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On reflection and non-reflection of countable list-chromatic number of graphs *

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Abstract

It is known that the reflection cardinal of countable chromatic number of graphs is fairly large. This stands in contrast with the situation of the countable coloring number whose reflection cardinal is less or equal to that of the Fodor-type Reflection Principle and hence can be consistently $\aleph_2$.

Applying a theorem of Peter Komjáth, it can be shown that the reflection of countable list-chromatic number behaves consistently similarly to the reflection of countable chromatic number but it can also behave consistently like the reflection of countable coloring number. Moreover, the Fodor-type Reflection Principle does not decide in which way the reflection of countable list-chromatic number behaves.

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1 Introduction

For a class $\mathcal{C}$ of structures and a property P the reflection cardinal of $\langle \mathcal{C}, P\rangle$ is the minimal cardinal κ such that, for any $M \in \mathcal{C}$, if there are club many substructures $N \in \mathcal{C}$ of M of cardinality $< \kappa$ with the property P then M also has the property P [(1)]. If κ is the reflection cardinal of $\langle \mathcal{C}, P\rangle$, we shall write $\kappa = \mathfrak{Refl}(\mathcal{C}, P)$.

In some cases non-existence of reflection cardinal for certain pairs $\langle \mathcal{C}, P\rangle$ can be proved already in ZFC. We shall denote the non-existence of the reflection cardinal for $\mathcal{C}$ and P by $\mathfrak{Refl}(\mathcal{C}, P) = \infty$. (1) of the next examples is one of such instances.

Examples 1. (1) (Hajnal and Juhász [9])
Let κ be a cardinal of uncountable cofinality. Let $X = \langle X, \mathcal{O}\rangle$ be the topological space defined by $X = \kappa+1$ with the open base for $\mathcal{O}$: $\{\{\alpha\} : \alpha < \kappa\} \cup \{u \cup \{\kappa\} : u \subseteq \kappa, \mid \kappa \backslash u \mid < \kappa\}$. All subspaces of X of cardinality $< \kappa$ are discrete and hence metrizable. But X itself is not metrizable since the character of the point κ is $\kappa > \aleph_0$. Thus ZFC *proves that $\mathfrak{Refl}(\mathcal{C}, P)$ does not exist for "$\mathcal{C}$ = topological spaces" and "$P \equiv$ metrizable".*

(2) (Dow [1])
For "$\mathcal{C}$ = compact spaces" and "$P \equiv$ metrizable", ZFC *proves* $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$.

(3) (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [5], Fuchino, Sakai, Soukup and Usuba [6])
For "$\mathcal{C}$ = locally compact spaces" and "$P \equiv$ metrizable", $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ is consistent with ZFC *(modulo some fairly large cardinal) and it is equivalent to the Fodor-type Reflection Principle.* □

In the following, we survey known facts and check some of the proofs in connection with reflection of countability of some of the characteristics about coloring of infinite graphs; namely, chromatic number, coloring number and list-chromatic number (see the next section for definition).

It is known that the reflection cardinal of countable chromatic number of graphs is fairly large (Erdős and Hajnal, see Theorem 3.1 below). This stands in contrast with the situation of the countable coloring number whose reflection cardinal is less or equal to that of the Fodor-type Reflection Principle (see Corollary 4.4) and hence in particular it can be consistently $\aleph_2$ (Fuchino, Sakai,

(1) We are mainly considering porperties P which transfer to arbitrary substructures. For such P, "club many" may be simply replaced by "all".

Soukup and Usuba [6]). In Section 4 we give an upper bound for this reflection cardinal.

Applying a theorem of Peter Komjáth, it can be shown that the reflection of countable list-chromatic number behaves consistently similarly to the reflection of countable chromatic number but it can also behave consistently like the reflection of countable coloring number. Moreover, the Fodor-type Reflection Principle does not decide in which way the reflection of countable list-chromatic number behaves (see Theorem 5.1).

This note is intended as a preliminary work toward [7]. More results including some more details of Theorem 5.1 and more related discussions should be found there.

2 Graph coloring

First, let us recall some basic notions about graphs and the cardinal characteristics in terms of coloring of graphs we are going to discuss.

A *graph* is a structure $G = \langle G, K\rangle$ such that G is a non empty set and K a binary relation which is non-reflective and symmetric. Intuitively E is a set of vertices and a pair $\{x, y\}$ of vertices with $K(x, y)$ represents an edge connecting x and y. If $K(x, y)$ we say that x and y are adjacent or x and y are connected in G.

We sometimes identify K with $\{\{x,y\} : \langle x,y\rangle \in K\}$ and write $\{x,y\} \in K$ instead of $K(x,y)$.

A *subgraph* H of a graph G is always an induced subgraph, that is, $H = \langle H, L\rangle$ is a subgraph of $G = \langle G, K\rangle$ if $H \subseteq G$ and $L = K \cap H^2$. If I is a subset of (the underlying set of) G then $G \restriction I$ denotes the subgraph $\langle I, K \cap I^2\rangle$ of $G = \langle G, K\rangle$. We often misuse the notation deliberately and write $G \restriction I = \langle I, K\rangle$ instead of $G \restriction I = \langle I, K \cap I^2\rangle$.

For a graph $G = \langle G, K\rangle$, a mapping $\phi : G \to \kappa$ is said to be a *good coloring* for G if $\phi(x) \neq \phi(y)$ holds whenever x and y are adjacent in G. The *chromatic number* of G is defined as:

$$chr(G) = \min\{\kappa \in \text{Card} : \text{there is a good coloring } f : G \to \kappa\}.$$

The *list-chromatic number* of G is defined by:

$$\begin{aligned} \textit{list-chr}(G) = \min\{\kappa \in \text{Card} :\ & \text{for } \mu = |\,G\,| \text{ and for any } l : G \to [\mu]^\kappa \\ & \text{there is a good coloring } f : G \to \mu \\ & \text{such that } f(x) \in l(x) \text{ for all } x \in G\}. \end{aligned}$$

The following notation is convenient in connection with coloring number we introduce next. For a graph $G = \langle G, K\rangle$, $x \in G$ and $I \subseteq G$, let

$$K_I^x = \{y \in I \,:\, K(x,y)\}.$$

For an ordering $\Subset$ on G and $x \in G$

$$K_{\Subset}^x = \{y \in G \,:\, K(x,y) \text{ and } y \Subset x\}.$$

Thus if $I = \{y \in I \,:\, y \Subset x\}$ we have $K_{\Subset}^x = K_I^x$.

Using this notation, *coloring number* of a graph $G = \langle G, K\rangle$ is defined as:

$$\begin{aligned} col(G) = \min\{\kappa \in \text{Card} \,:\, & \text{ there is a well-ordering } \Subset \text{ of } G \\ & \text{ such that } |\, K_{\Subset}^x \,| < \kappa \text{ for all } x \in G\} \end{aligned}$$

It is easy to see that $chr(G) \le list\text{-}chr(G) \le col(G)$ for any graph G. The inequality can be also rigid (for both finite and infinite graphs). Coloring number of graphs enjoys several quite useful characterizations. For a graph $\langle G, K\rangle$, a mapping $f : G \to [G]^{<\kappa}$ is a *κ-coloring mapping* if for any a, $b \in G$ with $K(a,b)$, at least one of $a \in f(b)$ and $b \in f(a)$ holds. A subgraph H of a graph $\langle G, K\rangle$ is a *κ-subgraph* (notation: $H \sqsubseteq_\kappa G$) if for any $a \in G \setminus H$ we have $|\, K_H^a \,| < \kappa$.

Theorem 2.1 (Erdős and Hajnal [2], see also [6] and [4]).
For any infinite cardinal κ and any graph G the following are equivalent:

(a) $col(G) \le \kappa$;

(b) *There is a κ-coloring mapping on G;*

(c) *There is a continuously increasing sequence $\langle G_\alpha \,:\, \alpha < \delta\rangle$ of subalgebras of G such that $col(G_\alpha) \le \kappa$ and $G_\alpha \sqsubseteq_\kappa G$ for all $\alpha < \kappa$.* □

We shall write $\mathfrak{Refl}_{col}$ to denote $\mathfrak{Refl}(\mathcal{C}, P)$ for "$\mathcal{C}$ = graphs" and "$P \equiv$ of countable coloring number". We have a relatively good picture of what $\mathfrak{Refl}_{col}$ can be. For the definition of the Fodor-type Reflection Principle and the reflection cardinal $\mathfrak{Refl}_{FRP}$ see Section 4.

Theorem 2.2. (1) (Fuchino, Sakai, Soukup and Usuba [6])
$\mathfrak{Refl}_{col} = \aleph_2 \Leftrightarrow$ *Fodor-type Reflection Principle holds.*

(2) $\mathfrak{Refl}_{col} = \infty$ *is consistent.*

(3) $\mathfrak{Refl}_{col} \le \mathfrak{Refl}_{\mathrm{FRP}}$.

Proof. For (1) see [6]. (3) will be proved in Section 4 (see Corollary 4.4).

(2): The next lemma shows that, for example, $V = L$ implies $\mathfrak{Refl}_{col} = \infty$.

□ (Theorem 2.2)

Lemma 2.3. *For a regular cardinal κ, suppose that there exists a non-reflecting stationary set $E \subseteq E^\kappa_\omega$. Then there is a graph G of cardinality κ such that $col(G) > \aleph_0$ but $col(H) = \aleph_0$ for all subgraphs H of G of cardinality $< \kappa$.*

Proof. Suppose that $E \subseteq E^\kappa_\omega$ is a non-reflecting stationary set. Let $g : E \to [E]^{\aleph_0}$ be any ladder system on E and let $G = \langle \kappa, K \rangle$ where

(2.1) $\{\alpha, \beta\} \in K$ for $\alpha < \beta < \kappa$ if $\beta \in E$ and $\alpha \in g(\beta)$.

The next two claims show that this G is as desired.

Claim 2.3.1. $col(G) > \aleph_0$.

$\vdash$ If $col(G) \leq \aleph_0$, then, by Theorem 2.1, there is a filtration $\langle G_\alpha : \alpha < \kappa \rangle$ such that $G_\alpha \sqsubseteq_\omega G$ for all $\alpha < \kappa$. Since E is stationary, there is an $\alpha \in E$ such that $G_\alpha = G \restriction \alpha$. But $K^\alpha_\alpha = g(\alpha)$ is infinite. This is a contradiction. $\dashv$ (Claim 2.3.1)

Claim 2.3.2. $col(G \restriction \alpha) \leq \aleph_0$ *for all* $\alpha < \kappa$.

$\vdash$ We prove the assertion by induction on α.

If $\alpha < \omega_1$ the claim is trivial. Successor steps are also trivial. So assume that α is a limit and we have shown $col(G \restriction \beta) \leq \aleph_0$. Since $E \cap \alpha$ is not stationary in α, there is a continuously increasing sequence $\langle \alpha_\xi : \xi < \delta \rangle$ of elements of α such that $\delta = \mathrm{cf}(\alpha)$ and $\alpha = \sup_{\xi<\delta} \alpha_\xi$ and $\alpha_\xi \notin E$ for all $\xi < \delta$. Then it is easy to see that $G \restriction \alpha_\xi \sqsubseteq_\omega G \restriction \alpha$ for all $\xi < \delta$. By Theorem 2.1, (c) it follows that $col(G \restriction \alpha) = \aleph_0$. $\dashv$ (Claim 2.3.2)

❑ (Lemma 2.3)

For "$\mathcal{C}$ = graphs" and "$P \equiv$ of countable chromatic number", let us denote $\mathfrak{Refl}(\mathcal{C}, P)$ by $\mathfrak{Refl}_{chr}$.

The picture we have for $\mathfrak{Refl}_{chr}$ is a less satisfactory one as we only have the following inequalities:

Theorem 2.4.

(1) (Erdős and Hajnal [3]) $\mathfrak{Refl}_{chr} \geq \beth_\omega$.

(2) *If κ is a strongly compact cardinal then* $\mathfrak{Refl}_{chr} \leq \kappa$.

Proof. (1) follows from Theorem 3.1 in the next section.

(2) follows easily from the characterization of strongly compact cardinals in terms of compactness of $\mathcal{L}_{\kappa,\kappa}$. ❑ (Theorem 2.4)

If we replace "graphs" in the definition of $\mathfrak{Refl}_{chr}$ by the class of graphs whose vertices are intervals of a given linear ordering and two vertices are adjacent if and only if they intersect, we obtain the reflection cardinal $\mathfrak{Refl}_{RC}$ which is connected to Rado's Conjecture: Rado's Conjecture is characterized by $\mathfrak{Refl}_{RC} = \aleph_2$ (for basic facts about Rado's Conjecture see e.g. [12], [13]). In [7], we prove among other things that $\mathfrak{Refl}_{RC} = \aleph_2$ implies $\mathfrak{Refl}_{col} = \aleph_2$.

In Section 5, we present a result on the reflection cardinal $\mathfrak{Refl}_{list\text{-}chr}$ which is $\mathfrak{Refl}(\mathcal{C}, P)$ for "$\mathcal{C}$ = graphs" and "$P \equiv$ of countable list-chromatic number".

3 Non-reflection of countable chromatic number

In this section, we reconstruct the details of a proof of the following theorem following the sketch of a proof given in [14]:

Theorem 3.1 (P. Erdős and Hajnal [3]).
For any $n \in \omega \setminus 1$, there is a graph G of cardinality $\geq (\beth_n)^+$ (actually of any cardinality $\geq (\beth_n)^+$) such that, for any subgraph H of G of cardinality $\leq \beth_n$, we have $chr(H) \leq \aleph_0$ while $chr(G) > \aleph_0$.

In the notation of the previous section, Theorem 3.1 implies $\mathfrak{Refl}_{chr} \geq \beth_\omega$.

This theorem is well-known. For example, it is cited in recent papers by Hajnal ([8]) and Todorčević ([14]). [8] contains a proof for the case $n = 1$ and [14] a rough sketch of the whole proof. It seems however that the original paper [3] cited in [8] and [14] proves the theorem only under GCH.

Here, we identify the set X^n of all n-tuples of elements of X with

$$^nX = \{f \ : \ f \text{ is a mapping from } n = \{0, ..., n-1\} \text{ to } X\}.$$

In particular, if $\vec{t} \in X^n$ is such that $\vec{t} = \langle t_0, ..., t_{n-1}\rangle$, we say $Im(\vec{t}) = \{t_0, ..., t_{n-1}\}$ and $Dom(\vec{t}) = n$. Also, for $\vec{t}$ as above, we write $\vec{t}(i) = t_i$. For a set X and $n \in \omega \setminus 2$, let shift_n be the binary relation on X^n, defined by

$$\text{(3.1)} \quad \mathsf{shift}_n(\vec{u}, \vec{v}) \ \Leftrightarrow \ \begin{array}{l} \text{(a) } \vec{u} \neq \vec{v} \text{ and} \\ \text{(b) } \vec{u}(i) = \vec{v}(i+1) \text{ for all } i < n-1 \text{ or} \\ \quad\ \ \vec{v}(i+1) = \vec{u}(i) \text{ for all } i < n-1 \end{array}$$

for $\vec{u}$, $\vec{v} \in X^n$.

In the following we show that the graph $G = \langle X^{n+1}, \mathsf{shift}_{n+1}\rangle$ for any set X of cardinality $\geq (\beth_n)^+$ is as in Theorem 3.1.

Lemma 3.2. *For $n \in \omega \setminus 1$ and for any set X with $|X| \leq \beth_n$, we have $chr(\langle X^{n+1}, \mathsf{shift}_{n+1}\rangle) \leq \aleph_0$.*

Proof. We prove the lemma by induction on n. First, let us prove the assertion for $n = 1$. Let F be a family of subsets of ω such that $|F| = \beth_1$ $(= 2^{\aleph_0})$ and such that elements of F are pairwise incomparable (with respect to $\subseteq$). For $\vec{t} \in F^2$, let

$$(3.2) \quad n_{\vec{t}} = \min(\vec{t}(0) \setminus \vec{t}(1)).$$

It is enough to prove the following:

Claim 3.2.1. *The mapping $\phi : F^2 \to \omega$; $\vec{t} \mapsto n_{\vec{t}}$ is a good coloring for the graph $\langle F^2, \mathsf{shift}_2\rangle$.*

$\vdash$ Suppose that $\vec{u}$, $\vec{v} \in F^2$ are such that $\mathsf{shift}_2(\vec{u}, \vec{v})$, say, with $\vec{u}(1) = \vec{v}(0)$. Then, since $n_{\vec{u}} \notin \vec{u}(1)$ but $n_{\vec{v}} \in \vec{v}(0) = \vec{u}(1)$ by (3.2), we have $\phi(\vec{u}) \neq \phi(\vec{v})$. $\dashv$ (Claim 3.2.1)

The next claim completes the induction proof of Lemma 3.2.

Claim 3.2.2. *For $n \geq 2$, suppose that X and Y are infinite sets such that $chr(\langle X^n, \mathsf{shift}_n\rangle) \leq \aleph_0$ and $|Y| \leq 2^{|X|}$. Then we have $chr(\langle Y^{n+1}, \mathsf{shift}_{n+1}\rangle) \leq \aleph_0$.*

$\vdash$ We may assume that X is a cardinal κ and $Y \subseteq \mathcal{P}(\kappa)$ and elements of Y are pairwise incomparable (with respect to $\subseteq$). Let $\phi : X^n \to \omega$ be a good coloring for $\langle X^n, \mathsf{shift}_n\rangle$. For u, $v \in Y$, let

$$(3.3) \quad \alpha_{u,v} = \begin{cases} \min(u \setminus v) + 1, & \text{if } u \neq v, \\ 0, & \text{otherwise.} \end{cases}$$

For $\vec{u} \in Y^{n+1}$, let

$$(3.4) \quad \vec{\alpha}_{\vec{u}} = \langle \alpha_{\vec{u}(0),\vec{u}(1)}, \alpha_{\vec{u}(1),\vec{u}(2)}, \dots, \alpha_{\vec{u}(n-1),\vec{u}(n)}\rangle.$$

Note that we have $\vec{\alpha}_{\vec{u}} \in X^n$ for $\vec{u} \in Y^{n+1}$. Note also that if $\vec{u}$ is not a constant function then neither is $\vec{\alpha}_{\vec{u}}$. If $\mathsf{shift}_{n+1}(\vec{u}, \vec{v})$ for $\vec{u}$, $\vec{v} \in Y^{n+1}$, then at least one of $\vec{u}$ and $\vec{v}$ is not constant. It follows that at least one of $\vec{\alpha}_{\vec{u}}$ and $\vec{\alpha}_{\vec{v}}$ is not constant. Since it is clear that $\vec{\alpha}_{\vec{u}}$ and $\vec{\alpha}_{\vec{v}}$ still satisfy (3.1), (b) it follows that $\vec{\alpha}_{\vec{u}} \neq \vec{\alpha}_{\vec{v}}$ and hence we have $\mathsf{shift}_n(\vec{\alpha}_{\vec{u}}, \vec{\alpha}_{\vec{v}})$.

Now let $\phi^* : Y^{n+1} \to \omega$ be defined by

$$(3.5)\qquad \phi^*(\vec{u}) = \phi(\vec{\alpha}_{\vec{u}})$$

for $\vec{u} \in Y^{n+1}$. ϕ^* is then a good coloring for $\langle Y^{n+1}, \mathsf{shift}_{n+1}\rangle$: Suppose that $\vec{u}$, $\vec{v} \in Y^{n+1}$ and $\mathsf{shift}_{n+1}(\vec{u}, \vec{v})$. Then we have $\mathsf{shift}_n(\vec{\alpha}_{\vec{u}}, \vec{\alpha}_{\vec{v}})$ as was already seen above. Since ϕ is a good coloring, it follows that $\phi^*(\vec{u}) = \phi(\vec{\alpha}_{\vec{u}}) \neq \phi(\vec{\alpha}_{\vec{v}}) = \phi^*(\vec{v})$.

$\dashv$ (Claim 3.2.2)

□ (Lemma 3.2)

Now, let $\lambda \geq (\beth_n)^+$. Together with Lemma 3.2, the next lemma completes the proof of Theorem 3.1 by showing that $\langle \lambda^{n+1}, \mathsf{shift}_{n+1}\rangle$ is as desired in Theorem 3.1.

Lemma 3.3. *chr*$(\langle \lambda^{n+1}, \mathsf{shift}_{n+1}\rangle) > \aleph_0$.

Proof. Let

$$(3.6)\qquad \langle\lambda\rangle^{n+1} = \{\vec{u} \in \lambda^{n+1} : \vec{u} \text{ is strictly increasing (as a mapping from } n+1 \text{ to } \lambda)\}.$$

Since $\langle\langle\lambda\rangle^{n+1}, \mathsf{shift}_{n+1}\rangle$ is a subgraph of $\langle \lambda^{n+1}, \mathsf{shift}_{n+1}\rangle$, it is enough to to show that *chr*$(\langle\langle\lambda\rangle^{n+1}, \mathsf{shift}_{n+1}\rangle) > \aleph_0$.

Suppose otherwise. Then there is a good coloring $\phi : \langle\lambda\rangle^{n+1} \to \omega$ for the graph $\langle\langle\lambda\rangle^{n+1}, \mathsf{shift}_{n+1}\rangle$. By Erdős-Rado theorem there is a set $H \in [\lambda]^{\aleph_1}$ such that ϕ is constant on $\langle H\rangle^{n+1}$. If $\alpha_0 < \alpha_1 < \cdots < \alpha_{n+1}$ are $n+2$ elements of H, then, letting $\vec{u} = \langle \alpha_0, \alpha_1, ..., \alpha_n\rangle$ and $\vec{v} = \langle \alpha_1, \alpha_2, ..., \alpha_{n+1}\rangle$, we have $\mathsf{shift}_{n+1}(\vec{u}, \vec{v})$ but $\phi(\vec{u}) = \phi(\vec{v})$. This is a contradiction. □ (Lemma 3.3)

4 Reflection cardinal of Fodor-type Reflection

The Fodor-type Reflection Principle (FRP, [5], see [6] for the formulation we give below) states that the following $(4.1)_\lambda$ holds for all regular $\lambda > \aleph_1$:

$(4.1)_\lambda$ For any stationary $E \subseteq E^\lambda_\omega$ and a mapping $g : E \to [\lambda]^{\aleph_0}$ such that $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$, there is $\alpha^* \in E^\lambda_{\omega_1}$ such that

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E,\ g(\sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

Let us consider here the following generalization. For any cardinal $\kappa > \aleph_1$ let $\mathrm{FRP}_{<\kappa}$ be the assertion stipulating that the following $(4.2)_\lambda$ holds for all regular cardinal $\lambda \geq \kappa$:

(4.2)$_\lambda$ For any stationary $E \subseteq E^\lambda_\omega$ and a mapping $g : E \rightarrow [\lambda]^{\aleph_0}$ such that $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$, there is $\alpha^* \in \lambda$ such that $\omega_1 \leq \mathrm{cf}(\alpha^*) < \kappa$ and

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E,\ g(\sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

Then FRP is equivalent to $\mathrm{FRP}_{<\aleph_2}$. For cardinals $\kappa < \kappa'$, if $\mathrm{FRP}_{<\kappa}$ holds, then $\mathrm{FRP}_{<\kappa'}$ holds.

Let

$$\mathfrak{Refl}_{\mathrm{FRP}} = \min\{\kappa \in \mathrm{Card} : \mathrm{FRP}_{<\kappa} \text{ holds}\}$$

if $\{\kappa \in \mathrm{Card} : \mathrm{FRP}_{<\kappa} \text{ holds}\}$ is nonempty. Otherwise we let $\mathfrak{Refl}_{\mathrm{FRP}} = \infty$.

The following reformulation of $\mathrm{FRP}_{<\kappa}$ shows that $\mathfrak{Refl}_{\mathrm{FRP}}$ is actually a reflection cardinal in line with the reflection cardinals of properties of classes of structures.

Proposition 4.1. *For a cardinal $\kappa > \aleph_1$, $\mathrm{FRP}_{<\kappa}$ is equivalent to the assertion that the following (4.3)$_\lambda$ holds for all regular cardinal $\lambda \geq \kappa$:*

(4.3)$_\lambda$ *For any stationary $E \subseteq E^\lambda_\omega$ and a mapping $g : E \rightarrow [\lambda]^{\aleph_0}$ such that $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$, there is a set I of regular uncountable cardinality $\mu < \kappa$ such that $\mathrm{cf}(\sup I) = \mu$, I is closed with respect to g and*

$$\{x \in [I]^{\aleph_0} : \sup(x) \in E,\ g(\sup(x)) \subseteq x\}$$

is stationary in $[I]^{\aleph_0}$.

The proposition follows immediately from the next lemma:

Lemma 4.2. *For regular uncountable cardinals λ, μ with $\mu < \lambda$, a stationary $E \subseteq E^\lambda_\omega$, a mapping $g : E \rightarrow [\lambda]^{\aleph_0}$ such that $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$, and $\alpha^* \in E^\kappa_\mu$, the following are equivalent:*

(a) *There is $I \in [\alpha^*]^\mu$ such that $\sup(I) = \alpha^*$, I is closed with respect to g and*

$$Z_I = \{x \in [I]^{\aleph_0} : \sup(x) \in E \text{ and } g(\sup(x)) \subseteq x\}$$

is stationary;

(a') *For any $I \in [\alpha^*]^\mu$ such that $\sup(I) = \alpha^*$ and I is closed with respect to g as well as with respect to the order topology of α^*, we have that*

$$Z_I = \{x \in [I]^{\aleph_0} \,:\, \sup(x) \in E \text{ and } g(\sup(x)) \subseteq x\}$$

is stationary;

(b) *The set*

$$Z_{\alpha^*} = \{x \in [\alpha^*]^{\aleph_0} \,:\, \sup(x) \in E \text{ and } g(\sup(x)) \subseteq x\}$$

is stationary.

Proof. (a') $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (b): Suppose that Z_{α^*} is not stationary and let $C \subseteq [\alpha^*]^{\aleph_0}$ be a club disjoint from Z_{α^*}. Let $I \in [\alpha^*]^{\mu}$ be such that I is cofinal in α^* and closed with respect to g. Let

(4.4) $\quad C' = \{x \cap I \,:\, x \in C \text{ and } \sup(x) = sup(x \cap I)\}$.

Then we can find a $C'' \subseteq C'$ which is a club in $[I]^{\aleph_0}$. C'' is still disjoint from Z_{α^*} and hence also from Z_I. Thus Z_I is not stationary.

(b) $\Rightarrow$ (a'): Assume that Z_{α^*} is stationary. Let $I \in [\alpha^*]^{\mu}$ be such that $\sup(I) = \alpha^*$ and I is closed with respect to g as well as with respect to the order topology of α^*. We have to show that Z_I is stationary in $[I]^{\aleph_0}$. Suppose that $C \subseteq [I]^{\aleph_0}$ is a club. Let

(4.5) $\quad \tilde{C} = \{x \cup y \,:\, x \in C,\, y \in [\alpha^* \setminus I]^{\aleph_0},\, \sup(x) \geq \sup(y)\}$.

Then $\tilde{C}$ is a club in $[\alpha^*]^{\aleph_0}$. Hence, by the assumption, there is $z \in Z_{\alpha^*} \cap \tilde{C}$. Let $x = z \cap I$. By (4.5) and since I is closed with respect to the order topology of α^*, we have $\sup(z) = \sup(x) \in E \cap I$. By closedness of I with respect to g, it follows that $g(\sup(x)) \subseteq I$. Hence $g(\sup(x)) \subseteq z \cap I = x$. Thus we have $x \in Z_I \cap C$. This shows that Z_I is stationary. ❑ (Lemma 4.2)

Theorem 4.3. *Assuume* $\mathrm{FRP}_{<\kappa}$ *for a cardinal* $> \aleph_1$. *For any graph* $G = \langle G, K\rangle$, *if*

(4.6) $\quad$ *col*$(G \restriction I) \leq \aleph_0$ *holds for all* $I \in [G]^{<\kappa}$,

then col$(G) \leq \aleph_0$.

Proof. The proof is almost identical with the one given in [6] for the case $\kappa = \aleph_2$.

We prove by induction on λ that the following assertion $(4.7)_\lambda$ holds for all cardinals λ:

$(4.7)_\lambda$ For any graph $G = \langle G, K\rangle$ of cardinality λ, if (4.6) holds, then *col*$(G) \leq \aleph_0$.

For $\lambda < \kappa$, $(4.7)_\lambda$ trivially holds.

Suppose that $\lambda \geq \kappa$ and we have proved $(4.7)_{\lambda'}$ for all $\lambda' < \lambda$.

If λ is singular, and G is as in $(4.7)_\lambda$, then we can conclude $col(G) \leq \aleph_0$ by the induction hypothesis and Shelah's Singular Compactness Theorem ([11] see also [6]).

Suppose now that λ is regular and assume, toward a contradiction, that there is a graph G of cardinality λ which satisfies (4.6) but $col(G) > \aleph_0$. Without loss of generality, we may assume that (the underlying set of) G is λ. Note that $col(G \restriction \alpha) \leq \aleph_0$ for all $\alpha < \lambda$ by the induction hypothesis. Hence

(4.8) $\quad E = \{\alpha \in \lambda : \text{there is } \beta \in \lambda \setminus \alpha \text{ such that } |K_\alpha^\beta| \geq \aleph_0\}.$

is stationary by Theorem 2.1. Let $E^* = E \cap E_\omega^\lambda$.

Claim 4.3.1. *E^* is stationary in λ.*

$\vdash$ Suppose otherwise. Then $E \cap E_{>\omega}^\lambda$ must be stationary. For each $\alpha \in E \cap E_{>\omega}^\lambda$, let $\beta_\alpha \in \lambda \setminus \alpha$ be such that $K_\alpha^{\beta_\alpha}$ is infinite. Let $c_\alpha \in [K_\alpha^{\beta_\alpha}]^{\aleph_0}$ and $\xi_\alpha = \sup(c_\alpha)$. Then $\xi_\alpha < \alpha$ since $\mathrm{cf}(\xi_\alpha) \leq \omega$. β_α and c_α witness that $[\xi_\alpha, \alpha) \cap E_\omega^\lambda \subseteq E^*$. By Fodor's theorem, there are $\xi^* \in E_\omega^\lambda$ and stationary $E^\dagger \subseteq E \cap E_{>\omega}^\lambda$ such that $\xi_\alpha = \xi^*$ for all $\alpha \in E^\dagger$. We have $E_\omega^\lambda \setminus \xi^* = \bigcup_{\alpha \in E^\dagger} [\xi^*, \alpha) \cap E_\omega^\lambda \subseteq E^*$. Thus E^* is stationary. This is a contradiction to the assumption. $\dashv$ (Claim 4.3.1)

For $\alpha \in E^*$, let $\beta_\alpha \in \lambda \setminus \alpha$ be such that $|K_\alpha^{\beta_\alpha}| \geq \aleph_0$ and $c_\alpha \in [K_\alpha^{\beta_\alpha}]^{\aleph_0}$. By thinning out E^* if necessary, we may assume that for any $\alpha \in E^*$, $\beta_\alpha < \min(E^* \setminus \alpha)$. Let $g : E^* \to [\lambda]^{\aleph_0}$ be such that $c_\alpha \subseteq g(\alpha) \subseteq \alpha$ and $g(\alpha)$ is cofinal in α for all $\alpha \in E^*$.

By $\mathrm{FRP}_{<\kappa}$, there is $I \in [\lambda]^{<\kappa}$ such that $\aleph_1 \leq |I| = \mathrm{cf}(\sup(I)) = \mu < \kappa$, I is closed with respect to g and Z as in Lemma 4.2, (a) (with E there replaced by E^*) is stationary in $[I]^{\aleph_0}$. By blowing up I if necessary without changing $\sup(I)$, we may assume that I is also closed with respect to the order topology of $\sup(I)$ as well as closed with respect to the mapping $\alpha \mapsto \beta_\alpha$.

Now, we have $col(G \restriction I) \leq \aleph_0$ by the assumption (4.6). Hence there is a $\aleph_0$-coloring mapping $f : I \to [I]^{<\aleph_0}$ for $G \restriction I$ by Theorem 2.1. Let

(4.9) $\quad C = \{x \in [I]^{\aleph_0} : x \text{ is closed with respect to } f\}.$

Since C is a club in $[I]^{\aleph_0}$, there is an $x \in Z \cap C$. By the definition of Z and g, $|K_x^\beta| \geq \aleph_0$ for $\beta = \beta_{\sup(x)}$. For any $\gamma \in K_x^\beta$, we have $f(\gamma) \subseteq x$ as x is closed with respect to f. Since $f(\beta)$ is finite, $K_x^\beta \setminus f(\beta)$ is nonempty. But then, for

any $\gamma^* \in K_x^\beta \setminus f(\beta)$, we have $\gamma^* \notin f(\beta)$ and $\beta \notin f(\gamma^*)$. This is a contradiction.
$\square$ (Theorem 4.3)

Theorem 4.3 can be reformulated in the following inequality of reflection cardinals:

Corollary 4.4. $\mathfrak{Refl}_{col} \le \mathfrak{Refl}_{\mathrm{FRP}}$. $\square$

This inequality can be further related to that of Theorem 2.4, (2) just by observing that $\mathfrak{Refl}_{\mathrm{FRP}}$ is less or equal to the first strongly compact cardinal (that is, if such a cardinal ever exists).

5 Reflection and Non-reflection of list-chromatic number

Theorem 5.1. *The statement "$\mathfrak{Refl}_{list\text{-}chr} = \aleph_2$" is independent from* ZFC + FRP.

For the proof of Theorem 5.1, we use the following reformulation of a theorem by P. Komjáth:

Theorem 5.2 (Komjáth [10]).

(1) (MA(Cohen)) *For any graph G of cardinality $\aleph_1$, we have $\aleph_0 < chr(G)$ $\Leftrightarrow$ $\aleph_0 < list\text{-}chr(G)$.*

(2) *For any graph G of cardinality $\aleph_1$, if $\aleph_0 < col(G)$ then, for the poset $\mathbb{P} = \mathrm{Fn}(\omega_2, 2, <\aleph_1)$, we have $\Vdash_{\mathbb{P}}$ "$\aleph_0 < list\text{-}chr(G)$".* $\square$

We also need the following facts:

Facts 5.3. (1) (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [5])
FRP *is preserved by any c.c.c. forcing.*

(2) (Fuchino, Sakai, Soukup and Usuba [6])
Suppose that κ is strongly compact. Then we have $\Vdash_{\mathbb{P}}$ "FRP" where $\mathbb{P}$ is the standard poset $\mathbb{P} = \mathrm{Col}(\aleph_1, <\kappa)$ collapsing κ to ω_2. $\square$

Proof of Theorem 5.1: First, start from a model V of ZFC + FRP and force MA(Cohen) by a c.c.c. poset. Let $V[\mathcal{G}]$ be the generic extension. By Fact 5.3, (1), we still have FRP in $V[\mathcal{G}]$. By Theorem 3.1, there is a graph G witnessing $\mathfrak{Refl}_{chr} > \aleph_2$ in $V[\mathcal{G}]$. By Theorem 5.2, (1), this G witnesses also $\mathfrak{Refl}_{list\text{-}chr} > \aleph_2$ in $V[\mathcal{G}]$.

Assume now that κ is a strongly compact cardinal in V and let $V[G]$ be the generic extension obtained by collapsing κ to $\aleph_2$ by $\mathbb{P} = \mathrm{Col}(\aleph_1, <\kappa)$. Then we have $V[\mathcal{G}] \models$ FRP by Fact 5.3, (2). In $V[\mathcal{G}]$ every graph H of cardinality $\aleph_1$ is contained in cofinally many intermediate models over each of which many Cohen subsets of ω_1 are added. Hence, by Theorem 5.2, (2), if H has countable list-chromatic number in $V[\mathcal{G}]$ then H also has countable coloring number. Now, in $V[\mathcal{G}]$, if G is a graph such that all subgraphs H of cardinality $\leq \aleph_1$ have countable list-chromatic number, then they all have countable coloring number. By FRP in $V[\mathcal{G}]$ it follows that $col(G) \leq \aleph_0$. Since $list\text{-}chr(G) \leq col(G)$ we obtain $list\text{-}chr(G) \leq \aleph_0$. This shows that $\mathfrak{Refl}_{list\text{-}chr} = \aleph_2$ in $V[\mathcal{G}]$.

□ (Theorem 5.1)

Note that by the same argument as in the first part of the proof of Theorem 5.1, we obtain that MM implies $\mathfrak{Refl}_{list\text{-}chr} > \aleph_2$. This is quite unexpected since it is usualy so that if we have certain reflection phenomena then we do have it under MM or some weakening of it. Here we have consistently the reflection of countable list-chromatic number but MM refutes it! This might suggest that $\mathfrak{Refl}_{list\text{-}chr} = \aleph_2$ underlies a new type of reflection phenomenon still to be studied.

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