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A SUPER ANALOG OF THE KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

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1. Introduction

In the conference, I reported a joint work [KKT] with Masaki Kashiwara (RIMS) and Seok-Jin Kang (SNU) that proposes a super analog of the Khovanov-Lauda-Rouquier algebras which we call quiver Hecke superalgebras. Our main results [KKT, Theorem 4.4,Theorem 5.4] establish a "Morita superequivalence" (see [KKT, §2.4]) between the cyclotomic quotient of the quiver Hecke superalgebras and the cyclotomic quotient of the affine Hecke-Clifford superalgebras[1] and its degeneration.

If you are interested in our work, I believe the best way to grasp the synopsis is reading the introduction of [KKT] since our motivations and results of the work is best summarized in it.

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2. KLR algebras and the symmetric groups

Recently, Khovanov-Lauda and Rouquier independently introduced a remarkable family of algebras (the KLR algebras, the quiver Hecke algebras) that categorifies the negative half of the quantized enveloping algebras associated with symmetrizable Kac-Moody Lie algebras [KL1, KL2, Rou] (see Definition 2.1 and Theorem 2.4). An application of the KLR algebras is the gradation of the symmetric group algebras [BK1, Rou] (see Theorem 2.5) which quantizes Ariki's categorification of the Kostant $\mathbb{Z}$-form of the basic $\widehat{\mathfrak{sl}}_p$-module $V(\Lambda_0)^{\mathbb{Z}} \cong \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Proj}(\mathbb{F}_p\mathfrak{S}_n))$. The story is also valid for its q-analog, the Iwahori-Heck algebra of type A.

Definition 2.1 ([KL1, KL2, Rou]). *Let* $\mathbf{k}$ *be a field and let* I *be a finite set. Take a matrix* $Q = (Q_{ij}(u,v)) \in \mathsf{Mat}_I(\mathbf{k}[u,v])$ *such that* $Q_{ii}(u,v) = 0, Q_{ij}(u,v) = Q_{ji}(v,u)$ *for all* $i,j \in I$.

(a) *The Khovanov-Lauda-Rouquier algebra (KLR algebra, for short)* $R_n(\mathbf{k};Q)$ *for* $n \geq 0$ *is a* $\mathbf{k}$*-algebra generated by* $\{x_p, \tau_a, e_\nu \mid 1 \leq p \leq n, 1 \leq a < n, \nu \in I^n\}$ *with the following defining relations for all* $\mu, \nu \in I^n, 1 \leq p,q \leq n,\ 1 \leq b < a \leq n-1$.

- $e_\mu e_\nu = \delta_{\mu\nu} e_\mu, 1 = \sum_{\mu\in I^n} e_\mu,\ x_p x_q = x_q x_p,\ x_p e_\mu = e_\mu x_p$, • $\tau_a\tau_b = \tau_b\tau_a$ *if* $|a-b| > 1$,
- $\tau_a^2 e_\nu = Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e_\nu,\ \tau_a e_\mu = e_{s_a(\mu)}\tau_a$, • $\tau_a x_p = x_p \tau_a$ *if* $p \neq a, a+1$,
- $(\tau_a x_{a+1} - x_a\tau_a)e_\nu = (x_{a+1}\tau_a - \tau_a x_a)e_\mu = \delta_{\nu_a,\nu_{a+1}} e_\nu$,
- $(\tau_{b+1}\tau_b\tau_{b+1} - \tau_b\tau_{b+1}\tau_b)e_\nu = \delta_{\nu_b,\nu_{b+2}}((x_{b+2}-x_b)^{-1}(Q_{\nu_b,\nu_{b+1}}(x_{b+2},x_{b+1}) - Q_{\nu_b,\nu_{b+1}}(x_b,x_{b+1})))e_\nu$.

(b) *For* $\beta = \sum_{i\in I}\beta_i \cdot i \in \mathbb{N}[I]$ *with* $n = \mathsf{ht}(\beta) := \sum_{i\in I}\beta_i$, *we define* $R_\beta(\mathbf{k};Q) = R_n(\mathbf{k};Q)e_\beta$ *where* $e_\beta = \sum_{\nu\in\mathsf{Seq}(\beta)} e_\nu$ *and* $\mathsf{Seq}(\beta) = \{(i_j)_{j=1}^n \in I^n \mid \sum_{j=1}^n i_j = \beta\}$.

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[1]They can be regarded as a superanalog of the Ariki-Koike algebras.

(c) *For $\lambda = \sum_{i\in I} \lambda_i \cdot i \in \mathbb{N}[I]$ and $\beta \in \mathbb{N}[I]$ with $n = \mathsf{ht}(\beta)$, we define*

$$R^\lambda_n(\mathbf{k};Q) = R_n(\mathbf{k};Q)/R_n(\mathbf{k};Q)(\textstyle\sum_{\nu\in I^n} x_1^{\lambda_{h_{\nu_1}}} e_\nu)R_n(\mathbf{k};Q),$$
$$R^\lambda_\beta(\mathbf{k};Q) = R_\beta(\mathbf{k};Q)/R_\beta(\mathbf{k};Q)(\textstyle\sum_{\nu\in\mathsf{Seq}(\beta)} x_1^{\lambda_{h_{\nu_1}}} e_\nu)R_\beta(\mathbf{k};Q).$$

As a consequence of PBW theorem for KLR algebras, we see that $\{e_\beta \mid \mathsf{ht}(\beta) = n\}$ exhausts all the primitive central idempotents of $R_n(\mathbf{k};Q)$. Thus, $R_n(\mathbf{k};Q) = \bigoplus_{\substack{\beta\in\mathbb{N}[I]\\ \mathsf{ht}(\beta)=n}} R_\beta(\mathbf{k};Q)$ is a decomposition into indecomposable factors. It is not difficult to see that both $R^\lambda_n(\mathbf{k};Q)$ and $R^\lambda_\beta(\mathbf{k};Q)$ are finite dimensional $\mathbf{k}$-algebras.

Definition 2.2 ([KL1, KL2, Rou]). *Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix with the symmetrization $d = (d_i)_{i\in I}$, i.e., a unique $d \in \mathbb{Z}^I_{\geq 1}$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$ and $\gcd(d_i)_{i\in I} = 1$. Take $Q^A = (Q^A_{ij}(u,v)) \in \mathsf{Mat}_I(\mathbf{k}[u,v])$ subject to*

$$Q^A_{ii}(u,v) = 0,\quad Q^A_{ij}(u,v) = Q^A_{ji}(v,u),\quad t_{i,j,-a_{ij},0} = t_{j,i,0,-a_{ij}} \neq 0$$

for all $i,j \in I$ where $Q^A_{ij}(u,v) = \sum_{\substack{p,q\geq 0\\ pd_i+qd_j=-d_ia_{ij}}} t_{ijpq}u^pv^q$.

For $n \geq 0$ and $\lambda, \beta \in \mathbb{N}[I]$ with $\mathsf{ht}(\beta) = n$, all of $R_n(\mathbf{k};Q^A)$, $R_\beta(\mathbf{k};Q^A)$, $R^\lambda_n(\mathbf{k};Q^A)$, $R^\lambda_\beta(\mathbf{k};Q^A)$ are $\mathbb{Z}$-graded via the assignment where $\nu \in I^n, 1 \leq p \leq n, 1 \leq a < n$.

$$\mathsf{deg}(e_\nu) = 0,\quad \mathsf{deg}(x_pe_\nu) = 2d_{\nu_p},\quad \mathsf{deg}(\tau_ae_\nu) = -d_{\nu_a}a_{\nu_a,\nu_{a+1}}.$$

Definition 2.3. *Let R a graded algebra. We denote by $\mathsf{Proj}_{\mathsf{gr}}(R)$ the category of finitely generated left graded projective R-modules and degree preserving R-homomorphisms.*

The grading shift autoequivalence $\langle -1\rangle : \mathsf{Proj}_{\mathsf{gr}}(R) \xrightarrow{\sim} \mathsf{Proj}_{\mathsf{gr}}(R)$ affords a $\mathbb{Z}[v,v^{-1}]$-module structure on $\mathsf{K}_0(\mathsf{Proj}_{\mathsf{gr}}(R))$ via $v = [\langle -1\rangle]$.

Theorem 2.4 ([KL1, KL2, Rou]). *Let A be a symmetrizable generalized Cartan matrix and let $\mathscr{A} = \mathbb{Z}[v,v^{-1}]$. Then, the following categorification results hold (though we don't explain how to define an algebra structure in (a) nor how to define a $U^{\mathscr{A}}_v(A)$-module structure in (b)).*

(a) *as an $\mathscr{A}$-algebra, we have $\bigoplus_{n\geq 0}\mathsf{K}_0(\mathsf{Proj}_{\mathsf{gr}}(R_n(\mathbf{k};Q^A))) \cong U^{-,\mathscr{A}}_v(A)$.*

(b) *as a $U^{\mathscr{A}}_v(A)$-module, we have $\bigoplus_{n\geq 0}\mathsf{K}_0(\mathsf{Proj}_{\mathsf{gr}}(R^\lambda_n(\mathbf{k};Q^A))) \cong V(\lambda)^{\mathscr{A}}$.*

Here $U^{\mathscr{A}}_v(A)$ (resp. $U^{-,\mathscr{A}}_v(A), V(\lambda)^{\mathscr{A}}$) is the Lusztig's $\mathscr{A}$-lattice of $U_v(A)$ (resp. $U^-_v(A), V(\lambda)$) and we identify $\lambda \in \mathcal{P}^+$ with $\sum_{i\in I}\lambda(h_i)\cdot i \in \mathbb{N}[I]$.

Recall that $A^{(1)}_{\ell-1} = (2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j\in\mathbb{Z}/\ell\mathbb{Z}}$ for $\ell \geq 2$ and $\widehat{\mathfrak{sl}}_\ell = \mathfrak{g}(A^{(1)}_{\ell-1})$.

Theorem 2.5 ([BK1, Rou]). *Let $\mathbf{k}$ be a field of characteristic $p > 0$. Then, as a $\mathbf{k}$-algebra we have $\mathbf{k}\mathfrak{S}_n \cong R^{\Lambda_0}_n(\mathbf{k};Q^{A^{(1)}_{p-1}})$ where $Q^{A^{(1)}_{p-1}}_{ij}(u,v) = \pm(u-v)^{-2\delta_{ij}+\delta_{i+1,j}+\delta_{i-1,j}}$ for $i \neq j \in \mathbb{Z}/p\mathbb{Z}$ (though we don't explain how to choose signs).*

You can find related topics to Theorem 2.5 in a well-written survey paper [Kl1] which can be seen as an update of [Kl2].

3. Super representations

We briefly recall our conventions and notations for superalgebras and supermodules following [BK2, §2-b] (see also the references therein). Although they are different from [KKT, §2], we review [BK2, §2-b] in order to cite [BK2, Tsu]. In this section, we always assume that in our field $\mathbf{k}$ we have $2 \neq 0$.

3.1. **Superspaces.** By a vector superspace, we mean a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ over $\mathbf{k}$ and denote the parity of a homogeneous vector $v \in V$ by $\overline{v} \in \mathbb{Z}/2\mathbb{Z}$. Given two vector superspaces V and W, an $\mathbf{k}$-linear map $f : V \to W$ is called homogeneous if there exists $p \in \mathbb{Z}/2\mathbb{Z}$ such that $f(V_i) \subseteq W_{p+i}$ for $i \in \mathbb{Z}/2\mathbb{Z}$. In this case we call p the parity of f and denote it by $\overline{f}$.

3.2. **Superalgebras.** A superalgebra A is a vector superspace which is an unital associative $\mathbf{k}$-algebra such that $A_iA_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. By an A-supermodule, we mean a vector superspace M which is a left A-module such that $A_iM_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$.

3.3. **Super categories.** In the rest of the paper, we only deal with finite-dimensional A-supermodules. Given two A-supermodules V and W, an A-homomorphism $f : V \to W$ is an $\mathbf{k}$-linear map such that $f(av) = (-1)^{\overline{f}\overline{a}} af(v)$ for $a \in A$ and $v \in V$. We denote the set of A-homomorphisms from V to W by $\mathrm{Hom}_A(V, W)$. By this, we can form a superadditive category A-smod whose hom-set is a vector superspace in a way that is compatible with composition. However, we adapt a slightly different definition of isomorphisms from the categorical one.

3.4. **Parity change functors.** Two A-supermodules V and W are called evenly isomorphic (and denoted by $V \simeq W$) if there exists an even A-homomorphism $f : V \to W$ which is an $\mathbf{k}$-vector space isomorphism. They are called isomorphic (and denoted by $V \cong W$) if $V \simeq W$ or $V \simeq \Pi W$. Here for an A-supermodule M, ΠM is an A-supermodule defined by the same but the opposite grading underlying vector superspace $(\Pi M)_i = M_{i+\overline{1}}$ for $i \in \mathbb{Z}/2\mathbb{Z}$ and a new action given as follows from the old one $a \cdot_{\mathsf{new}} m = (-1)^{\overline{a}} a \cdot_{\mathsf{old}} m$.

3.5. **Types of simple supermodules.** We denote the isomorphism class of an A-supermodule M by $[M]$ and denote the set of isomorphism classes of irreducible A-supermodules by $\mathsf{Irr}(A\text{-}\mathsf{smod})$. Let us assume that V is irreducible. We say that V is type Q if $V \simeq \Pi V$ otherwise type M.

3.6. **Super tensor products.** Given two superalgebras A and B, $A \otimes B$ with multiplication defined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\overline{b_1}\overline{a_2}}(a_1a_2) \otimes (b_1b_2)$ for $a_i \in A, b_j \in B$ is again a superalgebra[2]. Let V be an A-supermodule and let W be a B-supermodule. Their tensor product $V \otimes W$ is an $A \otimes B$-supermodule by the action given by $(a \otimes b)(v \otimes w) = (-1)^{\overline{b}\overline{v}}(av) \otimes (bw)$ for $a \in A, b \in B, v \in V, w \in W$. Let us assume that V and W are both irreducible. If V and W are both of type Q, then there exists a unique (up to odd isomorphism) irreducible $A \otimes B$-supermodule X of type M such that $V \otimes W \simeq X \oplus \Pi X$ as $A \otimes B$-supermodules. We denote X by $V \circledast W$. Otherwise $V \otimes W$ is irreducible but we also write it as $V \circledast W$. Note that $V \circledast W$ is defined only up to isomorphism in general and $V \circledast W$ is of type M if and only if V and W are of the same type.

3.7. **Grothendieck groups.** For a superalgebra A, we define the Grothendieck group $\mathsf{K}_0(A\text{-}\mathsf{smod})$ to be the quotient of the $\mathbb{Z}$-module freely generated by all finite-dimensional A-supermodules by the $\mathbb{Z}$-submodule generated by

- $V_1 - V_2 + V_3$ for every short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $A\text{-}\mathsf{smod}_{\overline{0}}$.
- $M - \Pi M$ for every A-supermodule M.

Here $A\text{-}\mathsf{smod}_{\overline{0}}$ is the abelian subcategory of A-smod whose objects are the same but morphisms are consisting of even A-homomorphisms. Clearly, $\mathsf{K}_0(A\text{-}\mathsf{smod})$ is a free $\mathbb{Z}$-module with basis $\mathsf{Irr}(A\text{-}\mathsf{smod})$. The importance of the operation $\circledast$ lies in the fact that it gives an isomorphism

$$\mathsf{K}_0(A\text{-}\mathsf{smod}) \otimes_{\mathbb{Z}} \mathsf{K}_0(B\text{-}\mathsf{smod}) \xrightarrow{\sim} \mathsf{K}_0(A \otimes B\text{-}\mathsf{smod}), \quad [V] \otimes [W] \longmapsto [V \circledast W] \tag{3.1}$$

for two superalgebras A and B.

[2]Note that in general we have $|A \otimes B| \ncong |A| \otimes |B|$ where for a superalgebra C we denote by $|C|$ the underlying unital associative algebra.

3.8. **Projective supermodules.** Let A be a superalgebra. A projective A-supermodule is, by definition, a projective object in $A\text{-smod}$ and it is equivalent to saying that it is a projective object in $A\text{-smod}_{\bar{0}}$ since there are canonical isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{A\text{-smod}}(V,W)_{\bar{0}} \cong \mathrm{Hom}_{A\text{-smod}_{\bar{0}}}(V,W),\\ &\mathrm{Hom}_{A\text{-smod}}(V,W)_{\bar{1}} \cong \mathrm{Hom}_{A\text{-smod}_{\bar{0}}}(V,\Pi W)(\cong \mathrm{Hom}_{A\text{-smod}_{\bar{0}}}(\Pi V,W)).\end{aligned}$$

We denote by $\mathsf{Proj}(A)$ the full subcategory of $A\text{-smod}$ consisting of all the projective A-supermodules.

3.9. **Cartan pairings.** Let us assume further that A is finite-dimensional. Then, as in the usual finite-dimensional algebras, every A-supermodule X has a (unique up to even isomorphism) projective cover P_X in $A\text{-smod}_{\bar{0}}$. If X is irreducible, then P_X is (evenly) isomorphic to a projective indecomposable A-supermodule. From this, we easily see $M \cong N$ if and only if $P_M \cong P_N$ for $M, N \in \mathsf{Irr}(A\text{-smod})$. Thus, $\mathsf{K}_0(\mathsf{Proj}(A))$ is identified with $\mathsf{K}_0(A\text{-smod})^* \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(\mathsf{K}_0(A\text{-smod}), \mathbb{Z})$ through the non-degenerate canonical pairing

$$\begin{aligned}\langle , \rangle_A : \mathsf{K}_0(\mathsf{Proj}(A)) \times \mathsf{K}_0(A\text{-smod}) &\longrightarrow \mathbb{Z},\\ ([P_M],[N]) &\longmapsto \begin{cases} \dim \mathrm{Hom}_A(P_M,N) & \text{if type } M = \mathsf{M},\\ \frac{1}{2}\dim \mathrm{Hom}_A(P_M,N) & \text{if type } M = \mathsf{Q},\end{cases}\end{aligned}$$

for all $M \in \mathsf{Irr}(A\text{-smod})$ and $N \in A\text{-smod}$. Note that the left hand side is nothing but the composition multiplicity $[N : M]$. We also reserve the symbol

$$\omega_A : \mathsf{K}_0(\mathsf{Proj}(A)) \longrightarrow \mathsf{K}_0(A\text{-smod})$$

for the natural Cartan map.

3.10. **Clifford superalgebras.** The Clifford superalgebra is defined as $\mathcal{C}_n = \mathcal{C}_1^{\otimes n}$ for $n \geq 0$ where $\mathcal{C}_1$ is a 2-dimensional superalgebra generated by the odd generator C with $C^2 = 1$. Assume $\sqrt{-1} \in \mathbf{k}$, then $\mathcal{C}_n$ is a split-simple superalgebra, but $|\mathcal{C}_n|$ is split-simple if and only if n is even. We denote by $U_n = \mathcal{C}_1^{\circledast n}$ the Clifford module, i.e., a $2^{\lfloor (n+1)/2 \rfloor}$-dimensional irreducible $\mathcal{C}_n$-supermodule (of type Q iff n is odd) characterized by $\mathsf{Irr}(\mathcal{C}_n\text{-smod}) = \{[U_n]\}$ noting (3.1).

3.11. **Morita superequivalences.** We must clarify our meaning of the terminology Morita superequivalence. Again we emphasize that our meaning of Morita superequivalence in this article is similar to [Kl2, BK2, Wan] and different from that of [KKT, §2.4].

Two superalgebras A and B are called Morita superequivalent of type M if there exist superadditive functors $F : A\text{-smod} \to B\text{-smod}$ and $G : B\text{-smod} \to A\text{-smod}$ such that $G \circ F \simeq \mathrm{id}, F \circ G \simeq \mathrm{id}$ and both $F|_{\mathsf{Irr}(A\text{-smod})} : \mathsf{Irr}(A\text{-smod}) \xrightarrow{\sim} \mathsf{Irr}(B\text{-smod}), G|_{\mathsf{Irr}(B\text{-smod})} : \mathsf{Irr}(B\text{-smod}) \xrightarrow{\sim} \mathsf{Irr}(A\text{-smod})$ are type preserving. We say that A and B are called Morita superequivalent of type Q if there exist superadditive functors $F : A\text{-smod} \to B\text{-smod}$ and $G : B\text{-smod} \to A\text{-smod}$ such that $G \circ F \simeq \mathrm{id} \oplus \Pi, F \circ G \simeq \mathrm{id} \oplus \Pi$ and induces type reversing bijections

$$\begin{aligned}\{[V] \in \mathsf{Irr}(A\text{-smod}) \mid \text{type } V = \mathsf{M}\} &\xrightarrow{\sim} \{[W] \in \mathsf{Irr}(B\text{-smod}) \mid \text{type } W = \mathsf{Q}\},\\ \{[V] \in \mathsf{Irr}(B\text{-smod}) \mid \text{type } V = \mathsf{M}\} &\xrightarrow{\sim} \{[W] \in \mathsf{Irr}(A\text{-smod}) \mid \text{type } W = \mathsf{Q}\}.\end{aligned}$$

We say that A and B are called Morita superequivalent if they are either Morita superequivalent of type M or type Q.

Example 3.1. Let A be a superalgebra and $e \in A$ a full even idempotent, i.e., $e \in A_{\bar{0}}, e^2 = e$ and $A = AeA \stackrel{\text{def}}{=} \{\sum_{i=1}^{n} a_i e b_i \mid a_i, b_i \in A, n \geq 0\}$. Then, A and eAe are Morita superequivalent of type M.

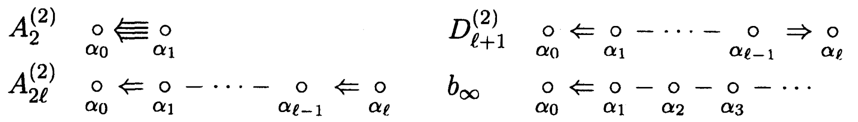


FIGURE 1. Dynkin diagrams of type $A_{2\ell}^{(2)}, D_{\ell+1}^{(2)}$ and b_∞.

Example 3.2. Let A and B superalgebras and suppose there exists a superalgebra isomorphism $A \otimes \mathcal{C}_n \xrightarrow{\sim} B$ for some $n \geq 0$. Then, A and B are Morita equivalent of type Q (resp. type M) if n is odd (resp. n is even) via

$$F : A\text{-smod} \longrightarrow B\text{-smod}, \quad V \longmapsto \mathrm{Hom}_{\mathcal{C}_n}(U_n, V),$$
$$G : B\text{-smod} \longrightarrow A\text{-smod}, \quad W \longmapsto W \otimes U_n.$$

4. PARTIAL CATEGORIFICATIONS USING HECKE-CLIFFORD SUPERALGEBRAS

From now on, we reserve a non-zero quantum parameter $q \in \mathbf{k}^\times$ and set $\xi = q - q^{-1}$ for convenience. Let us define the affine Hecke-Clifford superalgebra [JN, §3]. Although Jones and Nazarov introduced it under the name of affine Sergeev algebra, we call it affine Hecke-Clifford superalgebra following [BK2, §2-d].

Definition 4.1 ([JN]). *Let $n \geq 0$ be an integer. The affine Hecke-Clifford superalgebra $\mathcal{H}_n$ is defined by even generators $X_1^{\pm 1}, \cdots, X_n^{\pm 1}, T_1, \cdots, T_{n-1}$ and odd generators $C_1, \cdots, C_n$ with the following relations.*

(1) $X_iX_i^{-1} = X_i^{-1}X_i = 1, X_iX_j = X_iX_j$ *for all* $1 \leq i, j \leq n$.
(2) $C_i^2 = 1, C_iC_j + C_jC_i = 0$ *for all* $1 \leq i \neq j \leq n$.
(3) $T_i^2 = \xi T_i + 1, T_iT_j = T_jT_i, T_kT_{k+1}T_k = T_{k+1}T_kT_{k+1}$ *for all* $1 \leq k \leq n-2$ *and* $1 \leq i, j \leq n-1$ *with* $|i-j| \geq 2$.
(4) $C_iX_i^{\pm 1} = X_i^{\mp 1}C_i, C_iX_j^{\pm 1} = X_j^{\pm 1}C_i$ *for all* $1 \leq i \neq j \leq n$.
(5) $T_iC_i = C_{i+1}T_i, (T_i + \xi C_iC_{i+1})X_iT_i = X_{i+1}$ *for all* $1 \leq i \leq n-1$.
(6) $T_iC_j = C_jT_i, T_iX_j^{\pm 1} = X_j^{\pm 1}T_i$ *for all* $1 \leq i \leq n-1$ *and* $1 \leq j \leq n$ *with* $j \neq i, i+1$.

Definition 4.2 ([BK2, Tsu]). *Let $\mathbf{k}$ be a field whose characteristic different from 2 and take $q \in \mathbf{k}^\times$.*

(a) $\mathsf{Rep}\,\mathcal{H}_n$ *is a fullsubcategory of* $\mathcal{H}_n$-smod *consisting of $\mathcal{H}_\mu$-supermodule M such that the set of eigenvalues of $X_j + X_j^{-1}$ is a subset of $\{q(i) \mid i \in \mathbb{Z}\}$ for all $1 \leq j \leq n$*[3] *where* $q(i) = 2 \cdot (q^{2i+1} + q^{-(2i+1)})/(q + q^{-1})$.
(b) *Put I be the set of vertices of Dynkin diagram X (see Figure 1) where*

$$X = \begin{cases} A_{2\ell}^{(2)} & (\textit{if } q^2 \textit{ is a primitive } (2\ell+1)\textit{-the root of unity for some } \ell \geq 1) \\ D_{\ell+1}^{(2)} & (\textit{if } q^2 \textit{ is a primitive } 2(\ell+1)\textit{-the root of unity for some } \ell \geq 1) \\ b_\infty & (\textit{if otherwise and moreove we have } q^4 \neq 1). \end{cases}$$

We define for a dominant integral weight $\lambda \in \mathcal{P}^+$ of X a finite-dimensional quotient superalgebra $\mathcal{H}_n = \langle f^\lambda \rangle$ where $g^\lambda = \prod_{i \in I}(X_1^2 - q(i)X_1 + 1)^{\lambda(h_i)}$ and

$$f^\lambda = \begin{cases} g^\lambda/((X_1 - 1)^{\lambda(h_0)}(X_1 - 1)^{\lambda(h_\ell)}) & (\textit{if } X = D_{\ell+1}^{(2)}) \\ g^\lambda/(X_1 - 1)^{\lambda(h_0)} & (\textit{if } X = A_{2\ell}^{(2)}, b_\infty) \end{cases}$$

[3]It is equivalent to require only the set of eigenvalues of $X_1 + X_1^{-1}$ is a subset of $\{q(i) \mid i \in \mathbb{Z}\}$ by [BK2, Lemma 4.4].

Remark 4.3. In the setting of Definition 4.2 (b), for $M \in \mathcal{H}_n\text{-smod}$ we have $M \in \mathsf{Rep}\,\mathcal{H}_n \Leftrightarrow \exists\lambda \in \mathcal{P}^+, f^\lambda M = 0$

Theorem 4.4 ([BK2, Tsu]). *Let* $\mathbf{k}$ *be an algebraically closed field whose characteristic different from 2 and take* $q \in \mathbf{k}^\times$ *and* X *as in Definition 4.2 (b). Then, we have the following.*

(a) *the graded dual of* $K(\infty) = \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$ *is isomorphic to* $U_{\mathbb{Z}}^+$ *as graded* $\mathbb{Z}$*-Hopf algebra.*
(b) $K(\lambda)_{\mathbb{Q}} = \bigoplus_{n\geq 0} \mathbb{Q}\otimes \mathsf{K}_0(\mathcal{H}_n^\lambda\text{-smod})$ *has a left* $U_{\mathbb{Q}}$*-module structure which is isomorphic to the integrable highest weight* $U_{\mathbb{Q}}$*-module of highest weight* λ.
(c) $B(\infty) = \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ *is isomorphic to Kashiwara's crystal associated with* $U_v^-(\mathfrak{g}(X))$.
(d) $B(\lambda) = \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathcal{H}_n^\lambda\text{-smod})$ *is isomorphic to Kashiwara's crystal associated with the integrable* $U_v(\mathfrak{g}(X))$*-module of highest weight* λ.
(e) $K(\lambda)^* = \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Proj}(\mathcal{H})_n^\lambda)$ *and* $K(\lambda) \bigoplus_{n\geq 0} \mathsf{K}_0(\mathcal{H}_n^\lambda\text{-smod})$ *are two integral lattices of* $K(\lambda)_{\mathbb{Q}}$ *containing the trivial representation* $[\mathbf{1}_\lambda]$ *of* $\mathcal{H}_0^\lambda = \mathbf{k}$. *Moreover,* $K(\lambda)^*$ *is minimum lattice in the sense that* $K(\lambda)^* = U_{\mathbb{Z}}^-[\mathbf{1}_\lambda]$.

Here $U_{\mathbb{Z}}^\pm$ *is the* $\pm$*-part of the Kostant* $\mathbb{Z}$*-form of the universal enveloping algebra of* $\mathfrak{g}(X)$ *and* $U_{\mathbb{Q}}$ *is the* $\mathbb{Q}$*-subalgebra of the universal enveloping algebra of* $\mathfrak{g}(X)$ *generated by the Chevalley generators.*

Remark 4.5. Since A-smod is not necessarily an abelian category for a superalgebra A, Theorem 4.4 cannot be seen as a categorification result in the usual sense (see for example [KMS]). For example, in the identification Theorem 4.4 (b) neither the action of Chevalley generators e_i nor f_i are "exact" functors, of course. We just can assign for each simple module identified up to parity change (which is a basis of the Grothendieck groups (see 3.7)) a well-defined destination in a "module-theoretic" way.

Remark 4.6. Under the identification (b) and (e) of Theorem 4.4, the Cartan pairing on $K(\lambda)_{\mathbb{Q}}$ coincides with the Shapovalov form [BK2, Tsu]. It is expected but not proved so far[4] that the decomposition of $K(\lambda)_{\mathbb{Q}}$ comes from the block decomposition of $\{\mathcal{H}_n^\lambda \mid n \geq 0\}$ coincides with the weight space decomposition of the corresponding integrable highest weight module.

5. An expectation and two counterexamples

Considering both Theorem 2.4 and Theorem 4.4, it is reasonable to expect that in the setting of Definition 4.2 (b), $R_n^\lambda(X;Q^X)$ and $\mathcal{H}_n^\lambda$ has a "good relation" as Theorem 2.5. However, we believe that this expectation never achieved because of the following two facts.

5.1. **$X = D_2^{(2)}$ case.** Let $q = \exp(2\pi\sqrt{-1}/8) \in \mathbf{k}$ and let $\mathrm{char}\,\mathbf{k} = 0$. In virtue of Theorem 2.4 and Theorem 4.4, the family of (super)algebras $\{\mathcal{H}_n^{\Lambda_0}(q)\}_{n\geq 0}$ (resp. $\{R_n^{\Lambda_0}(\mathbf{k};Q^X)\}_{n\geq 0}$) categorifies $U(\mathfrak{g}(X))$-module (resp. $U_v(\mathfrak{g}(X))$-module) $V(\Lambda_0)$.

However, there is no Morita equivalence between $|\mathcal{H}_4^{\Lambda_0}(X)|$ and $R_4^{\Lambda_0}(\mathbf{k};Q^X)$ nor Morita superequivalence of type M between $\mathcal{H}_4^{\Lambda_0}(X)$ and $R_4^{\Lambda_0}(\mathbf{k};Q^X)$ whatever superalgebra structure we impose $R_4^{\Lambda_0}(\mathbf{k};Q^X)$ on and for any choice of parameters Q^X. This is because we have

$$\dim Z(|\mathcal{H}_4^{\Lambda_0}(q)|) = 4 \neq 5 = \dim Z(|R_4^{\Lambda_0}(\mathbf{k};Q^X)|).$$

Because $\#\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(R_4^{\Lambda_0}(\mathbf{k};Q^X))) = 2$ and $\mathsf{Irr}(\mathcal{H}_4^{\Lambda_0}(q)\text{-smod})$ consists of 2 irreducible supermodules of type M, there is no possibility that $\mathcal{H}_4^{\Lambda_0}(X)$ and $R_4^{\Lambda_0}(\mathbf{k};Q^X)$ get Morita superequivalence of type Q by defining a superalgebra structure on $R_4^{\Lambda_0}(\mathbf{k};Q^X)$ appropriately.

---

[4]For the degenerate case, some partial results are known [Ruf].

5.2. $X = A_2^{(2)}$ **and degenerate case.** Let us briefly recall the affine Sergeev superalgebra $\overline{\mathcal{H}}_n$ introduced by Nazarov in his study of spin Young symmetrizers for the symmetric groups [Naz].

Definition 5.1. (i) *The spin symmetric group superalgebra* $\mathbf{k}\mathfrak{S}_n^-$ *is defined by odd generators* $\{t_i \mid 1 \le i \le n-1\}$ *and the following relations*

$$t_a^2 = 1, \quad t_a t_b = -t_b t_a \text{ if } |a-b| > 1, \quad t_c t_{c+1} t_c = t_{c+1} t_c t_{c+1}.$$

(ii) *The Sergeev superalgebra is defined as* $\mathcal{Y}_n = \mathbf{k}\mathfrak{S}_n^- \otimes \mathcal{C}_n$ *(for super tensor product, see §3.6) where* $\mathcal{C}_n$ *is the Clifford superalgebra (see §3.10).*

(iii) *The affine Sergeev superalgebra* $\overline{\mathcal{H}}_n$ *is the* $\mathbf{k}$*-superalgebra generated by the even generators* $x_1, \dots, x_n, t_1, \dots, t_{n-1}$ *and the odd generators* $C_1, \dots, C_n$ *with the following relations.*
  - (i) $x_i x_j = x_j x_i$ *for all* $1 \le i, j \le n$,
  - (ii) $C_i^2 = 1$, $C_i C_j + C_j C_i = 0$ *for all* $1 \le i \ne j \le n$,
  - (iii) $t_i^2 = 1$, $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, $t_i t_j = t_j t_i$ $(|i-j| \ge 2)$,
  - (iv) $t_i C_j = C_{s_i(j)} t_i$,
  - (v) $C_i x_j = x_j C_i$ *for all* $1 \le i \ne j \le n$,
  - (vi) $C_i x_i = -x_i C_i$ *for all* $1 \le i \le n$,
  - (vii) $t_i x_i = x_{i+1} t_i - 1 - C_i C_{i+1}$, $t_i x_{i+1} = x_i t_i + 1 - C_i C_{i+1}$ *for all* $1 \le i \le n-1$,
  - (viii) $t_i x_j = x_j t_i$ *if* $j \ne i, i+1$.

$\overline{\mathcal{H}}_n$ is an affinization of the Sergeev superalgebra $\mathcal{Y}_n$ and $\overline{\mathcal{H}}_n$ has $\mathcal{Y}_n$ as its finite-dimensional quotient $\mathcal{Y}_n \cong \overline{\mathcal{H}}_n^{\Lambda_0} := \overline{\mathcal{H}}_n / \langle x_1 \rangle$ since there is a non-trivial superisomorphism

$$\text{(5.1)} \qquad \mathbf{k}\mathfrak{S}_n^- \otimes \mathcal{C}_n \xrightarrow{\sim} \mathbf{k}\mathfrak{S}_n^- \ltimes \mathcal{C}_n \quad 1 \otimes C_j \mapsto 1 \otimes C_j, \quad t_i \otimes 1 \mapsto \frac{1}{\sqrt{-2}} s_i \otimes (\dot{C}_i - C_{i+1}).$$

due to Sergeev and Yamaguchi [Ser, Yam]. Note that $\mathcal{Y}_n$ is Morita superequivalent to $\mathbf{k}\mathfrak{S}_n^-$ (see Example 3.2),

Modular representation theory of $\overline{\mathcal{H}}_n$ was considerably developed in [BK2] using the method of Grojnowski [Gro]. A consequence of [BK2] is that the category of finite-dimensional integral $\overline{\mathcal{H}}_n$-supermodules partially categorifies $U^-(\mathfrak{g}(b_\infty))$ (resp. $U^-(\mathfrak{g}(A_{2\ell}^{(2)}))$ when $\operatorname{char}\mathbf{k} = 0$ (resp. $\operatorname{char}\mathbf{k} = 2\ell + 1$ for $\ell \ge 1$) as Theorem 4.4.

Assume $\operatorname{char}\mathbf{k} = 3$ and put $X = A_2^{(2)}$ (see Figure 1). Take a block subsuperalgebra B of $\overline{\mathcal{H}}_{11}$ which categorifies $U^-(\mathfrak{g}(X))_{-\nu}$ where $\nu = 8\alpha_0 + 3\alpha_1$. Although $R_\nu(\mathbf{k}; Q^X)$ categorifies $U_v^-(\mathfrak{g}(X))_{-\nu}$, $\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(R_\nu(\mathbf{k}; Q^X)))$ and $\mathsf{Irr}(B\text{-}\mathsf{smod})$ correspond to different perfect basis at the specialization $v = 1$.

Let us explain in detail. By [BK2] (see also [Kl2, part II]), we have

$$\text{(5.2)} \qquad \bigoplus_{n \ge 0} \mathsf{K}_0(\overline{\mathcal{H}}_n^{\Lambda_0}\text{-}\mathsf{smod})_{\mathbb{C}} \cong V(\Lambda_0), \quad \bigsqcup_{n \ge 0} \mathsf{Irr}(\overline{\mathcal{H}}_n^{\Lambda_0}\text{-}\mathsf{smod}) \cong \mathsf{RP}_3 \cong B(\Lambda_0)$$

where the left isomorphism is as $U(\mathfrak{g}(X))$-modules and the right isomorphism is as $U_v(\mathfrak{g}(X))$-crystals. In virtue of (5.1) and Example 3.2, the same Lie-theoretic descriptions hold when we replace $\overline{\mathcal{H}}_n^{\Lambda_0}$ with $\mathbf{k}\mathfrak{S}_n^-$.

Recall RP_3 is the set of all 3-restricted 3-strict partitions. A partition $\lambda = (\lambda_1, \cdots, \lambda_r)$ is 3-restricted 3-strict if the following conditions are satisfied [Kan, Kl2, LT].

- $\lambda_k = \lambda_{k+1}$ implies $\lambda_k \in 3\mathbb{Z}$,
- $\lambda_k - \lambda_{k+1} < 3$ if $\lambda_k \in 3\mathbb{Z}$,
- $\lambda_k - \lambda_{k+1} \le 3$ if $\lambda_k \notin 3\mathbb{Z}$.

For each $\lambda \in \mathsf{RP}_3 \cong B(\Lambda_0)$, we denote by $V_\lambda^{\mathsf{spin}}$ the corresponding isomorphism class of irreducibles of $\mathbf{k}\mathfrak{S}_{|\lambda|}^-$. Note that $V_\lambda^{\mathsf{spin}}$ is of type Q if and only if $\gamma_1(\lambda) := \sum_{k \ge 1} \lfloor \frac{1+\lambda_k}{3} \rfloor$ is odd.

On the other hand, by [KK, LV] we have

$$\bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Mod}_{\mathsf{gr}}(R_n^{\Lambda_0}(\mathbf{k};Q^X)))_{\mathbb{C}} \cong V(\Lambda_0), \quad \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(R_n^{\Lambda_0}(\mathbf{k};Q^X)) \cong B(\Lambda_0)$$

where the left isomorphism is as $U_v(\mathfrak{g}(X))$-modules and the right isomorphism is as $U_v(\mathfrak{g}(X))$-crystals. For each $\lambda \in \mathsf{RP}_3 \cong B(\Lambda_0)$, we denote by V_λ^{KLR} the corresponding isomorphism class of irreducibles of $R_n^{\Lambda_0}(\mathbf{k};Q^X)$.

If both $\mathsf{Irr}(\mathsf{Mod}_{\mathsf{gr}}(R_\nu(\mathbf{k};Q^X)))$ and $\mathsf{Irr}(B\text{-}\mathsf{smod})$ correspond (after the specialization $v=1$) the same perfect basis in the sense of [BeKa] on $U(\mathfrak{g}(X))$-module $V(\Lambda_0)$, then we must have

$$\dim V_\lambda^{\mathsf{spin}} / \dim V_\lambda^{\mathsf{KLR}} = 2^{[(1+\gamma_1(\lambda))/2]}$$

for any $\lambda \in \mathsf{RP}_3$ (see [Kl2, Lemma 22.3.8]). A computer calculation shows that for $\lambda = (6,4,1)$, we have $\dim V_\lambda^{\mathsf{KLR}} = 648$ while it is known that $\dim V_\lambda^{\mathsf{spin}} = 2880$. It may be interesting to point out that in history this dimension $\dim V_\lambda^{\mathsf{spin}} = 2880$ was first miscalculated as $\dim V_\lambda^{\mathsf{spin}} = 2592$ in [MY]. If it were correct, observing such a direct discrepancy between the KLR algebras and the spin symmetric groups must become more difficult.

6. Quiver Hecke superalgebras

Definition 6.1 ([KKT, §3.1]). *Let $\mathbf{k}$ be a field such that $2 \neq 0$ and let I be a finite set with parity decomposition $I = I_{\mathrm{odd}} \sqcup I_{\mathrm{even}}$. For $i \in I$, we denote the parity of i by $\mathsf{par}(i) \in \mathbb{Z}/2\mathbb{Z}$, i.e., $\mathsf{par}(i) = \overline{1}$ if $i \in I_{\mathrm{odd}}$ otherwise $\overline{0}$. Take $Q = (Q_{ij}(u,v))$ such that*

- *$Q_{ij} \in \mathbf{k}\langle u,v\rangle/\langle uv - (-1)^{\mathsf{par}(i)\mathsf{par}(j)}vu\rangle$ for all $i,j \in I$,*
- *$Q_{ij}(u,v) = 0$ for all $i,j \in I$ with $i = j$,*
- *$Q_{ij}(u,v) = Q_{ji}(v,u)$ for all $i,j \in I$,*
- *$Q_{ij}(u,v) = Q_{ij}(-u,v)$ for all $i \in I_{\mathrm{odd}}, j \in I$.*

(a) *The quiver Hecke superalgebra*[5] *$R_n(\mathbf{k};Q)$ is the $\mathbf{k}$-superalgebra generated by $\{x_p, \tau_a, e_\nu \mid 1 \leq p \leq n, 1 \leq a < n, \nu \in I^n\}$ with parity $\overline{e(\nu)} = \overline{0}$, $\overline{x_p e(\nu)} = \mathsf{par}(\nu_p)$, $\overline{\tau_a e(\nu)} = \mathsf{par}(\nu_a)\,\mathsf{par}(\nu_{a+1})$ with the following defining relations*[6] *for all $\mu,\nu \in I^n, 1 \leq p,q \leq n$, $1 \leq b < a \leq n-1$.*

$$e_\mu e_\nu = \delta_{\mu\nu} e_\mu, 1 = \sum_{\mu\in I^n} e_\mu, x_p x_q e_\nu = (-1)^{\mathsf{par}(\nu_p)\mathsf{par}(\nu_q)} x_q x_p e_\nu,$$

$$x_p e_\nu = e_\nu x_p, \tau_a \tau_b e_\nu = (-1)^{\mathsf{par}(\nu_a)\mathsf{par}(\nu_{a+1})\mathsf{par}(\nu_b)\mathsf{par}(\nu_{b+1})} \tau_b \tau_a e_\nu \text{ if } |a-b| > 1,$$

$$\tau_a^2 e_\nu = Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}) e_\nu, \tau_a e_\mu = e_{s_a(\mu)} \tau_a, \tau_a x_p e_\nu = (-1)^{\mathsf{par}(\nu_p)\mathsf{par}(\nu_a)\mathsf{par}(\nu_{a+1})} x_p \tau_a e_\nu \text{ if } p \neq a, a+1,$$

$$(\tau_a x_{a+1} - (-1)^{\mathsf{par}(\nu_a)\mathsf{par}(\nu_{a+1})} x_a \tau_a) e_\nu = (x_{a+1}\tau_a - (-1)^{\mathsf{par}(\nu_a)\mathsf{par}(\nu_{a+1})} \tau_a x_a) e_\nu = \delta_{\nu_a,\nu_{a+1}} e_\nu,$$

$$(\tau_{b+1}\tau_b\tau_{b+1} - \tau_b\tau_{b+1}\tau_b) e_\nu =$$

$$\begin{cases} \frac{Q_{\nu_b,\nu_{b+1}}(x_{b+2},x_{b+1}) - Q_{\nu_b,\nu_{b+1}}(x_b,x_{b+1})}{x_{b+2}-x_b} e_\nu & \text{if } \nu_b = \nu_{b+2} \in I_{\mathrm{even}}, \\ (-1)^{\mathsf{par}(\nu_b)}(x_{b+2}-x_b)\frac{Q_{\nu_b,\nu_{b+1}}(x_{b+2},x_{b+1}) - Q_{\nu_b,\nu_{b+1}}(x_b,x_{b+1})}{x_{b+2}^2-x_b^2} e_\nu & \text{if } \nu_b = \nu_{b+2} \in I_{\mathrm{odd}}, \\ 0 & \text{otherwise} \end{cases}$$

(b) *For $\beta = \sum_{i\in I} \beta_i \cdot i \in \mathbb{N}[I]$ with $n = \mathsf{ht}(\beta) := \sum_{i\in I}\beta_i$, we define $R_\beta(\mathbf{k};Q) = R_n(\mathbf{k};Q)e_\beta$ where $e_\beta = \sum_{\nu\in\mathsf{Seq}(\beta)} e_\nu$,*

---

[5]Because when $I_{\mathrm{odd}} = \emptyset$ the quiver Hecke superalgebra is the same as the Khovanov-Lauda-Rouquier algebra, the notation $R_n(\mathbf{k};Q)$ for the quiver Hecke superalgebra is justified.

[6] When ν_b is odd, $Q_{\nu_b,\nu_{b+1}}(x_b,x_{b+1})$ belongs to the commutative ring $\mathbf{k}[x_b^2, x_{b+1}]$, and hence we can define $\frac{Q_{\nu_b,\nu_{b+1}}(x_{b+2},x_{b+1}) - Q_{\nu_b,\nu_{b+1}}(x_b,x_{b+1})}{x_{b+2}^2-x_b^2}$ as an element of $\mathbf{k}[x_b^2, x_{b+1}, x_{b+2}^2]$.

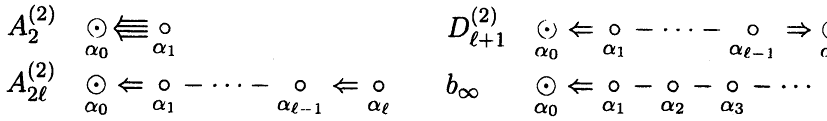


FIGURE 2. Dynkin diagrams of type $A_{2\ell}^{(2)}, D_{\ell+1}^{(2)}$ and b_∞ with parity. Here $\odot$ indicates an odd vertex.

(c) *For* $\lambda = \sum_{i \in I} \lambda_i \cdot i \in \mathbb{N}[I]$ *and* $\beta \in \mathbb{N}[I]$ *with* $n = \mathsf{ht}(\beta)$*, we define*

$$R_n^\lambda(\mathbf{k}; Q) = R_n(\mathbf{k}; Q)/R_n(\mathbf{k}; Q)(\textstyle\sum_{\nu \in I^n} x_1^{\lambda_{h_{\nu_1}}} e_\nu) R_n(\mathbf{k}; Q),$$

$$R_\beta^\lambda(\mathbf{k}; Q) = R_\beta(\mathbf{k}; Q)/R_\beta(\mathbf{k}; Q)(\textstyle\sum_{\nu \in \mathsf{Seq}(\beta)} x_1^{\lambda_{h_{\nu_1}}} e_\nu) R_\beta(\mathbf{k}; Q).$$

Definition 6.2 ([BKM, KKT]). *A generalized Cartan matrix (GCM) with parity is a GCM* $A = (a_{ij})_{i,j \in I}$ *with the parity decomposition* $I = I_{\mathsf{even}} \sqcup I_{\mathsf{odd}}$ *such that* $a_{ij} \in 2\mathbb{Z}$ *for all* $i \in I_{\mathsf{odd}}$ *and* $j \in I$*.*

Definition 6.3 ([KKT, §3.6]). *Let* $A = (a_{ij})_{i,j \in I}$ *be a symmetrizable GCM with parity. Take the symmetrization* $d = (d_i)_{i \in I}$*. For* $i, j \in I$*, let* S_{ij} *be the set of* (r, s) *where* r *and* s *are integers satisfying the following conditions. Note that* $S_{i,j} = \emptyset$ *when* $i = j$*.*

(i) $0 \le r \le -a_{ij}$, $0 \le s \le -a_{ji}$ *and* $d_i r + d_j s = -d_i a_{ij}$*,*
(ii) $r \in 2\mathbb{Z}$ *if* $i \in I_{\mathsf{odd}}$ *and* $s \in 2\mathbb{Z}$ *if* $j \in I_{\mathsf{odd}}$*.*

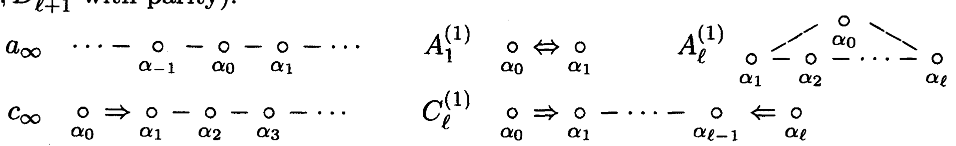
Take a sequence $(t_{i,j,r,s})_{(r,s) \in S_{ij}}$ *in* $\mathbf{k}$ *such that* $t_{i,j,r,s} = t_{j,i,s,r}$ *and* $t_{i,j,-a_{i,j},0} \neq 0$ *and put* $Q_{i,j}^A(u, v) = \sum_{(r,s) \in S_{ij}} t_{i,j,r,s} u^r v^s \in \mathbf{k}_A\langle w, z\rangle / \langle zw - (-1)^{\mathsf{par}(i)\,\mathsf{par}(j)} wz \rangle$*.*

For $n \geq 0$ and $\lambda, \beta \in \mathbb{N}[I]$ with $\mathsf{ht}(\beta) = n$, all of $R_n(\mathbf{k}; Q^A)$, $R_\beta(\mathbf{k}; Q^A)$, $R_n^\lambda(\mathbf{k}; Q^A)$, $R_\beta^\lambda(\mathbf{k}; Q^A)$ are $(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-graded via the assignment where $\nu \in I^n, 1 \le p \le n, 1 \le a < n$.

$$\mathsf{deg}(e_\nu) = (0, \bar{0}), \quad \mathsf{deg}(x_p e_\nu) = (2d_{\nu_p}, \mathsf{par}(\nu_p)), \quad \mathsf{deg}(\tau_a e_\nu) = (-d_{\nu_a} a_{\nu_a, \nu_{a+1}}, \mathsf{par}(\nu_a)\mathsf{par}(\nu_{a+1})).$$

Theorem 6.4 ([KKT, Corollary 4.8,Theorem 3.13]). *Let* $\mathbf{k}$ *be an algebraically closed field whose characteristic different from 2 and take* $q \in \mathbf{k}^\times$ *and* $X \in \mathsf{Mat}_I(\mathbb{Z})$ *as in Definition 4.2 (b) and make* X *a GCM with parity as in Figure 2. Then,* $\mathcal{H}_n^\lambda$ *and* $R_n^\lambda(X; Q^X)$ *are Morita superequivalent (see §3.11) for all* $\lambda \in \mathcal{P}^+$ *where we identify* $\lambda \in \mathcal{P}^+$ *and* $\sum_{i \in I} \lambda(h_i) \cdot i \in \mathbb{N}[I]$*.*

Remark 6.5. Actually, in [KKT, Theorem 4.4] we also treat other blocks of $\mathcal{H}_n$-smod than $\mathsf{Rep}\,\mathcal{H}_n$ where Dynkin diagram without parity of type $a_\infty, c_\infty, A_\ell^{(1)}, C_\ell^{(1)}$ appear (in addition to $b_\infty, A_{2\ell}^{(2)}, D_{\ell+1}^{(2)}$ with parity).



Remark 6.6. We believe that $R_n^\lambda(\mathbf{k}; Q^X)$ has simpler representation theory than $\mathcal{H}_n^\lambda$ while they are Morita superequivalent. For example, we conjectured that all the simple supermodules of $R_n^\lambda(\mathbf{k}; Q^X)$ are of type M. This "type M phenomenon" are verified in [HW, §6.5]. Moreover, Hill and Wang claims that $R_n(\mathbf{k}; Q^A)$ categorifies the half of quantum Kac-Moody superalgebra introduced by Benkart-Kang-Melville [BKM].

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