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Critical exponent for semilinear wave equation with time-dependent damping

早稲田大学・政治経済学術院 西原 健二 (Kenji Nishihara) [1]
Faculty of Political Science and Economics,
Waseda University

1 Introduction

In this note we consider the Cauchy problems for the wave equations with time-dependent damping

$$(P)\qquad \begin{cases} u_{tt}-\Delta u+b(t)u_t=f(u), & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^N,\end{cases}$$

where

$$b(t)=b_0(t+1)^{-\beta},\quad b_0>0\,(b_0=:1\text{ WLOG}),$$

$$|f(u)|\sim|u|^\rho,\quad 1<\rho<\frac{N+2}{[N-2]_+}=\begin{cases}\infty & (N=1,2)\\ \frac{N+2}{N-2} & (N\geq 3),\end{cases}$$

and the data $(u_0,u_1)\in H^1\times L^2$ are compactly supported. Then there exists a unique weak solution $u\in C([0,T];\,H^1)\cap C^1([0,T];\,L^2)$ for some $T>0$ with compact support by the finite propagation property of the wave equation. Our concern is with an asymptotic behavior of the solution as $t\to\infty$. In particular, our aim is to determine the critical exponent for the semilinear problem.

When $\beta=0$, (P) is reduced to

$$(1.1)\qquad \begin{cases} u_{tt}-\Delta u+u_t=f(u), & (t,x)\in\mathbf{R}_+\times\mathbf{R}^N\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^N.\end{cases}$$

If the semilinear term in (P) is

$$(1.2)\qquad f(u)=-|u|^{\rho-1}u,$$

then it works as absorbing, and for any large data there uniquely exists the solution $u\in C([0,\infty);\,H^1)\cap C^1([0,\infty);\,L^2)$, whose behaviors will be classified to three cases:

(i) In the case $\rho>\rho_F(N):=1+\frac{2}{N}$, the solution u behaves like $\theta_0G(t,x)$ as $t\to\infty$ for a suitable constant θ_0 and the Gauss kernel $G(t,x)=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$, which is the fundamental solution of the corresponding linear parabolic equation

$$\phi_t-\Delta\phi=0.$$

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(ii) In the case $\rho = \rho_F(N)$, the solution behaves like the approximate Gauss kernel $G(t,x)(\log t)^{-\frac{N}{2}}$.

(iii) In the case $\rho < \rho_F(N)$, the solution u behaves like the self-similar solution $w(t,x) := (t+1)^{\frac{\rho-1}{2}} f(|x|/\sqrt{t+1})$ of the corresponding semilinear parabolic equation

$$\phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0.$$

Therefore, the exponent $\rho_F(N)$ is critical, which is called the Fujita exponent named after his pioneering work [1].

While

$$f(u) = |u|^{\rho-1}u, \ |u|^\rho \ etc.$$

works as the source term, and the behaviors of the solution u to (1.1) are classified as follows:

(iv) If $\rho > \rho_F(N)$, then for suitably small data (u_0, u_1) there exists a time-global solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$, whose asymptotic profile is $\theta_0 G(t,x)$ for suitable constant θ_0.

(v,vi) If $\rho \le \rho_F(N)$, then the time-local solution $u(t)$ cannot be extended time-globally for some data (u_0, u_1). Depending on (v) $\rho = \rho_F(N)$ and (vi) $\rho < \rho_F(N)$, the estimates of its life span are different from each other.

(For (i) $\sim$ (vi) see [2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 15, 16, 21, 22, 28] and the references therein. Many parts are already solved, but some are still expected.)

Thus the Fujita exponent $\rho_F(N)$ is critical in both the absorbing and source semilinear problems. These imply so called the diffusion phenomenon of the damped wave equation.

We now consider the time-dependent damping problem (P). Wirth [24, 25] analyzed the linear equation of (P)

$$v_{tt} - \Delta v + b(t)v_t = 0, \quad b(t) = (t+1)^{-\beta}. \tag{1.4}$$

If $\beta > 1$, then the damping become weaker and the solution v behaves as the corresponding wave equation, when the damping is called non-effective. If $-1 < \beta < 1$, then the damping is called effective, that is, the solution behaves like that of the corresponding parabolic equation. The rest case $\beta < -1$ is called over-damping (For $\beta \ge 0$ see also Yamazaki [26, 27]).

In this note we consider the case of effective damping, and show that

Conclusion. When $-1 < \beta < 1$ (effective damping) and $f(u) = |u|^\rho$ (source semilinear term) with $1 < \rho < \frac{N+2}{[N-2]_+}$, the Fujita exponent $\rho_F(N)$ is still critical even in the time-dependent damping case.

In the case of semilinear absorbing term $f(u) = -|u|^{\rho-1}u$ we had the following theorem.

Theorem 1.1 *Suppose* $-1 < \beta < 1$ *and* $f(u) = -|u|^{\rho-1}u$ *with* $1 < \rho < \frac{N+2}{[N-2]_+}$. *Then the following assertions hold.*

(*I*) ([20]) *When* $\rho \geq \rho_F(N)$, *the time-global solution* $u \in C([0,\infty);\, H^1) \cap C^1([0,\infty);\, L^2)$ *decays with rate*

$$\int_{\mathbf{R}^N} e^{2\psi} u(t,x)^2\, dx \leq C I_0^2 (t+1)^{-\frac{(1+\beta)N}{2}+\varepsilon}, \tag{1.5}$$

where $\psi(t,x) = \frac{(1+\beta)|x|^2}{4(2+\delta)(t+1)^{1+\beta}}$ $(0 < \delta \ll 1)$ *with* $\varepsilon = \varepsilon(\delta) > 0$, $\varepsilon(\delta) \to 0\, (\delta \to 0)$ *and*

$$I_0^2 = \int_{\mathbf{R}^N} e^{2\psi(0,x)} (u_1^2 + |\nabla u_0|^2 + |u_0|^{\rho+1} + u_0^2)\, dx < \infty.$$

(*II*) ([19]) *Moreover, assume* $N = 1$ *and* $\rho > 3 = \rho_F(1)$. *Then*

$$\|u(t,\cdot) - \theta_0 G_B(t,\cdot)\|_{L^p} = o(t^{-\frac{1+\beta}{2}(1-\frac{1}{p})}), \tag{1.6}$$

for suitable constant θ_0, *where*

$$G_B(t,x) = (4\pi B(t))^{-\frac{N}{2}} e^{-\frac{|x|^2}{4B(t)}}, \quad B(t) = \int_0^t \frac{1}{b(\tau)} d\tau.$$

(*III*) ([20]) *When* $\rho \leq \rho_F(N)$, *the solution decays with*

$$\int_{\mathbf{R}^N} e^{2\psi} u(t,x)^2\, dx \leq C I_0^2 (t+1)^{-(1+\beta)(\frac{2}{\rho-1}-\frac{N}{2})} \tag{1.7}$$

where $\psi(t,x) = \frac{a|x|^2}{(t+t_0)^{1+\beta}}$ $(0 < a \ll 1,\ t_0 \gg 1)$.

Note that decay rates in both (1.5) and (1.7) are available for $1 < \rho < (N+2)/[N-2]_+$. But, the decay rate in (1.5) with $\varepsilon = 0$ is equal to that in (1.7) when $\rho = \rho_F(N)$, and so (1.5) is effective for $\rho \geq \rho_F(N)$ and (1.7) for $\rho \leq \rho_F(N)$. Also, note that the solution u in the case of $\rho = \rho_F(N)$ is expected to decay a little bit faster than $G_B(t,x)$ like the case (ii), but it remains open.

Let us discuss about the decay rates obtained in Theorem 1.1. Our equation is

$$u_{tt} - \Delta u + b(t)u_t + |u|^{\rho-1}u = 0, \tag{1.8}$$

whose corresponding linear and nonlinear parabolic equations are, respectively,

$$b(t)\phi_t - \Delta\phi = 0 \quad \text{or} \quad \phi_t = \frac{1}{b(t)}\Delta\phi, \tag{1.9}$$

and

$$b(t)\phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0. \tag{1.10}$$

The solution ϕ of (1.9) with $\phi(0,x) = \phi_0(x)$ is given by

$$\phi(t,x) = \int_{\mathbf{R}^N} G_B(t, x-y)\phi_0(y)\,dy$$

thanks to the fundamental solution $G_B(t,x)$, so that

$$\|\phi(t)\|_{L^2} \leq C\|\phi_0\|_{L^q} t^{-\frac{(1+\beta)N}{4}}, \quad t > 0. \tag{1.11}$$

While, (1.10) has the similarity solution of the form

$$w_0(t,x) = (c+ct)^{-\frac{1+\beta}{\rho-1}} f\left(\frac{|x|}{(c+ct)^{\frac{1+\beta}{2}}}\right), \quad c^{1+\beta}(1+\beta) = 1,$$

(see [20]) and its decay rate is

$$\|w_0(t,\cdot)\|_{L^2} = O(t^{-(\frac{1}{\rho-1}-\frac{N}{4})(1+\beta)}). \tag{1.12}$$

The decay rate (1.5) is the almost same as (1.11) and the rate (1.7) is the same as (1.12). Therefore, from the viewpoint of the diffusion phenomenon, the decay rate (1.5) implies almost optimal and (1.7) does optimal, which suggest the Fujita exponent $\rho_F(N)$ will be critical. The behavior (1.6) means that the decay rate $\|u(t)\|_{L^2} = O(t^{-\frac{\beta+1}{4}})$ is completely optimal and that the Fujita exponent is actually critical, when $N = 1$. However, the (almost) optimalities of (1.5) and (1.7) are not shown when $N \geq 2$ and so we cannot say that $\rho_F(N)$ is completely critical.

In the source semilinear problem we have the following two theorems, which derives our **Conclusion**.

Theorem 1.2 (Small data global existence) *Suppose that $-1 < \beta < 1$ and $\rho_F(N) < \rho < \frac{N+2}{[N-2]_+}$. If $(u_0, u_1) \in H^1 \times L^2$ is compactly supported and*

$$I_0^2 := \int_{\mathbf{R}^N} e^{\frac{(1+\beta)|x|^2}{2(2+\delta)}}(|u_1|^2 + |\nabla u_0|^2 + |u_0|^{\rho+1})dx \ll 1$$

for some small $\delta > 0$, then there exists a unique global solution $u \in C([0,\infty);\, H^1) \cap C^1([0,\infty);\, L^2)$ to (P), which satisfies

$$\|u(t)\|_{L^2} \leq C_\delta I_0 (t+1)^{-\frac{N}{4}(1+\beta)+\frac{\varepsilon}{2}}$$

for $\varepsilon = \varepsilon(\delta) > 0,\ C_\delta > 0$ with $\varepsilon \to 0,\ C_\delta \to \infty$ as $\delta \to 0$.

Theorem 1.3 (Blow-up in critical and subcritical exponents) *Suppose that $-1 < \beta < 1$ and $(u_0, u_1) \in H^1 \times L^2$ are compactly supported with*

$$\int_{\mathbf{R}^N} (u_1(x) + \hat{b}_1 u_0(x))dx > 0, \quad \hat{b}_1^{-1} = \int_0^\infty e^{-\int_0^t (\tau+1)^{-\beta} d\tau}\,dt. \tag{1.13}$$

Then the global solution $u \in C([0,\infty);\, H^1) \cap C^1([0,\infty);\, L^2)$ to (P) does not exist provided that $1 < \rho \leq \rho_F(N)$.

Two theorems are shown in Nishihara [18] and Lin, Nishihara and Zhai [11]. In the next section we only sketch the proof of Theorem 1.3. The proof of Theorem 1.2 is given by the weighted energy method, originally developed in Todorova and Yordanov [22], which is omitted in this note.

2 Nonexistence of time-global solution

To prove Theorem 1.3 we apply the test function method developed by Qi S. Zhang [28].
First we remember his method in [28] for the wave equation with damping of constant coefficient

$$u_{tt} - \Delta u + u_t = |u|^\rho \tag{2.1}$$

with data (u_0, u_1) satisfying

$$\int_{\mathbf{R}^N} (u_0 + u_1)(x)\, dx > 0. \tag{2.2}$$

Note that (1.13) is reduced to (2.2) since $\hat{b}_1 = 1$ when $b(t) = 1$. Assume that u is a non-trivial global solution to (2.1) with (2.2). To derive the contradiction, we set

$$I_R = \int_{Q_R} |u|^\rho \cdot (\psi_R)^{\rho'}(t,x)dxdt, \quad \frac{1}{\rho} + \frac{1}{\rho'} = 1$$

for large constant $R > 0$, where $Q_R = [0, R^2] \times B_R(0)$, $B_R = B_R(0) = \{|x| \leq R\}$ and

$$\psi_R(t,x) = \eta_R(t)\phi_R(r) = \eta(\frac{t}{R^2})\phi(\frac{r}{R}), \ r = |x|$$

for the functions η, $\phi \in C_0^\infty$ satisfying

$$\begin{aligned}
&0 \leq \eta \leq 1, \ \eta(t) = \begin{cases} 1 & t \in [0, 1/4] \\ 0 & t \in [1, \infty) \end{cases}, \ |\eta'(t)|, |\eta''(t)| \leq C, \\
&0 \leq \phi \leq 1, \ \phi(r) = \begin{cases} 1 & r \in [0, 1/2] \\ 0 & r \in [1, \infty) \end{cases}, \ |\phi'(r)|, |\phi''(r)| \leq C, \\
&(\eta')^2/\eta' \leq C \ (0 \leq t \leq 1), \ \ |\nabla\phi|^2/|\phi| \leq C \ (0 \leq r \leq 1).
\end{aligned}$$

Then, by (2.1)

$$I_R = \int_{Q_R} (u_{tt} - \Delta u + u_t) \cdot (\psi_R)^{\rho'} dxdt =: J_1 + J_2 + J_3.$$

By the integral by parts, for example,

$$\begin{aligned}
&J_3 \left(= \int_{Q_R} u_t(\psi_R)^{\rho'}\, dx\, dt) \right) \\
&= -\int_{B_R} u_0(x)dx - \int_{\hat{Q}_{R,t}} u \cdot \rho'(\psi_R)^{\rho'-1} \cdot \frac{1}{R^2}\eta'(\frac{t}{R^2})\psi(\frac{|x|}{R})dx\, dt \\
&\leq -\int_{B_R} u_0(x)\, dx + \left(\int_{\hat{Q}_{R,t}} |u|^\rho(\psi_R)^{\rho'} dx\, dt \right)^{\frac{1}{\rho}} \left(\int_{\hat{Q}_{R,t}} \{\eta'(\frac{t}{R^2})\phi(\frac{|x|}{R})\}^{\rho'} dx\, dt \right)^{\frac{1}{\rho'}} \frac{C}{R^2} \\
&\leq -\int_{B_R} u_0(x)\, dx + C(\hat{I}_{R,t})^{\frac{1}{\rho}} R^{(2+N)\frac{1}{\rho'}-2}.
\end{aligned}$$

Here we have used the Hölder inequality with $\frac{1}{\rho} = \frac{\rho'-1}{\rho'}$, $\rho' = (\rho' - 1)\rho$ and denoted

$$\hat{I}_{R,t} := \int_{\hat{Q}_{R,t}} |u|^\rho (\psi_R)^{\rho'} dx\, dt = \int_{R^2/4}^{R^2} \int_{B_R} |u|^\rho (\psi_R)^{\rho'} dx\, dt.$$

By the similar way to J_1, J_2, we have

$$\begin{array}{rcl} I_R & \leq & -\int_{B_R} (u_0 + u_1)(x)\, dx + C(\hat{I}_{R,t} + \hat{I}_{R,|x|})^{\frac{1}{\rho}} R^{\frac{N+2}{\rho'}-2} \\ & \leq & -\int_{B_R} (u_0 + u_1)(x)\, dx + C(I_R)^{\frac{1}{\rho}} R^{(N+2)(1-\frac{1}{\rho})-2}, \end{array}$$

where $\hat{I}_{R,|x|} = \int_0^t \int_{R/2 \leq |x| \leq R} |u|^\rho (\psi_R)^{\rho'} dx\, dt$. Moreover, $(N+2)(1-\frac{1}{\rho}) - 2 = N - \frac{N+2}{\rho} < 0$ is equivalent to $\rho < 1 + \frac{2}{N} = \rho_F(N)$. Hence if $\rho < \rho_F(N)$, then $(I_R)^{1-\frac{1}{\rho}} \leq CR^{(N+2)(1-\frac{1}{\rho})-2}$ and $I_R \to 0$ as $R \to \infty$, which contradicts to the non-triviality of the solution u. If $\rho = \rho_F(N)$, then $I_R \leq C$ and $\int_{\mathbf{R}^N} |u|^\rho dx\, dt < \infty$ as $R \to \infty$. Hence,

$$I_R \leq -\int_{B_R} (u_0 + u_1) + C\underbrace{(\hat{I}_{R,t} + \hat{I}_{R,|x|})}_{\to 0 \text{ as } R \to \infty}{}^{\frac{1}{\rho}} < 0 \quad \text{as} \quad R \to \infty,$$

which is also the contradiction. Thus we could show the non-existence of global solution in the case of the damping of constant coefficient. In the proof both the divergence form of the left-hand side of (2.1) and the positivity of the right-hand side were key points.

We now back to our equation

$$u_{tt} - \Delta u + b(t) u_t = |u|^\rho, \quad b(t) = (t+1)^{-\beta} \ (-1 < \beta < 1), \tag{2.3}$$

whose left-hand side is not in the divergence form. To change (2.3) to the divergence form, we multiply (2.3) by some function $g(t)$ to get

$$(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + (-g'(t) + b(t)g(t))u_t = g(t)|u|^\rho. \tag{2.4}$$

If the coefficient of u_t is constant, then the left-hand side of (2.4) becomes the divergent form. Since the positivity of $g(t)$ is also necessary, we define $g(t)|u|^\rho$ by the solution to the initial value problem for the first order ordinary differential equation

$$\begin{cases} -g'(t) + b(t)g(t) = 1, \quad t > 0, \\ g(0) = 1/\hat{b}_1, \quad \hat{b}_1 = (\int_0^\infty e^{-\int_0^t b(s)ds} dt)^{-1}. \end{cases} \tag{2.5}$$

Explicitly,

$$g(t) = e^{\int_0^t b(s)ds} (\int_0^\infty e^{-\int_0^\tau b(s)ds} d\tau - \int_0^t e^{-\int_0^\tau b(s)ds} d\tau) \ (> 0). \tag{2.6}$$

We note that

$$\lim_{t \to \infty} b(t)g(t) = 1 \tag{2.7}$$

and that $C^{-1}/b(t) \le g(t) \le C/b(t)$ for any $t \in [0,\infty)$. In fact, by the l'Hôpital's rule

$$\begin{aligned}
\lim_{t\to\infty} b(t)g(t) &= \lim_{t\to\infty} \frac{\int_0^\infty e^{-\int_0^\tau b(s)ds}d\tau - \int_0^t e^{-\int_0^\tau b(s)ds}d\tau}{\frac{1}{b(t)}e^{-\int_0^t b(s)ds}} \\
&= \lim_{t\to\infty} \frac{-e^{-\int_0^\tau b(s)ds}}{-\frac{b'(t)}{b(t)^2}e^{-\int_0^\tau b(s)ds} - e^{-\int_0^\tau b(s)ds}} \\
&= 1, \quad \text{since } \lim_{t\to\infty}\frac{b'(t)}{b(t)^2} = -\lim_{t\to\infty}\beta(1+t)^{-1+\beta} = 0.
\end{aligned}$$

Thus, (2.4) is changed to

$$(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + u_t = g(t)|u|^\rho. \tag{2.8}$$

We can now apply the test function method to (2.8) and set

$$I_R = \int_{Q_R} g(t)|u|^\rho \cdot (\psi_R)^{\rho'}(t,x)\,dxdt \tag{2.9}$$

for large constant $R > 0$, where $Q_R = [0, R^{2/(1+\beta)}] \cdot B_R(0)$ and

$$\psi_R(t,x) = \eta_R(t)\cdot\phi_R(r) = \eta(\frac{t}{R^{2/(1+\beta)}})\cdot\phi(\frac{|x|}{R}).$$

Same as above, we can easily derive

$$\begin{aligned}
I_R &\le -\frac{1}{\hat{b}_1}\int_{B_R(0)}(u_1 + \hat{b}_1 u_0)(x)\,dx + C(\hat{I}_{R,t}^{1/\rho} + \hat{I}_{R,|x|}^{1/\rho})R^{\frac{N+2}{\rho'}-2} \\
&\le -\frac{1}{\hat{b}_1}\int_{B_R(0)}(u_1 + \hat{b}_1 u_0)(x)\,dx + CI_R^{1/\rho}R^{(N+2)(1-\frac{1}{\rho})-2}.
\end{aligned}$$

Hence we have contradictions in both cases $\rho < \rho_F(N)$ and $\rho = \rho_F(N)$.

We have now completed the sketch of the proof of Theorem 1.3.

Remark 1. Corresponding parabolic equation to (2.3) is

$$-\Delta u + b(t)u_t = |u|^\rho, \quad \text{or} \quad u_t - \Delta(b(t)^{-1}u) = b(t)^{-1}|u|^\rho, \tag{2.10}$$

which is itself in the divergence form. Hence we can apply the test function method to (2.10) by taking

$$I_R = \int_{Q_R} b(t)^{-1}|u|^\rho(\psi_R)^{\rho'}(t,x)dx\,dt, \quad \frac{1}{\rho} + \frac{1}{\rho'} = 1,$$

which corresponds to (2.9) by (2.7).

Remark 2. In the space-dependent damping case

$$u_{tt} - \Delta u + a(x)u_t = |u|^\rho, \quad a(x) = (1+|x|^2)^{-\alpha/2} \ (0 \le \alpha < 1), \tag{2.11}$$

the equation is in the divergence form. Hence we can apply the test function method to (2.11). Ikehata, Todorova and Yordanov [7] have recently treated this equation and obtained the critical exponent

$$\rho_c(N, \alpha) = 1 + \frac{2}{N - \alpha}. \tag{2.12}$$

For the absorbed semilinear problem see Nishihara [17] and references therein.

Remark 3. Related to Remark 2, we also want to have the critical exponent $\rho_F(N, \alpha, \beta)$ for the space and time-dependent damping case

$$u_{tt} - \Delta u + a(x)b(t)u_t = |u|^\rho. \tag{2.13}$$

The existence of time global solution for suitably small data will be shown by the weighted energy method (cf. Lin, Nishihara and Zhai [10] and Wakasugi [23]). The key point is to obtain the blow-up result. Our method adopted in the proof of Theorem 1.3 does not seem to be applicable to (2.13). Our conjecture of the critical exponent $\rho_c(N, \alpha, \beta)$ is

$$\rho_c(N, \alpha, \beta) = 1 + \frac{2}{N - \alpha}, \tag{2.14}$$

which still remains open.

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