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Mild solutions to the Navier-Stokes equations in unbounded domains with unbounded boundary

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Abstract

It is mathematically investigated the incompressible viscous flows in domains $\Omega \subset \mathbb{R}^n$ with nonslip boundary conditions in the framework of $L^p_\sigma(\Omega)$, where Ω has a possibly non-compact uniform C^3-boundary and boundedness of the Helmholtz projection $\mathbb{P}_p$ onto $L^p_\sigma(\Omega)$ with some $1 < p < \infty$. The key is to show that the Stokes operator generates an analytic semigroup on $L^p_\sigma(\Omega)$ admitting the maximal L^q-L^p-regularity estimates. Moreover, the local-in-time existence and the uniqueness of mild solutions to the Navier-Stokes equation in such Ω and $p \in (n, \infty)$ are proved, when the initial data belong to $L^p_\sigma(\Omega)$.

1 Introduction

This is a brief survey of the results related to [18], mainly.

For any open set $\Omega \subset \mathbb{R}^n$, it is well-known that the Stokes operator $A_2 := -\mathbb{P}_2\Delta$ (with nonslip boundary conditions) is a self adjoint operator in $L^2_\sigma(\Omega)$ by Masuda [28]. Hence, $-A_2$ is the generator of an analytic contraction semigroup $\{e^{-tA_2}\}_{t\geq 0}$ onto $L^2_\sigma(\Omega)$. Here, $L^2_\sigma(\Omega)$ is defined by the solenoidal part of the Helmholtz decomposition of $L^2(\Omega)$ into $L^2_\sigma(\Omega) \oplus G^2(\Omega)$, where $\oplus$ denotes the direct sum, and $\mathbb{P}_2$ denotes the Helmholtz projection from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$. It seems to be natural to investigate whether this technique can be applicable in general L^p-setting, that is, $\{e^{-tA_p}\}_{t\geq 0}$ extends to an analytic semigroup on an L^p_σ-space for some $1 < p < \infty$, and that there are the maximal L^q-L^p-regularity estimates for the solution of the associated Stokes equations. Once we obtain the above semigroup theory, we have a chance to construct the local-in-time mild solutions to the Navier-Stokes equations in $L^p_\sigma(\Omega)$ for $n \le p < \infty$ by the fixed point argument of Kato [26] or Giga-Miyakawa [22]. The notion of a mild solution was first introduced by Fujita-Kato [14, 27] when the initial velocity belongs to $H^{1/2}_\sigma(\Omega)$ with smooth bounded domains $\Omega \subset \mathbb{R}^3$ via Duhamel's principle at the almost same years of Browder [6] to study some equations of parabolic type.

It is clear to have the affirmative answer of the above question when Ω is the whole space or the half space (see Ukai [33] and Desch-Hieber-Prüß [7]) for any $p \in (1, \infty)$. For bounded or exterior domains with smooth boundaries, the maximal L^q-L^p-regularity estimates were firstly shown by Solonnikov [30]. His proof makes use of potential theoretic arguments. Later on, Giga [19, 20] also established the Stokes semigroup theory due to the bounded imaginary powers of the Stokes operator, Giga-Sohr [23] applied the Dore-Venni theorem in two-dimension case, Grubb-Solonnikov [24] used the pseudo-differential techniques, and Fröhlich [13] made use of the concept of weighted estimates with respect to Muckenhoupt weights. The reader can find related results in the list of reference in Farwig-Sohr [11]. Furthermore, the case of a perturbed half space is treated by e.g. Noll-Saal [29]. For results concerning infinite layers-like domains, we refer to the works of Abe-Shibata [1], Abels [2] and Abels-Wiegner [3]. Franzke [12] and Hishida [25] considered the case of aperture domains. Farwig-Ri [10] derived the maximal L^q-L^p-regularity estimates in infinite tube-like domains. In the domains listed-up above the Helmholtz decomposition is valid.

The key of this approach is to show the boundedness of the Helmholtz projection $\mathbb{P}_p$ on $L^p(\Omega)$ into its solenoidal subspace. For example, if Ω is bounded, then the boundedness of $\mathbb{P}_p$; this fact was first proved by Fujiwara-Morimoto [15].

On the other hand, in the case of general domains Ω, it is not clear whether the Helmholtz decomposition makes sense, that is, $L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$ or not, in general, unless $p = 2$. Indeed, Bogovskiĭ [4, 5] gave examples of unbounded domains Ω with smooth boundaries in which it is not enable to have the Helmholtz decomposition of $L^p(\Omega)$ for certain p. For details, see also [16]. To overcome the difficulties, Farwig-Kozono-Sohr [9] introduced

$$\tilde{L}^p(\Omega) := \begin{cases} L^2(\Omega) \cap L^p(\Omega), & 2 \le p < \infty, \\ L^2(\Omega) + L^p(\Omega), & 1 < p < 2. \end{cases}$$

for domains $\Omega \subset \mathbb{R}^3$ with uniform C^2-boundaries, proved the existence of the Helmholtz projection $\tilde{\mathbb{P}}$ in $\tilde{L}^p$ (assisted by L^2), and obtained the useful properties as usual in $L^p(\Omega)$. Moreover, they proved that the Stokes operator $A_p := -\tilde{\mathbb{P}}\Delta$ with nonslip boundary conditions is well-defined in $\tilde{L}^p_\sigma$, and generates an analytic semigroup onto $\tilde{L}^p_\sigma(\Omega)$ as well as the maximal L^q-$\tilde{L}^p$-regularity estimates in the class $L^q(\tilde{L}^p) := L^q((0,T); \tilde{L}^p(\Omega))$ for $T > 0$

$$\|u_t\|_{L^q(\tilde{L}^p)} + \|u\|_{L^q(\tilde{L}^p)} + \|\nabla^2 u\|_{L^q(\tilde{L}^p)} + \|\nabla\tilde{\pi}\|_{L^q(\tilde{L}^p)} \le C\|f\|_{L^q(\tilde{L}^p)}$$

with some constant $C > 0$ independent of $f \in L^q(\tilde{L}^p)$. Here $(u, \tilde{\pi})$ is a solution to the Stokes equations in domains Ω with $f \in L^q(\tilde{L}^p)$:

$$\begin{aligned} u_t - \Delta u + \nabla\tilde{\pi} &= f \quad && \text{in } \Omega \times (0,T), \\ \nabla \cdot u &= 0 && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u|_{t=0} &= 0 && \text{in } \Omega. \end{aligned} \tag{1.1}$$

In the paper [18] they however employed a different approach to [9]. For $\Omega \subset \mathbb{R}^n$ having a uniformly C^3-boundary with $p \in (1, \infty)$, it is assumed that the Helmholtz projection $\mathbb{P}_p$ exists bounded in $L^p(\Omega)$. They actually showed that $-A_p$ generates an analytic semigroup onto usual $L^p_\sigma(\Omega)$, which comes from the fact that solutions to the Stokes equation satisfies the maximal L^q-L^p-regularity estimates in $L^q((0,T); L^p(\Omega))$. They also obtained the local-in-time existence of a unique mild solution to the Navier-Stokes equations in $L^p_\sigma(\Omega)$ with $p > n$ under the assumption of the existence of the Helmholtz projection. Although it seems to be an interesting problem in the framework of $L^n_\sigma(\Omega)$ which is excluded by [18], the author has no idea to overcome the difficulties (for example, it is not clear whether $\mathbb{P}_p = \mathbb{P}_q$ if $p \neq q$) so far.

This paper is organized as follows. In Sections 2 we will state the main results of [18]. In Section 3 the strategy of their approach is explained.

2 Main Results

In this section we mention the main results in [18]. Here and hereafter, let $n \geq 2$. The definition of uniform C^k-domain for $k \in \mathbb{N}$ will be given in the next section. For any open set $\Omega \subset \mathbb{R}^n$ and for $p \in (1,\infty)$, we set

$$G^p(\Omega) := \{u \in L^p(\Omega); u = \nabla\tilde{\pi} \text{ for some } \tilde{\pi} \in W^{1,p}_{loc}(\Omega)\},$$
$$L^p_\sigma(\Omega) := \overline{\{u \in C^\infty_c(\Omega); \nabla\cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_p}.$$

We say that the Helmholtz projection $\mathbb{P} := \mathbb{P}_p$ exists for $L^p(\Omega)$, whenever $L^p(\Omega)$ can be decomposed into

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega).$$

In this case, there naturally exists a unique projection $\mathbb{P}_p : L^p(\Omega) \to L^p_\sigma(\Omega)$ having the properties $\mathbb{P}_p^2 = \mathbb{P}_p$ and $G^p(\Omega)$ as its null space. A well-known fact by e.g. [16] is that the Helmholtz projection exists for $L^p(\Omega)$ for $p \in (1,\infty)$ if and only if for every $f \in L^p(\Omega)$, there exists a unique function $u \in \widehat{W}^{1,p}(\Omega)$ satisfying

$$\langle\nabla u, \nabla\varphi\rangle = \langle f, \nabla\varphi\rangle, \qquad \varphi \in \widehat{W}^{1,p'}(\Omega).$$

Thus the Helmholtz projection exists for $L^p(\Omega)$ if and only if for every $f \in L^p(\Omega)$ the above weak Neumann problem is uniquely solvable within the class $\widehat{W}^{1,p}(\Omega)$. We now state the maximal L^q-L^p-regularity estimate for solutions to the Stokes equations, which is one of the main results of [18].

Theorem 2.1. *Let $n \geq 2$, $p, q \in (1,\infty)$ and $T > 0$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection $\mathbb{P}_p$ exists for $L^p(\Omega)$. Let $f \in L^q((0,T); L^p(\Omega))$. Then equation* (1.1) *admits a unique solution $(u, \tilde{\pi})$ in the class*

$$u \in W^{1,q}(L^p) \cap L^q(W^{2,p} \cap W^{1,p}_0 \cap L^p_\sigma) \quad \text{and} \quad \tilde{\pi} \in L^q(\widehat{W}^{1,p}),$$

and there exists a constant $C > 0$ such that

$$\|u_t\|_{L^q(L^p)} + \|u\|_{L^q(L^p)} + \|\nabla^2 u\|_{L^q(L^p)} + \|\nabla\tilde{\pi}\|_{L^q(L^p)} \leq C\|f\|_{L^q(L^p)}.$$

Assuming as in the above theorem that the Helmholtz projection $\mathbb{P}_p$ exists for $L^p(\Omega)$, we may define the Stokes operator $A = A_p$ in $L^p_\sigma(\Omega)$ as

$$
\begin{aligned}
D(A_p) &:= W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega), \\
A_p u &:= -\mathbb{P}_p \Delta u \qquad \text{for} \quad u \in D(A_p).
\end{aligned} \tag{2.1}
$$

The definition of the function spaces are found in e.g. the book of Triebel [32]. Concerning the Cauchy problem in $L^p_\sigma(\Omega)$, the following corollary holds true for the abstract equation with valued in the solenoidal subspace

$$
\begin{aligned}
u'(t) + A_p u(t) &= f(t), \quad t > 0, \\
u(0) &= u_0.
\end{aligned} \tag{2.2}
$$

Corollary 2.2. *Let $n \geq 2$, $p, q \in (1, \infty)$ and $T > 0$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection $\mathbb{P}_p$ exists for $L^p(\Omega)$. Then $-A_p$ defined as in* (2.1) *generates an analytic C_0-semigroup $\{e^{-tA_p}\}_{t \geq 0}$ onto $L^p_\sigma(\Omega)$. Moreover, the solution u to the problem* (2.2) *satisfies*

$$
\|u'\|_{L^q(L^p)} + \|A_p u\|_{L^q(L^p)} \leq C \left(\|f\|_{L^q(L^p)} + \|u_0\|_{B^{2-2/q}_{p,q}} \right)
$$

with some constant $C > 0$ independent of $f \in L^q((0,T); L^p_\sigma(\Omega))$ and $u_0 \in B^{2-2/q}_{p,q}(\Omega) \cap L^p_\sigma(\Omega)$.

Setting $\nabla\tilde{\pi} = (\mathbb{I} - \mathbb{P})\Delta R(\lambda, A) f$ for $f \in L^p(\Omega)$, where $\mathbb{I}$ denotes the identity matrix and $R(\lambda, A) := (\lambda + A)^{-1}$, we can also obtain the following results for the Stokes resolvent problem

$$
\begin{aligned}
\lambda u - \Delta u + \nabla\tilde{\pi} &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned} \tag{2.3}
$$

for $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}; \lambda \neq 0, |\arg \lambda| < \theta\}$ for some $\theta \in (0, \pi)$.

Corollary 2.3. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ as above and $\theta \in (0, \pi)$. Then there exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \lambda_0 + \Sigma_\theta$ and $f \in L^p(\Omega)$ there exists a unique solution $(u, \tilde{\pi}) \in (W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega)) \times \widehat{W}^{1,p}(\Omega)$ satisfying* (2.3). *Moreover, there exists $C > 0$ such that*

$$
|\lambda| \|u\|_{L^p(\Omega)} + \|\nabla^2 u\|_{L^p(\Omega)} + \|\nabla\tilde{\pi}\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \quad \lambda \in \lambda_0 + \Sigma_\theta, \ f \in L^p(\Omega).
$$

The semigroup $\{e^{-tA_p}\}_{t\geq 0}$ on $L^p_\sigma(\Omega)$ described in Corollary 2.2 admits the following L^p-L^q smoothing properties, which are well known for the situation of bounded or exterior domains.

Proposition 2.4. *Let $p, r, s \in (1, \infty)$ such that $s \leq p \leq r$, $f \in L^s(\Omega)^n$, $F \in L^s(\Omega)^{n\times n}$ and $T > 0$. Then there exists a $C > 0$ such that for $t \in (0, T)$*

$$\|e^{-tA_p}\mathbb{P}_p f\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})}\|f\|_s \qquad \text{for} \quad \frac{1}{p}-\frac{2}{n} \leq \frac{1}{r}, \quad \frac{1}{s} \leq \frac{1}{p}+\frac{2}{n}.$$

$$\|\nabla e^{-tA_p}\mathbb{P}_p f\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}\|f\|_s \qquad \text{for} \quad \frac{1}{p}-\frac{1}{n} \leq \frac{1}{r}, \quad \frac{1}{s} \leq \frac{1}{p}+\frac{1}{n}.$$

$$\|e^{-tA_p}\mathbb{P}_p \nabla\cdot F\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}\|F\|_s \qquad \text{for} \quad \frac{1}{p}-\frac{1}{n} \leq \frac{1}{r}, \quad \frac{1}{s} \leq \frac{1}{p}+\frac{1}{n}.$$

The proof of this proposition can be found in [18]. So, we omit it in here. We finally consider the Navier-Stokes equations

$$\begin{aligned} u_t - \Delta u + (u\cdot\nabla)u + \nabla\tilde{\pi} &= 0 && \text{in } \Omega\times(0,T), \\ \nabla\cdot u &= 0 && \text{in } \Omega\times(0,T), \\ u &= 0 && \text{on } \partial\Omega\times(0,T), \\ u|_{t=0} &= u_0 && \text{in } \Omega. \end{aligned} \tag{2.4}$$

We prove the following local well-posedness results for (2.4). To this end, assume that $\Omega \subset \mathbb{R}^n$ is a domain such that the Helmholtz projection $\mathbb{P}_p$ exists for $L^p(\Omega)$. Then, by the notion of a mild solution of (2.4), it is understood a function $u \in C([0,T); L^p_\sigma(\Omega))$ for some $T > 0$ satisfying the integral equation

$$u(t) = e^{-tA_p}u_0 - \int_0^t e^{-(t-s)A_p}\mathbb{P}_p\nabla\cdot(u(s)\otimes u(s))ds, \quad 0 \leq t < T.$$

Theorem 2.5. *Let $n \geq 2$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection $\mathbb{P}_p$ exists for $L^p(\Omega)$ for some $p > n$. Let $u_0 \in L^p_\sigma(\Omega)$. Then there exist $T_0 > 0$ and a unique mild solution.*

The proof follows the lines of the well-known iteration procedure described in [21, 22, 26] with Proposition 2.4. We will not give a detailed proof here.

3 Outline of the proof

In this section we give the outline of the proof of Theorem 2.1. We refer to the localization procedure and the divergence equation. Starting from the corresponding results for the half space $\mathbb{R}^n_+$, the main problem is that the usual localization procedure known from elliptic problem does not transfer to the situation of the Stokes equation. Indeed, the usual localization procedure does not respect the condition on the divergence. In [17], a new localization procedure for the Stokes resolvent problem (2.3) respecting the condition on the divergence was introduced.

Throughout this section, let Ω be an unbounded domain. For given $k \in \mathbb{N}$, a domain $\Omega \subset \mathbb{R}^n$ is called a uniform C^k-domain, if there exist constants $K, \alpha, \beta > 0$ such that for each $x_0 \in \partial\Omega$ there exists a Cartesian coordinate system with origin at x_0, coordinates $y = (y', y_n)$ and $h \in C^k((-\alpha, \alpha)^{n-1})$ with $\|h\|_{C^k} \leq K$ such that the neighborhood

$$U(x_0) := \{(y', y_n) \in \mathbb{R}^n; h(y') - \beta < y_n < h(y') + \beta,\ |y'| < \alpha\}$$

of x_0 satisfying $\partial\Omega \cap U(x_0) = \{(y', h(y')); |y'| < \alpha\}$ and

$$U^-(x_0) := \{(y', y_n) \in \mathbb{R}^n; h(y') - \beta < y_n < h(y'),\ |y'| < \alpha\} = U(x_0) \cap \Omega.$$

Let us note that our assumption implies that one may choose for some $r \in (0, \alpha)$, depending only on α, β, K, balls $B_j := B_r(x_j)$ with centers $x_j \in \overline{\Omega}$ for $j \in \mathbb{N}$ and C^3-functions h_j if $x_j \in \partial\Omega$ such that

$$\overline{\Omega} \subset \cup_{j=1}^{\infty} B_j, \quad \overline{B_j} \subset U(x_j) \text{ if } x_j \in \partial\Omega, \quad \overline{B_j} \subset \Omega \text{ if } x_j \in \Omega.$$

Moreover, we may construct this covering in such a way that not more than a finite fixed number $N_0 \in \mathbb{N}$ of these balls can have a nonempty intersection. Thus, choosing $N_0 + 1$ different balls $B_1, B_2, \ldots$, their common intersection is empty. For given the covering $\{B_j\}_{j=1}^{\infty}$, there exists a partition of unity $\varphi_j \in C_c^{\infty}(\mathbb{R}^n)$, $\sum_j \varphi_j \equiv 1$ in Ω, satisfying $\operatorname{supp} \varphi_j \subset B_j$ and $0 \leq \varphi_j \leq 1$.

(i) Compact Boundary. We now consider the case when $\partial\Omega$ is compact. In order to explain the main idea of [17], let us consider

$$\tilde{u} := \sum_{j=1}^{\infty} \varphi_j u_j \quad \text{and} \quad \tilde{\pi} := \sum_{j=1}^{\infty} \varphi_j \pi_j.$$

Here (u_j, π_j) is the solution to the Stokes resolvent equations (2.3) in the whole space with $\psi_j f$ in the right hand side if $x_j \in \Omega$, and (u_j, π_j) is the push-forward of the solution $(\hat{u}_j, \hat{\pi}_j)$ to the Stokes resolvent equations in the half space

$$
\begin{aligned}
\lambda \hat{u}_j - \Delta \hat{u}_j + \nabla \hat{\pi}_j = \hat{f}_j \quad &\text{in} \quad \mathbb{R}^n_+, \\
\nabla \cdot \hat{u}_j = 0 \quad &\text{in} \quad \mathbb{R}^n_+, \\
\hat{u} = 0 \quad &\text{on} \quad \partial\mathbb{R}^n_+
\end{aligned}
\tag{3.1}
$$

with the right hand side $\hat{f}_j$ defined by a suitable affine transformation of $\psi_j f$ if $x_j \in \partial\Omega$, where $\psi_j \in C_c^\infty(\mathbb{R}^n)$ satisfying $\psi_j \equiv 1$ in B_j and $\operatorname{supp} \psi_j \subset D_j := B_{2r}(x_j)$. Define the solution operator $\widehat{U}_\lambda$ and $\widehat{\Pi}_\lambda$ by

$$
(\widehat{U}_\lambda \hat{f}_j, \widehat{\Pi}_\lambda \hat{f}_j) := (\hat{u}_j, \hat{\pi}_j).
$$

Since we assume that Ω has boundary of class C^3, we may construct the pull-back and push-forward mappings in such a way that they preserve the condition on the divergence. Hence, u_j is solenoidal by construction. However, $\tilde{u}$ is not solenoidal, in general, since

$$
\nabla \cdot \tilde{u} = \sum_{j=1}^{\infty} (\nabla \varphi_j) \cdot u_j \neq 0.
$$

Therefore, we use the modified ansatz

$$
\bar{u} := \sum_{j=1}^{\infty} \left(\varphi_j u_j + \mathbb{B}_j(\nabla \cdot (\varphi_j u_j)) \right),
\tag{3.2}
$$

where $\mathbb{B}_j$ denotes the Bogovskiĭ operator on $U_j^- := B_j \cap \Omega$ such that $\operatorname{supp} \nabla \varphi_j \subset \overline{U_j^-} = B_j \cap \overline{\Omega}$. Inserting $(\bar{u}, \bar{\pi})$ in (2.3), we thus obtain

$$
\begin{aligned}
\lambda \bar{u} - \Delta \bar{u} + \nabla \bar{\pi} = f + T_\lambda f \quad &\text{in}\, \Omega, \\
\nabla \cdot \bar{u} = 0 \quad &\text{in}\, \Omega, \\
\bar{u} = 0 \quad &\text{on}\, \partial\Omega,
\end{aligned}
$$

where T_λ denotes the correction terms. In order to show that T_λ is small for λ large, it is crucial to estimate the correction terms involving the pressure $\bar{\pi}$

and the Bogovskiĭ operator. Note that, for domains with compact boundary it is enough to consider the divergence problem on suitable bounded domains, since one can get the convergence of the right hand side of (3.2). If the domain does not have a compact boundary it seems to be necessary to correct the divergence term on an unbounded domain, because it is not clear how to prove the convergence of (3.2).

(ii) Non-compact Boundary. We now consider the case when $\partial\Omega$ is not compact. In order to circumvent these difficulties, we present an approach to the Stokes problem on domains which non-compact boundaries which relies on the above localization procedure where, however, the Bogovskiĭ correction term (3.2) is replaced by the solution v_j of the weak Neumann problem:

$$\begin{aligned} \Delta v &= \nabla \cdot f \quad &&\text{in } \Omega, \\ \frac{\partial v}{\partial \nu} &= f \cdot \nu \quad &&\text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

To be more precise, we use the other ansatz

$$u := \sum_{j=1}^{\infty} \varphi_j u_j + \nabla v_j$$

with v_j which solves the weak Neumann problem (3.3) with $f = \varphi_j u_j$. Note that the existence and uniqueness of v_j is guaranteed since the Helmholtz projection exists by assumption. By construction we then obtain

$$\nabla \cdot u = \sum_{j=1}^{\infty} \nabla \cdot (\varphi_j u_j) + \Delta v_j = 0.$$

However, the tangential component of u does not vanish at the boundary anymore. This leads to additional correction terms. In our main linear result we show that (2.3) has a unique solution for any $f \in L^q_\sigma(\Omega)$ satisfying the usual resolvent estimates. Replacing norm bounds by $\mathcal{R}$-bounds (see e.g. [8]) in the arguments above, we even obtain the maximal L^p-L^q-estimate in view of the vector-valued version of Mihklin's theorem due to Weis [34].

To explain more details, we prepare the notation. For each $x_j \in \partial\Omega$, the local coordinate corresponding to x_j is defined as a coordinate obtained from the original ones by some affine transform which moves x_j to the origin and after which the positive x_n-axis has the direction of the interior normal to $\partial\Omega$ at x_j. Let $x_j \in \partial\Omega$ and choose local coordinates corresponding to x_j. By definition of a uniform C^3-boundary, there exists an open neighborhood $U := U_j := V_1 \times V_2 \subset \mathbb{R}^n$ containing $x_j = 0$ with $V_1 \subset \mathbb{R}^{n-1}$ and $V_2 \subset \mathbb{R}$ open, and a height function $h_j \in C^3(\overline{V_1})$ satisfying $\partial\Omega \cap U = \{x = (x', x_n) \in U; x_n = h_j(x')\}$ and $\Omega \cap U = \{x \in U; x_n > h_j(x')\}$. Note that choosing the radius of V_1 small, we may assume that $\|h_j\|_\infty + \|\nabla h_j\|_\infty$ (independent of j) is as small as we like. Next we define

$$g_j(x) := (g_j^1(x), \ldots, g_j^n(x)) := (x', x_n - h_j(x')), \quad x \in U. \tag{3.4}$$

We obtain an injection $g_j \in C^3(\overline{U}, \mathbb{R}^n)$ satisfying $\Omega \cap U = \{x \in U; g_j^n(x) > 0\}$ and $\partial\Omega \cap U = \{x \in U : g_j^n(x) = 0\}$. Since $\partial\Omega$ is a uniform C^3-boundary, all derivatives of g_j and of g_j^{-1} (defined on $\hat{U}_j := g_j(U_j)$) up to order 3 may be assumed to be bounded by a constant independent of x_j.

For a function $u : U_j \cap \Omega \to \mathbb{R}$, we call the push-forward $v = \mathcal{G}u$ on $\hat{U}_j \cap \mathbb{R}^n_+$ defined by $v(y) := u(g_j^{-1}(y))$, locally. Due to the regularity of the boundary, this transformation is an isomorphism $W^{s,p}(U_j \cap \Omega) \to W^{s,p}(\hat{U}_j \cap \mathbb{R}^n_+)$ for all $p \in (1, \infty)$ and $s \in [-2, 2]$. Similarly, for a vector-valued function $u : U \cap \Omega \to \mathbb{R}^n$ we define the push-forward $v_\sigma = \mathcal{G}_\sigma u$ for the solenoidal spaces by $v_\sigma(y) := J_g(u(g^{-1}(y)))$, where J_g denotes the Jacobian of g. In fact, the linear transformation $\mathcal{G}_\sigma$ is an isomorphisms from $L^p_\sigma(U_j \cap \Omega)$ to $L^p_\sigma(\hat{U}_j \cap \mathbb{R}^n_+)$. Furthermore, it is an isomorphism from $W^{s,p}(U_j \cap \Omega) \to W^{s,p}(\hat{U}_j \cap \mathbb{R}^n_+)$ for all $p \in (1, \infty)$ and $s \in [-2, 2]$. The corresponding pull-back mappings $\mathcal{G}^{-1}$ and $\mathcal{G}_\sigma^{-1}$ are defined in a similar way. Note, that we may choose $h = 0$ if $U_j \cap \partial\Omega = \emptyset$, that is, $x_j \in \Omega$.

For any $\varepsilon \in (0, 1)$, let $\{\Omega_j^\varepsilon\}_{j\in\mathbb{N}}$ be a family of locally finite covers of Ω such that $U_j \subset \Omega_j^\varepsilon$, $\partial\Omega_j^\varepsilon$ has C^3-regularity,

$$\|\nabla h_j^\varepsilon\|_\infty < \varepsilon, \tag{3.5}$$

$$\sum_{j\in\mathbb{N}} \chi_{\Omega_j^\varepsilon} \leq C, \tag{3.6}$$

where $\chi_{\Omega_j^\varepsilon}$ is the characteristic function on Ω_j^ε for each j, h_j^ε is the height function corresponding to Ω_j^ε, and $C > 0$ is a constant independent of ε. For each such covering $\{\Omega_j^\varepsilon\}_{j\in\mathbb{N}}$, we choose a partition of unity $\{\varphi_j^\varepsilon\}_{j\in\mathbb{N}}$ subordinate to this covering. Furthermore, denote by $\mathcal{G}_j^\varepsilon$, $\mathcal{G}_{\sigma,j}^\varepsilon$, $\mathcal{G}_j^{-1,\varepsilon}$ and $\mathcal{G}_{\sigma,j}^{-1,\varepsilon}$ the corresponding push-forward mappings and pull-back mappings.

The commutator $[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]\hat{u}_j$ for $\hat{u}_j \in W^{2,p}(\mathbb{R}_+^n)$ of Δ and $\mathcal{G}_{j,\sigma}^{-1,\varepsilon}$ can be split into two parts: $[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_h\hat{u}_j$ contains second order terms (highest) of $\hat{u}_j$ only and $[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_l\hat{u}_j$ contains all lower order terms. In particular, by (3.5) there exists a constant $C > 0$ such that

$$\|[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_h\hat{u}_j\|_{L^p(\Omega_j)} \leq C\varepsilon\|\hat{u}_j\|_{\widehat{W}^{2,p}(\hat{\Omega}_j^\varepsilon)}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ \hat{u}_j \in W^{2,p}(\hat{\Omega}_j),$$
$$\|[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_l\hat{u}_j\|_{L^p(\Omega_j)} \leq C\|\hat{u}_j\|_{W^{1,p}(\hat{\Omega}_j^\varepsilon)}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ \hat{u}_j \in W^{2,p}(\hat{\Omega}_j).$$

Here and in the following, $\hat{\Omega}_j^\varepsilon$ denotes the transformation by the j-th push forward map of Ω_j^ε. In the same way $\hat{u}_j^\varepsilon$ denotes the function living on the half space $\mathbb{R}_+^n$ which is connected with u_j^ε through the j-th push forward map. Similarly, there exists a constant $C > 0$ such that

$$\|[\nabla, \mathcal{G}_j^{-1,\varepsilon}]\hat{\pi}_j\|_{L^p(\Omega_j)} \leq C\varepsilon\|\hat{\pi}_j\|_{\widehat{W}^{1,p}(\mathbb{R}_+^n)}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ \hat{\pi}_j \in \widehat{W}^{1,p}(\hat{\Omega}_j^\varepsilon).$$

As in [17], we use Bogovskiĭ's operator to construct localized data for our localization procedure. For a bounded Lipschitz domain $\Omega' \subset \Omega$ and $g \in L^p(\Omega')$ with $\int_{\Omega'} g = 0$ Bogovskiĭ's operator $\mathbb{B}_{\Omega'}$ is a solution operator to the divergence equation as follows

$$\left\{\begin{array}{rcll} \operatorname{div} u &=& g & \text{in } \Omega', \\ u &=& 0 & \text{on } \partial\Omega'. \end{array}\right. \tag{3.7}$$

By the definition of Ω_j^ε, there exists $C > 0$ independent of $j \in \mathbb{N}$ such that

$$\|\mathbb{B}_{\Omega_j^\varepsilon} f\|_{L^p(\Omega_j^\varepsilon)} \leq C\|f\|_{L^p(\Omega_j^\varepsilon)}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ f \in L^p(\Omega_j^\varepsilon).$$

We finally choose cut-off functions $\psi_j^\varepsilon \in C_c^\infty(\Omega_j^\varepsilon)$ such that $0 \leq \psi_j^\varepsilon \leq 1$ and $\psi_j^\varepsilon \equiv 1$ on $\operatorname{supp}\varphi_j^\varepsilon$. For $f \in X_f := L_\sigma^p(\Omega)$, we define the local external force terms by

$$f_j^\varepsilon := \psi_j^\varepsilon f - \mathbb{B}_{\Omega_j^\varepsilon}\left((\nabla\psi_j^\varepsilon)f\right),$$

and let $\hat{f}_j^\varepsilon$ denote the extension to $\mathbb{R}_+^n$ by 0 of the push-forward $\mathcal{G}_{\sigma,j}^\varepsilon f_j^\varepsilon$. By the uniform boundedness of Bogovskiĭ operator, we obtain $\hat{f}_j^\varepsilon \in L_\sigma^p(\mathbb{R}_+^n)$ and

$$\|\hat{f}_j^\varepsilon\|_{L^p(\mathbb{R}_+^n)} \leq C\|f\|_{L^p(\Omega_j^\varepsilon)} \tag{3.8}$$

with some $C > 0$ independent of ε, j and f. Hence, (3.6) yields that

$$\Big((S_j^{1,\varepsilon})_{j\in\mathbb{N}}\Big)_{\varepsilon\in(0,1)} \subset \mathcal{L}(X_f, \ell^p(\widehat{X}_f)) \tag{3.9}$$

is uniformly bounded, where $S_j^{1,\varepsilon} f := \hat{f}_j^\varepsilon$. Similarly, for $(a,b) \in X_{a,b} := \{a \in W^{1-1/p,p}(\partial\Omega); a\cdot\nu = 0\} \times \{b \in W^{2-1/p,p}(\partial\Omega); b\cdot\nu = 0\}$, we define the local boundary data $a_j^\varepsilon := \psi_j^\varepsilon a$, $b_j^\varepsilon := \psi_j^\varepsilon b$, $\hat{a}_j^\varepsilon := \mathcal{G}_{j,\sigma}^{\partial\Omega,\varepsilon}\psi_j^\varepsilon a$ and $\hat{b}_j^\varepsilon := \mathcal{G}_{j,\sigma}^{\partial\Omega,\varepsilon}\psi_j^\varepsilon b$. Here, $\mathcal{G}_{j,\sigma}^{\partial\Omega,\varepsilon}$ is the restriction of $\mathcal{G}_{j,\sigma}^\varepsilon$ to the boundary of Ω. Again, we see

$$\Big((S_j^{2,\varepsilon}(a,b))_{j\in\mathbb{N}}\Big)_{\varepsilon\in(0,1)} \subset \mathcal{L}(X_{a,b}, \ell^p(\widehat{X}_{a,b})) \tag{3.10}$$

is uniformly bounded, where $S_j^{2,\varepsilon} := S_j^{2,\varepsilon}(a,b) := (\hat{a}_j^\varepsilon, \hat{b}_j^\varepsilon)$. We now set

$$U_\lambda^\varepsilon(f,a,b) := \sum_{j\in\mathbb{N}} \varphi_j^\varepsilon \mathcal{G}_{j,\sigma}^{-1,\varepsilon} \hat{U}_\lambda S_j^\varepsilon(f,a,b) - \nabla\mathcal{N}(\sum_{j\in\mathbb{N}} \varphi_j^\varepsilon \mathcal{G}_{j,\sigma}^{-1,\varepsilon} \hat{U}_\lambda S_j^\varepsilon(f,a,b))$$

where $\mathcal{N}$ is the solution operator of the weak Neumann problem and $S_j^\varepsilon := S_j^\varepsilon(f,a,b) := (S_j^{1,\varepsilon} f, S_j^{2,\varepsilon}(a,b))$. Here, similarly to (3.2), we add a correction term in order to have a solenoidal ansatz U_λ^ε. However, in contrast to the case (i), the correction term is based on the solution operator of the weak Neumann problem instead of Bogovskiĭ's operator. Inserting $u := U_\lambda^\varepsilon(f,a,a)$, we calculate

$$\begin{aligned} \lambda u - \mathbb{P}\Delta u &= f + \mathcal{T}_\lambda^{1,\varepsilon}(f,a,a) && \text{in } \Omega, \\ \nabla\cdot u &= 0 && \text{in } \Omega, \\ u &= a + \mathcal{T}_\lambda^{2,\varepsilon}(f,a,a) && \text{on } \partial\Omega, \end{aligned} \tag{3.11}$$

where

$$\mathcal{T}_\lambda^\varepsilon(f,a,b) := (\mathcal{T}_\lambda^{1,\varepsilon}(f,a,b), \mathcal{T}_\lambda^{2,\varepsilon}(f,a,b)) := T_{1,\lambda}^\varepsilon(f,a,b) + \cdots + T_{6,\lambda}^\varepsilon(f,a,b)$$

with

$$\begin{aligned} T_{1,\lambda}^\varepsilon(f,a,b) &:= \Big(\mathbb{P}\sum_{j\in\mathbb{N}} \varphi_j^\varepsilon[\nabla, \mathcal{G}_j^{-1,\varepsilon}]\widehat{\Pi}_\lambda S_j^\varepsilon(f,a,b), 0, 0\Big), \\ T_{2,\lambda}^\varepsilon(f,a,b) &:= \Big(\mathbb{P}\sum_{j\in\mathbb{N}} (\nabla\varphi_j^\varepsilon)\mathcal{G}_j^{-1,\varepsilon}\widehat{\Pi}_\lambda S_j^\varepsilon(f,a,b), 0, 0\Big), \end{aligned}$$

$$T^{\varepsilon}_{3,\lambda}(f,a,b) := \Big(-\mathbb{P}\sum_{j\in\mathbb{N}}[\varphi^{\varepsilon}_j,\Delta]\mathcal{G}^{-1,\varepsilon}_{j,\sigma}\widehat{U}_\lambda S^{\varepsilon}_j(f,a,b),0,0\Big),$$
$$T^{\varepsilon}_{4,\lambda}(f,a,b) := \Big(-\mathbb{P}\sum_{j\in\mathbb{N}}\varphi^{\varepsilon}_j[\Delta,\mathcal{G}^{-1,\varepsilon}_{j}]_h\widehat{U}_\lambda S^{\varepsilon}_j(f,a,b),0,0\Big),$$
$$T^{\varepsilon}_{5,\lambda}(f,a,b) := \Big(-\mathbb{P}\sum_{j\in\mathbb{N}}\varphi^{\varepsilon}_j[\Delta,\mathcal{G}^{-1,\varepsilon}_{j}]_l\widehat{U}_\lambda S^{\varepsilon}_j(f,a,b),0,0\Big),$$
$$T^{\varepsilon}_{6,\lambda}(f,a,b) := \Big(0,-\nabla\mathcal{N}V^{\varepsilon},-\nabla\mathcal{N}V^{\varepsilon}\Big).$$

Here $V^{\varepsilon} := \sum_{j\in\mathbb{N}}\varphi^{\varepsilon}_j\mathcal{G}^{-1,\varepsilon}_{j,\sigma}\widehat{U}_\lambda S^{\varepsilon}_j(f,a,b)|_{\partial\Omega}$. This means that we obtain a solution of the Stokes resolvent problem in Ω which is given by

$$R^{\varepsilon}(\lambda)f := U^{\varepsilon}_\lambda(1+\mathcal{T}^{\varepsilon}_\lambda)^{-1}(f,0,0) = U^{\varepsilon}_\lambda\sum_{k\in\mathbb{N}_0}(\mathcal{T}^{\varepsilon}_\lambda)^k(f,0,0), \tag{3.12}$$

provided if the above Neumann series converges.

In the following we show that the Neumann series exists for some $\varepsilon\in(0,1)$, which hence yields the existence of a solution to (3.11). The uniqueness of the solution follows from a standard duality argument. Hence, we finally obtain $R^{\varepsilon}(\lambda) := (\lambda+A_p)^{-1}$. In order to estimate it, we set $X := X_f\times X_{a,b}$. Then, the representation formula (3.12) can be written as

$$\begin{aligned} R^{\varepsilon}(\lambda)f &= U^{\varepsilon}_\lambda\sum_{k\in\mathbb{N}_0}(\mathcal{T}^{\varepsilon}_\lambda)^k(f,0,0) = U^{\varepsilon}_\lambda K_\lambda^{-1}\sum_{k\in\mathbb{N}_0}(K_\lambda\mathcal{T}^{\varepsilon}_\lambda K_\lambda^{-1})^k K_\lambda(f,0,0)\\ &= U^{\varepsilon}_\lambda K_\lambda^{-1}\sum_{k\in\mathbb{N}_0}(K_\lambda\mathcal{T}^{\varepsilon}_\lambda K_\lambda^{-1})^k(f,0,0), \end{aligned}$$

provided if the above series converges. Here

$$K_\lambda := \begin{pmatrix} 1 & 0 & 0\\ 0 & \lambda^{1-\frac{1}{3p}} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

In the following lemma we show that

$$\mathcal{R}_X\{K_\lambda\mathcal{T}^{\varepsilon}_\lambda K_\lambda^{-1};\lambda\in\lambda_0+\Sigma_\theta\} < 1 \tag{3.13}$$

for sufficient large $\lambda_0>0$. Hence, $R^{\varepsilon}(\lambda)$ is well defined for some $\varepsilon\in(0,1)$ and all $\lambda\in\lambda_0+\Sigma_\theta$ with large λ_0. The following lemma is crucial.

Lemma 3.1. *For $\alpha \in (0, 1/2p')$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that*

$$\begin{aligned}
\mathcal{R}_X\{K_\lambda \mathcal{T}^\varepsilon_{1,\lambda} K_\lambda^{-1}; \lambda \in 1 + \Sigma_\theta\} &\leq 1/4, \\
\mathcal{R}_X\{\lambda^\alpha K_\lambda \mathcal{T}^\varepsilon_{2,\lambda} K_\lambda^{-1}; \lambda \in 1 + \Sigma_\theta\} &\leq C, \\
\mathcal{R}_X\{\lambda^{1/2} K_\lambda \mathcal{T}^\varepsilon_{3,\lambda} K_\lambda^{-1}; \lambda \in 1 + \Sigma_\theta\} &\leq C, \\
\mathcal{R}_X\{K_\lambda \mathcal{T}^\varepsilon_{4,\lambda} K_\lambda^{-1}; \lambda \in 1 + \Sigma_\theta\} &\leq 1/4, \\
\mathcal{R}_X\{\lambda^{1/2} K_\lambda \mathcal{T}^\varepsilon_{5,\lambda} K_\lambda^{-1}; \lambda \in 1 + \Sigma_\theta\} &\leq C, \\
\mathcal{R}_X\{\lambda^{1/2p} K_\lambda \mathcal{T}^\varepsilon_{6,\lambda} K_\lambda^{-1}; \lambda \in 1 + \Sigma_\theta\} &\leq C.
\end{aligned}$$

The reader can find the proof of the above lemma in [18]. This lemma leads us to (3.13) if λ_0 is taken sufficient large. That is the outline of proof of Theorem 2.1.

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