorem and Fubini's theorem, we have

$$\begin{aligned}
\|v(t)\|_2^2 = \|\hat{v}(t)\|_2^2 &\geq \int_{|\xi_1|\leq\gamma,\, |\xi'|\leq\delta} e^{-2t|\xi|^2} |\hat{a}(\xi)|^2 \, d\xi \\
&\geq \alpha^2 \int_{|\xi_n|\leq\gamma,\, |\xi'|\leq\delta} e^{-2t|\xi|^2} |\xi_n|^{2m} \, d\xi \\
&= \alpha^2 \left(\int_{|\xi_n|\leq\gamma} e^{-2t\xi_n^2} |\xi_n|^{2m} \, d\xi_n \right) \left(\int_{|\xi'|\leq\delta} e^{-2t|\xi'|^2} \, d\xi' \right) \\
&=: \alpha^2 I_1 \cdot I_2,
\end{aligned}$$

for all $t \geq 0$. By changing variables we have

$$\begin{aligned}
I_1 &= 2 \int_0^\gamma e^{-2t\xi_n^2} \xi_n^{2m} \, d\xi_n \\
&= 2 \int_0^{\sqrt{t}\gamma} e^{-2\rho^2} \left(\frac{\rho}{\sqrt{t}} \right)^{2m} \frac{d\rho}{\sqrt{t}} \\
&\geq 2t^{-\frac{2m+1}{2}} \int_0^\gamma e^{-2\rho^2} \rho^{2m} \, d\rho
\end{aligned}$$

for all $t \geq 1$. Similarly by polar coordinates $\xi' = \varrho\omega \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
I_2 &= (n-1)\omega_{n-1} \int_0^\delta e^{-2t\varrho^2} \varrho^{n-2} \, d\varrho \\
&= (n-1)\omega_{n-1} \int_0^{\sqrt{t}\delta} e^{-2\rho^2} \left(\frac{\rho}{\sqrt{t}} \right)^{n-2} \frac{d\rho}{\sqrt{t}} \\
&\geq (n-1)\omega_{n-1} t^{-\frac{n-1}{2}} \int_0^\delta e^{-2\rho^2} \rho^{n-2} \, d\rho,
\end{aligned}$$

for all $t \geq 1$, where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Therefore, we obtain (3.3) with a constant

$$C^2 = 2\alpha^2(n-1)\omega_{n-1}\left(\int_0^\gamma e^{-2\rho^2}\rho^{2m}\,d\rho\right)\left(\int_0^\delta e^{-2\rho^2}\rho^{n-2}\,d\rho\right).$$

This completes the proof of Lemma 3.1 □

Remark 3.2. We note that Lemma 3.1 still holds, if we replace $a \in L^2_\sigma(\mathbb{R}^n) \cap T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ and e^{-tA} by $a \in T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ and $e^{t\Delta}$ respectively.

Finally, with this lemma and theorem for the linear Stokes flow, we obtain main theorem for the nonlinear Navier-Stokes flow in the half-space.

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