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A Survey of Conditional Expectations on an Analogue of Wiener Space

By

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Abstract

Let $C^r[0,t]$ denote the space of $\mathbb{R}^r$-valued continuous functions on the interval $[0,t]$, where r is a positive integer, and for a partition $0=t_0<t_1<\cdots<t_n<t_{n+1}=t$ of $[0,t]$, let $X_n\colon C^r[0,t]\to\mathbb{R}^{(n+1)r}$ and $X_{n+1}: C^r[0,t]\to\mathbb{R}^{(n+2)r}$ be given by $X_n(x):=(x(t_0),x(t_1),\ldots,x(t_n))$ and $X_{n+1}(x)=(X_n(x),x(t_{n+1}))=(x(t_0),\ldots,x(t_n),x(t_{n+1}))$.

In this survey paper, with the conditioning function X_n and X_{n+1}, we introduce two simple formulas for conditional expectations of functions defined on $C^r[0,t]$ which is a generalization of the r-dimensional Wiener space. As applications of the formulas, we evaluate the conditional expectations of functions defined on $C^r[0,t]$. Finally, we investigate that the conditional analytic Feynman integrals of some functionals can be applied to solve an integral equation which is formally equivalent to the Schrödinger partial differential equation and to express the operator-valued Feynman integrals in terms the conditional Feynman integrals.

§ 1. Introduction and Preliminaries

Let $C_0[0,t]$ be the space of real-valued continuous functions x on $[0,t]$ with $x(0)=0$. It is well-known that the space $C_0[0,t]$ equipped with the Wiener measure which is a probability space.. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by Fourier-transform [19], and also, in [20], [21], he obtained very useful results including Kac-Feynman integral equation and the conditional Cameron-Martin translation theorem using the inversion formula. However Yeh's inversion formula is very complicated in its applications when the conditioning function is vector-valued. In [17], Park and Skoug derived a simple formula for conditional Wiener integrals on $C_0[0,t]$ with the conditioning function $X\colon C_0[0,t]\to\mathbb{R}^{n+1}$ given by

$$X(x)=(x(t_1),\ldots,x(t_{n+1}))$$

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where $0 = t_0 < t_1 < \cdots < t_{n+1} = t$ is a partition of the interval $[0,t]$. In their simple formula they expressed the conditional Wiener integral directly in terms of ordinary Wiener integral. Using the formula, they generalized Kac-Feynman formula and obtained a Cameron-Martin type translation theorem for conditional Wiener integrals.

Let $C^r[0,t]$ denote the space of $\mathbb{R}^r$-valued continuous functions on the interval $[0,t]$, where r is a positive integer, and let $X_n\colon C^r[0,t] \to \mathbb{R}^{(n+1)r}$ and $X_{n+1}\colon C^r[0,t] \to \mathbb{R}^{(n+2)r}$ be given by

$$X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n)) \text{ and } X_{n+1}(x) = (x(t_0), \ldots, x(t_n), x(t_{n+1})).$$

In this survey paper, with the conditioning function X_n and X_{n+1}, we introduce two simple formulas for conditional expectations of functions defined on $C^r[0,t]$ which is a generalization of the r-dimensional Wiener space. As applications of the formulas, we evaluate the conditional expectations of functions defined on $C^r[0,t]$. Finally, we investigate that the conditional analytic Feynman integral of the functional

$$\exp\left\{\int_0^t \theta(s, x(s))\, dm_L(s)\right\}$$

can be applied to solve an integral equation which is formally equivalent to the Schrödinger partial differential equation and to express the operator-valued Feynman integrals in terms the conditional Feynman integrals.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of the complex numbers, the complex numbers with positive real parts and the nonzero complex numbers with nonnegative real parts, respectively. Let m_L be the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.

Now, we begin with introducing the probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$. For a positive real t, let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}}\colon C[0,t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n)).$$

For $B_j \in \mathcal{B}(\mathbb{R})$ $(j = 0, 1, \cdots, n)$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let

$$\begin{aligned} &m_\varphi\left[J_{\vec{t}}^{-1}\left(\prod_{j=0}^{n} B_j\right)\right] \\ &= \left[\prod_{j=1}^{n} \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^{n} B_j} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} dm_L^n(u_1, \cdots, u_n)\, d\varphi(u_0). \end{aligned}$$

The Borel σ-algebra $\mathcal{B}(C[0,t])$ on $C[0,t]$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $w_\varphi(I) = m_\varphi(I)$

for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [11], [18]. Let r be a positive integer and $C^r = C^r[0,t]$ the product space of $C[0,t]$ with the product measure w_φ^r. Since $C[0,t]$ is a separable Banach space, we have $\mathcal{B}(C^r[0,t]) = \prod_{j=1}^{r} \mathcal{B}(C[0,t])$. This probability measure space $(C^r[0,t], \mathcal{B}(C^r[0,t]), w_\varphi^r)$ is called an analogue of the r-dimensional Wiener space.

Lemma 1.1 ([11, Lemma 2.1]). *If $f\colon \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then*

$$\int_C f(x(t_0), x(t_1), \ldots, x(t_n))\, dw_\varphi(x)$$

$$\stackrel{*}{=} \left[\prod_{j=1}^{n} \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0, u_1, \ldots, u_n) \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\}$$

$$\cdot\, dm_L^n(u_1, \ldots, u_n)\, d\varphi(u_0),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $\{e_k; k = 1, 2, \ldots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_k is of bounded variation. For $f \in L_2[0,t]$ and $x \in C[0,t]$, let

$$(f, x) = \lim_{n\to\infty} \sum_{k=1}^{n} \langle f, e_k\rangle \int_0^t e_k(s)\, dx(s),$$

if the limit exists. Here $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. (f,x) is called the Paley-Wiener-Zygmund integral of f according to x. Note that $\langle\cdot,\cdot\rangle$ also denotes the inner product over Euclidean space unless otherwise specified.

Applying [11, Theorem 3.5], we can easily prove the following theorem.

Theorem 1.2. *Let $\{h_1, h_2, \ldots, h_n\}$ be an orthonormal system of $L_2[0,t]$. For $l = 1, 2, \cdots, n$, let $Z_l(x) = (h_l, x)$ on $C[0,t]$. Then $Z_1, Z_2, \cdots, Z_n$ are independent and each Z_l has the standard normal distribution. Moreover, if $f\colon \mathbb{R}^n \to \mathbb{R}$ is Borel measurable, then*

$$\int_C f(Z_1(x), Z_2(x), \ldots, Z_n(x))\, dw_\varphi(x)$$

$$\stackrel{*}{=} \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \ldots, u_n) \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} u_j^2\right\} dm_L^n(u_1, u_2, \ldots, u_n),$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

§ 2. Evaluation Formulas for Feynman Integrals

In this section, we establish the evaluation formulas for the analytic Feynman w_φ^r-integrals of several kinds of functionals on the analogue of the r-dimensional Wiener space. For a function

$F: C^r[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$. If $E[F^\lambda]$ has the analytic extension $J^*_\lambda(F)$ on $\mathbb{C}_+$, then we call $J^*_\lambda(F)$ the analytic Wiener w^r_φ-integral of F over $C^r[0,t]$ with the parameter λ and it is denoted by

$$E^{\mathrm{anw}_\lambda}[F] = J^*_\lambda(F).$$

Further, if for a nonzero real q, $E^{\mathrm{anw}_\lambda}[F]$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic Feynman w^r_φ-integral of F over $C^r[0,t]$ with the parameter q and denoted by

$$E^{\mathrm{anf}_q}[F] = \lim_{\lambda \to -iq} E^{\mathrm{anw}_\lambda}[F].$$

Now, we have the following theorem [15].

Theorem 2.1. *Let $m_1, \cdots, m_r$ be positive integers and suppose that*

$$\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty,$$

where $m = \max\{m_1, \ldots, m_r\}$. For $x = (x_1, \ldots, x_r) \in C^r[0,t]$, let

$$F(x) = \sum_{j=1}^{r} \int_0^t (x_j(s))^{m_j} dm_L(s).$$

Then for $\lambda \in \mathbb{C}_+$, $E^{\mathrm{anw}_\lambda}[F]$ exists and it is given by

$$E^{\mathrm{anw}_\lambda}[F] = \sum_{j=1}^{r} \lambda^{-\frac{m_j}{2}} \sum_{k=0}^{\left[\frac{m_j}{2}\right]} \frac{t^{k+1}}{k+1} \frac{m_j!}{2^k k!(m_j-2k)!} \int_{\mathbb{R}} v_j^{m_j-2k} d\varphi(v_j),$$

where $[\cdot]$ denotes the greatest integer function. Furthermore, for a nonzero real q, $E^{\mathrm{anf}_q}[F]$ exists and it is given by the right hand side of the equality above where λ is replaced by $-iq$.

Example 2.2. Suppose that $\int_{\mathbb{R}} |u|^2 d\varphi(u) < \infty$. For $x = (x_1, \ldots, x_r) \in C^r[0,t]$, let

$$F_l(x) = \sum_{j=1}^{r} \int_0^t (x_j(s))^l dm_L(s), \quad l = 1, 2.$$

For a nonzero real q, we have by Theorem 2.1

$$E^{\mathrm{anf}_q}[F_1] = t\left(\frac{i}{q}\right)^{\frac{1}{2}} \sum_{j=1}^{r} \int_{\mathbb{R}} v_j d\varphi(v_j)$$

and

$$E^{\mathrm{anf}_q}[F_2] = \frac{i}{q}\left(t \sum_{j=1}^{r} \int_{\mathbb{R}} u_j^2 d\varphi(v_j) + \frac{rt^2}{2}\right).$$

Let $\mathcal{M} = \mathcal{M}(L_2^r[0,t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2^r[0,t]$ and $\mathcal{S}_{w_\varphi}^r$ the space of all functions F which for $\sigma \in \mathcal{M}$ have the form

$$F(x) = \int_{L_2^r[0,t]} \exp\left\{ i \sum_{j=1}^{r} (v_j, x_j) \right\} d\sigma(v_1, \cdots, v_r) \tag{2.1}$$

for w_φ^r-a.e. $x = (x_1, \ldots, x_r) \in C^r[0,t]$. Note that $\mathcal{S}_{w_\varphi}^r$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ [11].

Now, we have the following theorem [15].

Theorem 2.3. *Let $F \in \mathcal{S}_{w_\varphi}^r$ be given by* (2.1). *Then for $\lambda \in \mathbb{C}_+$, $E^{\mathrm{anw}_\lambda}[F]$ exists and it is given by*

$$E^{\mathrm{anw}_\lambda}[F] = \int_{L_2^r[0,t]} \exp\left\{ -\frac{1}{2\lambda} \sum_{j=1}^{r} \|v_j\|^2 \right\} d\sigma(\vec{v}),$$

where $\vec{v} = (v_1, \ldots, v_r)$. Furthermore, for nonzero real q, $E^{\mathrm{anf}_q}[F]$ exists and it is given by the right hand side of the equality above where λ is replaced by $-iq$.

Let $r = 1$ and $\{e_1, \ldots, e_l\}$ be an orthonormal subset of $L_2[0,t]$. For $1 \le p \le \infty$, let $\mathcal{A}_l^{(p)}$ be the set of cylinder type functions having the form

$$F_l(x) = f((e_1, x), \ldots, (e_l, x)) \tag{2.2}$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R}^l)$ is Borel measurable on $\mathbb{R}^l$. We now have the following theorem [15].

Theorem 2.4. *Let $F_l \in \mathcal{A}_l^{(p)}$ $(1 \le p \le \infty)$ be given by* (2.2). *Then for $\lambda \in \mathbb{C}_+$, $E^{\mathrm{anw}_\lambda}[F_l]$ exists and it is given by*

$$E^{\mathrm{anw}_\lambda}[F_l] = \left(\frac{\lambda}{2\pi} \right)^{\frac{l}{2}} \int_{\mathbb{R}^l} f(\vec{u}) \exp\{ -\frac{\lambda}{2} \|\vec{u}\|^2 \} dm_L^l(\vec{u}).$$

Furthermore, for nonzero real q, $E^{\mathrm{anf}_q}[F_l]$ exists if $p = 1$ and it is given by the right hand side the above equality where λ is replaced by $-iq$.

For a positive integer l, let $\mathcal{M}(\mathbb{R}^l)$ be the class of all complex Borel measures on $\mathbb{R}^l$ and $\hat{\mathrm{M}}(\mathbb{R}^l)$ the set of all functions ϕ on $\mathbb{R}^l$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^l} \exp\{ i \langle \vec{u}, \vec{z} \rangle \} d\rho(\vec{z}), \tag{2.3}$$

where $\rho \in \mathcal{M}(\mathbb{R}^l)$.

Theorem 2.5. *Let* $\Phi(x)=\phi((e_1,x),\ldots,(e_l,x))$ *for* w_φ*-a.e.* $x\in C[0,t]$*, where* ϕ *is given by* (2.3)*. Then for* $\lambda\in\mathbb{C}_+$*,* $E^{\mathrm{anw}_\lambda}[\Phi]$ *exists and it is given by*

$$E^{\mathrm{anw}_\lambda}[\Phi]=\int_{\mathbb{R}^l}\exp\{-\frac{1}{2\lambda}\|\vec{z}\|^2\}d\rho(\vec{z}).$$

Furthermore, for nonzero real q*,* $E^{\mathrm{anf}_q}[\Phi]$ *exists and it is given by the right hand side of the above equality where* λ *is replaced by* $-iq$*.*

Theorem 2.6. *Let* $0=t_0<t_1<\cdots<t_n\le t$ *and* G *be given by*

$$G(x)=f(x(t_0),x(t_1),\ldots,x(t_n)),\ \textit{for}\ w_\varphi\textit{-a.e.}\ x\in C[0,t],$$

where $f\in L_p(\mathbb{R}^{n+1},\mathcal{B}(\mathbb{R}^{n+1}),m_L^n\otimes\varphi)(1\le p\le\infty)$*. Then for* $\lambda\in\mathbb{C}_+$*,* $E^{\mathrm{anw}_\lambda}[G]$ *exists and it is given by*

$$E^{\mathrm{anw}_\lambda}[G]=\Big(\frac{\lambda}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}^n}f(u_0,u_1,\cdots,u_n)\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j-u_{j-1})^2\Big\}dm_L^n(u_1,\ldots,u_n)d\varphi(u_0).$$

Furthermore, for nonzero real q*,* $E^{\mathrm{anf}_q}[G]$ *exists if* $p=1$ *and it is given by the right hand side the above equality where* λ *is replaced by* $-iq$*.*

Let η be a complex valued Borel measure on $[0,t]$. Then $\eta=\mu+\nu$ can be decomposed uniquely into the sum of a continuous measure μ and a discrete measure ν. Further, let δ_{p_j} denote the Dirac measure with total mass 1 concentrated at p_j. Let r be a positive integer and let $\mathcal{G}^*$ be the set of all $\mathbb{C}$-valued functions θ on $[0,\infty)\times\mathbb{R}^r$ which have the form

$$\theta(s,\vec{u})=\int_{\mathbb{R}^r}\exp\{i\langle\vec{u},\vec{v}\rangle\}d\sigma_s(\vec{v})\tag{2.4}$$

where $\{\sigma_s;\,s\in[0,t]\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the following conditions:

1. for each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,t]$,
2. $\|\sigma_s\|\in L_1([0,t],\mathcal{B}([0,t]),|\eta|)$.

Now, we have the following theorems [9].

Theorem 2.7. *Let* m *and* k *be two positive integers, and*

$$\eta=\mu+\sum_{j=1}^{m}w_j\delta_{p_j},$$

where $0<p_1<\cdots<p_m<t$ *and* $w_j\in\mathbb{C}$*. Let* $\theta\in\mathcal{G}^*$ *be given by* (2.4) *and for* $x\in C^r[0,t]$*, set*

$$F_k(x)=\left[\int_0^t\theta(s,x(s))d\eta(s)\right]^k.$$

Then for $\lambda > 0$,

$$E[F_k^\lambda] = \sum_{\substack{q_0+q_1+\cdots+q_m=k \\ j_0+\cdots+j_m=q_0}} H(k,\lambda,\vec{0},q_0,q_1,\ldots,q_m;j_0,\ldots,j_m)$$

where for $\vec{v}_{m,j_m+1} \in \mathbb{R}^r$

$$H(k,\lambda,\vec{v}_{m,j_m+1},q_0,q_1,\ldots,q_m;j_0,\ldots,j_m)$$
$$= k!\left(\prod_{\alpha=1}^{m}\frac{w_\alpha^{q_\alpha}}{q_\alpha!}\right)\int_{\Delta_{q_0;j_0,\cdots,j_m}}\int_{\mathbb{R}^{kr}}\exp\left\{-\frac{1}{2\lambda}\sum_{\alpha=0}^{m}\sum_{\beta=1}^{j_\alpha+1}(s_{\alpha,\beta}-s_{\alpha,\beta-1})\left\|\sum_{\gamma=\beta}^{j_\alpha+1}\vec{v}_{\alpha,\gamma}+\right.\right.$$
$$\left.\left.\sum_{l=\alpha+1}^{m}\sum_{\gamma=1}^{j_l+1}\vec{v}_{l,\gamma}\right\|^2\right\}\int_{\mathbb{R}^r}\exp\left\{i\lambda^{-\frac{1}{2}}\left\langle\vec{\eta},\sum_{\alpha=0}^{m}\sum_{\beta=1}^{j_\alpha+1}\vec{v}_{\alpha,\beta}\right\rangle\right\}d\varphi^r(\vec{\eta})d\left(\prod_{\alpha=0}^{m}\prod_{\beta=1}^{j_\alpha}\sigma_{s_{\alpha,\beta}}\times\right.$$
$$\left.\prod_{\alpha=1}^{m}\sigma_{p_\alpha}^{q_\alpha}\right)(\vec{v},\vec{z})d\mu^{q_0}(\vec{s})$$

with

$$s_{0,0}=0,\quad \vec{s}=(s_{0,1},\ldots,s_{0,j_0},s_{1,1},\ldots,s_{1,j_1},\ldots,s_{m,1},\cdots,s_{m,j_m}),$$
$$s_{\alpha,0}=s_{\alpha-1,j_{\alpha-1}+1}=p_\alpha \quad \textit{for } \alpha=1,\cdots,m,\quad s_{m,j_m+1}=t,$$
$$\Delta_{q_0;j_0,\cdots,j_m}=\{\vec{s}:0<s_{0,1}<\cdots<s_{0,j_0}<p_1<s_{1,1}<\cdots<s_{1,j_1}<p_2$$
$$<\cdots<p_{m-1}<s_{m-1,1}<\cdots<s_{m-1,j_{m-1}}<p_m<s_{m,1}<\cdots<s_{m,j_m}<t\},$$
$$\vec{v}=(\vec{v}_{0,1},\ldots,\vec{v}_{0,j_0},\vec{v}_{1,1},\cdots,\vec{v}_{1,j_1},\cdots,\vec{v}_{m,1},\ldots,\vec{v}_{m,j_m}) \textit{ and}$$
$$\vec{z}=(\vec{z}_{1,1},\ldots,\vec{z}_{1,q_1},\vec{z}_{2,1},\ldots,\vec{z}_{2,q_2},\ldots,\vec{z}_{m,1},\ldots,\vec{z}_{m,q_m});$$
$$\vec{v}_{\alpha-1,j_{\alpha-1}+1}=\sum_{l=1}^{q_\alpha}\vec{z}_{\alpha,l}\quad \textit{for } \alpha=1,\cdots,m.$$

Corollary 2.8. *Under the assumptions as given in Theorem 2.7 with one exception $\eta=\mu$, that is, assuming that η has no discrete part, we have*

$$E[F_k^\lambda]=k!\int_{\Delta_k}\int_{\mathbb{R}^{kr}}\exp\left\{-\frac{1}{2\lambda}\sum_{l=1}^{k}(s_l-s_{l-1})\left\|\sum_{\gamma=l}^{k}\vec{v}_\gamma\right\|^2\right\}\int_{\mathbb{R}^r}\exp\left\{i\lambda^{-\frac{1}{2}}\left\langle\vec{\eta},\sum_{l=1}^{k}\vec{v}_l\right\rangle\right\}$$
$$d\varphi^r(\vec{\eta})d\left(\prod_{l=1}^{k}\sigma_{s_l}\right)(\vec{v})d\mu^k(\vec{s})$$

where $s_0=0$, $\vec{s}=(s_1,\cdots,s_k)$, $\vec{v}=(\vec{v}_1,\cdots,\vec{v}_k)$ *and* $\Delta_k=\{\vec{s}:0<s_1<\cdots<s_k<t\}$.

Corollary 2.9. *Under the assumptions as given in Theorem 2.7 with one exception $\eta=\sum_{j=1}^{m}w_j\delta_{p_j}$, that is, assuming that η has no continuous part, we have*

$$E[F_k^\lambda]=k!\sum_{q_1+\cdots+q_m=k}\left(\prod_{\alpha=1}^{m}\frac{w_\alpha^{q_\alpha}}{q_\alpha!}\right)\int_{\mathbb{R}^{kr}}\exp\left\{-\frac{1}{2\lambda}\sum_{\alpha=1}^{m}(p_\alpha-p_{\alpha-1})\left\|\sum_{\gamma=\alpha}^{m}\sum_{l=1}^{q_\gamma}\vec{z}_{\gamma,l}\right\|^2\right\}$$

$$\times \int_{\mathbb{R}^r} \exp\left\{ i\lambda^{-\frac{1}{2}} \left\langle \vec{\eta}, \sum_{\alpha=1}^{m} \sum_{l=1}^{q_\alpha} \vec{z}_{\alpha,l} \right\rangle \right\} d\varphi^r(\vec{\eta}) d\left(\prod_{\alpha=1}^{m} \sigma_{p_\alpha}^{q_\alpha} \right)(\vec{z})$$

where $p_0 = 0$, $\vec{z} = (\vec{z}_{1,1}, \cdots, \vec{z}_{1,q_1}, \vec{z}_{2,1}, \cdots, \vec{z}_{2,q_2}, \cdots, \vec{z}_{m,1}, \cdots, \vec{z}_{m,q_m})$.

Theorem 2.10. *Let* φ^r *be normally distributed with mean vector* $\vec{0}$ *and variance covariance matrix* $\sigma^2 I_r$, *where* I_r *is the r-dimensional identity matrix. Then, under the assumptions and notations as given in Theorem* 2.7, $E^{anw_\lambda}[F_k]$ *exists for* $\lambda \in \mathbb{C}_+$ *and it is given by*

$$E^{anw_\lambda}[F_k] = \sum_{q_0+q_1+\cdots+q_m=k} \sum_{j_0+\cdots+j_m=q_0} T(k, \lambda, \sigma, \vec{0}, q_0, \cdots, q_m; j_0, \cdots, j_m)$$

where for $\vec{v}_{m,j_m+1} \in \mathbb{R}^r$, $T(k, \lambda, \sigma, \vec{v}_{m,j_m+1}, q_0, \cdots, q_m; j_0, \cdots, j_m)$ *is given by the expression of* $H(k, \lambda, \vec{v}_{m,j_m+1}, q_0, q_1, \cdots, q_m; j_0, \cdots, j_m)$ *replacing* $\int_{\mathbb{R}^r} \exp\{i\lambda^{-\frac{1}{2}} \langle \vec{\eta}, \sum_{\alpha=0}^{m} \sum_{\beta=1}^{j_\alpha+1} \vec{v}_{\alpha,\beta} \rangle\} d\varphi^r(\vec{\eta})$ *by* $\exp\{-\frac{\sigma^2}{2\lambda} \| \sum_{\alpha=0}^{m} \sum_{\beta=1}^{j_\alpha+1} \vec{v}_{\alpha,\beta} \|^2\}$. *Furthermore, for nonzero real q,* $E^{anf_q}[F_k]$ *exists and it is given by the above equality replacing* λ *by* $-iq$.

Theorem 2.11. *Let the assumptions and notations be as given in Theorem* 2.7 *and let*

$$F(x) = \exp\left\{ \int_0^t \theta(s, x(s)) d\eta(s) \right\}$$

for $x \in C^r[0,t]$. *Then for* $\lambda > 0$, *we have*

$$E[F^\lambda] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E[F_k^\lambda]$$

where $E[F_k^\lambda]$ *is as given in Theorem* 2.7. *Furthermore, under the assumptions as given in Theorem* 2.10, $E^{anf_q}[F]$ *is obtained by*

$$E^{anf_q}[F] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[F_k]$$

for nonzero real q, where $E^{anf_q}[F_k]$ *is as given in Theorem* 2.10.

Theorem 2.12. *Let the assumptions and notations be as given Theorem* 2.7 *and let* $G_k(x) = F_k(x)\psi(x(t))$ *for* $x \in C^r[0,t]$, *where* $\psi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v} \rangle\} d\nu(\vec{v})$ *for* $\nu \in \mathcal{M}(\mathbb{R}^r)$. *Let* $G(x) = \exp\{\int_0^t \theta(s, x(s)) d\eta(s)\}\psi(x(t))$. *Then for* $\lambda > 0$, *we have*

$$E[G^\lambda] = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp\left\{ i\lambda^{-\frac{1}{2}} \langle \vec{\eta}, \vec{v} \rangle - \frac{t}{2\lambda} \|\vec{v}\|^2 \right\} d\varphi^r(\vec{\eta}) d\nu(\vec{v}) + \sum_{k=1}^{\infty} \frac{1}{k!} E[G_k^\lambda]$$

where

$$E[G_k^\lambda] = \sum_{q_0+q_1+\cdots+q_m=k} \sum_{j_0+\cdots+j_m=q_0} \int_{\mathbb{R}^r} H(k, \lambda, \vec{v}_{m,j_m+1}, q_0, q_1, \cdots, q_m; j_0, \cdots, j_m) \, d\nu(\vec{v}_{m,j_m+1}).$$

Furthermore, under the assumptions and notations as given in Theorem 2.10, $E^{anf_q}[G]$ *can be obtained by*

$$E^{anf_q}[G] = \int_{\mathbb{R}^r} \exp\left\{\frac{t+\sigma^2}{2qi}\|\vec{v}\|^2\right\} d\nu(\vec{v}) + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[G_k]$$

for nonzero real q, where

$$E^{anf_q}[G_k] = \sum_{q_0+q_1+\cdots+q_m=k} \sum_{j_0+\cdots+j_m=q_0} \int_{\mathbb{R}^r} T(k,-iq,\sigma,\vec{v}_{m,j_m+1},q_0,\cdots,q_m;j_0,\cdots,j_m) \, d\nu(\vec{v}_{m,j_m+1}).$$

Corollary 2.13. *Under the assumptions and notations as given in Theorem* 2.12 *with one exception* $\varphi^r = \delta_{\vec{0}}$, *the Dirac measure concentrated at* $\vec{0} \in \mathbb{R}^r$, *we have for a nonzero real q*

$$E^{anf_q}[G] = \int_{\mathbb{R}^r} \exp\left\{\frac{t}{2qi}\|\vec{v}\|^2\right\} d\nu(\vec{v}) + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[G_k]$$

where

$$E^{anf_q}[G_k] = \sum_{q_0+q_1+\cdots+q_m=k} \sum_{j_0+\cdots+j_m=q_0} \int_{\mathbb{R}^r} T(k,-iq,0,\vec{v}_{m,j_m+1},q_0,\cdots,q_m;j_0,\cdots,j_m) \, d\nu(\vec{v}_{m,j_m+1})$$

which is a main result of [14].

Remark.

- Under the conditions as given in Corollaries 2.8 and 2.9, we can obtain more simple expressions in Theorems 2.10, 2.11, 2.12 and Corollary 2.13.
- If $\eta = \mu + \sum_{j=1}^{m} w_j \delta_{p_j}$, where $0 \le p_1 < \cdots < p_m \le t$, we can obtain all the results including η with minor modifications.
- If $\eta = \mu + \sum_{j=1}^{\infty} w_j \delta_{p_j}$, then using the following version of the $\aleph_0$-nomial formula [12, p.41]

$$\left(\sum_{p=0}^{\infty} b_p\right)^n = \sum_{h=0}^{\infty} \sum_{q_0+q_1+\cdots+q_h=n, q_h \ne 0} \frac{n!}{q_0! q_1! \cdots q_h!} b_0^{q_0} b_1^{q_1} \cdots b_h^{q_h}, \tag{2.5}$$

we can show that for $\lambda > 0$, $E[G^\lambda]$ exists in Theorem 2.12.

§ 3. Simple formulas for conditional expectations

Let $F : C^r[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C^r[0,t]$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory [16]. Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ^r-a.e. $x \in C^r[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ^r-integral of F given X and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of the interval $[0,t]$. Define the polygonal functions by

$$[x](s) = x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})), \quad t_{j-1} \le s \le t_j,$$

for $x \in C^r[0,t]$ and

$$[\vec{\xi}_{n+1}](s) = \xi_{j-1} + \frac{s - t_j}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}), \quad t_{j-1} \le s \le t_{j-1},$$

for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{(n+2)r}$, where $j = 1, \cdots, n+1$.

Now, we have two simples formulas for the conditional w_φ^r-integrals on $C^r[0,t]$ [4, 6].

Theorem 3.1. *Let $X_{n+1} : C^r[0,t] \to \mathbb{R}^{(n+2)r}$ be given by*

$$X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_{n+1})) \tag{3.1}$$

and let $F : C^r[0,t] \to \mathbb{C}$ be w_φ^r-integrable. Then for a Borel subset B of $\mathbb{R}^{(n+2)r}$ we have

$$\int_{X_{n+1}^{-1}(B)} F(x)\,dw_\varphi^r(x) = \int_B E[F(x - [x] + [\vec{\xi}_{n+1}])]\,dP_{X_{n+1}}(\vec{\xi}_{n+1}),$$

where $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on $(\mathbb{R}^{(n+2)r}, \mathcal{B}(\mathbb{R}^{(n+2)r}))$, so that we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+2)r}$

$$E[F|X_{n+1}](\vec{\xi}_{n+1}) = E[F(x - [x] + [\vec{\xi}_{n+1}])].$$

Theorem 3.2. *Let $X_n : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ be given by*

$$X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n)). \tag{3.2}$$

Moreover let F be integrable on $C^r[0,t]$ and P_{X_n} a probability distribution of X_n defind on $(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r}))$. Then we have for any $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{(n+1)r}$

$$E[F|X_n](\vec{\xi}_n) = \left[\frac{1}{2\pi(t - t_n)}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} E[F(x - [x] + [\vec{\xi}_{n+1}])] \exp\left\{-\frac{\|\xi_{n+1} - \xi_n\|^2}{2(t - t_n)}\right\} dm_L^r(\xi_{n+1})$$

for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$.

Theorem 3.3. *Let $r=1$, $F_m(x)=\int_0^t (x(u))^m dm_L(u)$ $(m\in\mathbb{N})$ for $x\in C[0,t]$ and suppose that $\int_{\mathbb{R}}|u|^m d\varphi(u)<\infty$. Then F_m is w_φ-integrable. Moreover, $E[F_m|X_{n+1}](\vec{\xi}_{n+1})$ exists for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})\in\mathbb{R}^{n+2}$ and it is given by*

$$E[F_m|X_{n+1}](\vec{\xi}_{n+1})=\sum_{j=1}^{n+1}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\frac{m!(l+k)!(t_j-t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j-\xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}$$

where $[\cdot]$ denotes the greatest integer function.

Example 3.4. For $m=1,2,3$, let $F_m(x)=\int_0^t (x(u))^m dm_L(u)$ for $x\in C[0,t]$ and suppose that $\int_{\mathbb{R}}|u|^m d\varphi(u)<\infty$. Then for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})\in\mathbb{R}^{n+2}$, we have by Theorem 3.3

$$E[F_1|X_{n+1}](\vec{\xi}_{n+1})=\frac{1}{2}\sum_{j=1}^{n+1}(t_j-t_{j-1})(\xi_j+\xi_{j-1})$$

which can be also obtained by an application of Corollary 4.5 in [11]. We also have

$$E[F_2|X_{n+1}](\vec{\xi}_{n+1})=\frac{1}{6}\sum_{j=1}^{n+1}(t_j-t_{j-1})(t_j-t_{j-1}+2\xi_j^2+2\xi_j\xi_{j-1}+2\xi_{j-1}^2)$$

which is the result given by Corollary 4.10 of [11]. Moreover we have

$$\begin{aligned}E[F_3|X_{n+1}](\vec{\xi}_{n+1})=&\frac{1}{4}\sum_{j=1}^{n+1}(t_j-t_{j-1})[(t_j-t_{j-1})(\xi_j+\xi_{j-1})\\&+\xi_j^3+\xi_j^2\xi_{j-1}+\xi_j\xi_{j-1}^2+\xi_{j-1}^3].\end{aligned}$$

Theorem 3.5. *Under the conditions and notations as given in Theorem 3.3, we have for P_{X_n}-a.e. $\vec{\xi}_n=(\xi_0,\xi_1,\cdots,\xi_n)\in\mathbb{R}^{n+1}$*

$$\begin{aligned}E[F_m|X_n](\vec{\xi}_n)=&\sum_{j=1}^{n}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\frac{m!(l+k)!(t_j-t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j-\xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}\\&+\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{[\frac{m-2k}{2}]}\frac{m!(2l+k)!\xi_n^{m-2k-2l}(t-t_n)^{l+k+1}}{2^{l+k}l!(m-2k-2l)!(2l+2k+1)!}.\end{aligned}$$

Example 3.6. For $m=1,2,3$, let $F_m(x)=\int_0^t (x(u))^m dm_L(u)$ for $x\in C[0,t]$ and suppose that $\int_{\mathbb{R}}|u|^m d\varphi(u)<\infty$. Moreover, let $Z_1(x)=\frac{1}{t}F_1(x)$ and

$$Z_2(x)=\sum_{j=1}^{n+1}\frac{1}{t_j-t_{j-1}}\int_{t_{j-1}}^{t_j}x(u)dm_L(u)$$

for $x \in C[0,t]$. Then for P_{X_n}-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, we have

$$E[F_1|X_n](\vec{\xi}_n) = \frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + (t - t_n)\xi_n.$$

Hence we have

$$E[Z_1|X_n](\vec{\xi}_n) = \frac{1}{2t}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \frac{1}{t}(t - t_n)\xi_n$$

and

$$E[Z_2|X_n](\vec{\xi}_n) = \frac{1}{2}\sum_{j=1}^{n}(\xi_j + \xi_{j-1}) + \xi_n$$

which are also given by Theorems 4.3 and 4.6 in [11], respectively. Further, we have

$$E[F_2|X_n](\vec{\xi}_n) = \frac{1}{6}\sum_{j=1}^{n}(t_j - t_{j-1})(t_j - t_{j-1} + 2\xi_j^2 + 2\xi_j\xi_{j-1} + 2\xi_{j-1}^2)$$
$$+(t - t_n)\xi_n^2 + \frac{1}{2}(t - t_n)^2$$

which is also given by Theorem 4.8 in [11]. Finally, we have

$$E[F_3|X_n](\vec{\xi}_n) = \frac{1}{4}\sum_{j=1}^{n}(t_j - t_{j-1})[(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \xi_j^3 + \xi_j^2\xi_{j-1} + \xi_j\xi_{j-1}^2$$
$$+\xi_{j-1}^3] + (t - t_n)\xi_n^3 + \frac{3}{2}(t - t_n)^2\xi_n.$$

§ 4. Evaluation formulas for an analogue of conditional Feynman integrals

For a function $F : C^r[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-1/2}x)$, $X_{n+1}^\lambda(x) = X_{n+1}(\lambda^{-1/2}x)$ and $X_n^\lambda(x) = X_n(\lambda^{-1/2}x)$, where X_{n+1} and X_n are given by (3.1) and (3.2), respectively. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of conditional w_φ^r-integral and Theorem 3.1, we have for $\lambda > 0$

$$E[F^\lambda|X_{n+1}^\lambda](\vec{\xi}_{n+1}) = E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}_{n+1}])]$$

for $P_{X_{n+1}^\lambda}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+2)r}$, where $P_{X_{n+1}^\lambda}$ is the probability distribution of X_{n+1}^λ defined on $(\mathbb{R}^{(n+2)r}, \mathcal{B}(\mathbb{R}^{(n+2)r}))$. Moreover, we can obtain from Theorem 3.2

$$\begin{aligned} &E[F^\lambda|X_n^\lambda](\vec{\xi}_n) \\ (4.1)\qquad &= \left[\frac{\lambda}{2\pi(t - t_n)}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]\exp\left\{-\frac{\lambda\|\xi_{n+1} - \xi_n\|^2}{2(t - t_n)}\right\} \\ &dm_L^r(\xi_{n+1}) \end{aligned}$$

for $P_{X_n^\lambda}$-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{(n+1)r}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}^r$ and $P_{X_n^\lambda}$ is the probability distribution of X_n^λ on $(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r}))$. If, for $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+2)r}$, $E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}_{n+1}])]$ has an analytic extension $J_\lambda^*(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ^r-integral of F given X_{n+1} with parameter λ and denoted by

$$E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F)(\vec{\xi}_{n+1})$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+2)r}$. If for a non-zero real q, $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ^r-integral of F given X_{n+1} with parameter q and denoted by

$$E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}).$$

Similarly, we define $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ and $E^{anf_q}[F|X_n](\vec{\xi}_n)$ using (4.1).

Throughout the remainder of this section, we will assume $r = 1$.

Theorem 4.1. *Let X_{n+1} be given by* (3.1)*. Then, under the assumptions and notations as given in Theorem* 3.3*, $E^{anw_\lambda}[F_m|X_{n+1}](\vec{\xi}_{n+1})$ exists for $\lambda \in \mathbb{C}_+$ and for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, for a non-zero real q, $E^{anf_q}[F_m|X_{n+1}](\vec{\xi}_{n+1})$ exists and it is given by*

$$\begin{aligned} &E^{anf_q}[F_m|X_{n+1}](\vec{\xi}_{n+1}) \\ &= \sum_{j=1}^{n+1} \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{m-2k} \left(\frac{i}{q}\right)^k \frac{m!(l+k)!(t_j - t_{j-1})^{k+1} \xi_{j-1}^{m-2k-l} (\xi_j - \xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}. \end{aligned}$$

Theorem 4.2. *Let X_n be given by* (3.2)*. Then, under the assumptions and notations as given in Theorem* 3.5*, $E^{anw_\lambda}[F_m|X_n](\vec{\xi}_n)$ exists for $\lambda \in \mathbb{C}_+$ and for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$. Moreover, for a non-zero real q, $E^{anf_q}[F_m|X_n](\vec{\xi}_n)$ exists and it is given by*

$$\begin{aligned} &E^{anf_q}[F_m|X_n](\vec{\xi}_n) \\ &= \sum_{j=1}^{n} \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{m-2k} \left(\frac{i}{q}\right)^k \frac{m!(l+k)!(t_j - t_{j-1})^{k+1} \xi_{j-1}^{m-2k-l} (\xi_j - \xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!} \\ &+ \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-2k}{2}]} \left(\frac{i}{q}\right)^{k+l} \frac{m!(2l+k)! \xi_n^{m-2k-2l} (t - t_n)^{l+k+1}}{2^{l+k} l!(m-2k-2l)!(2l+2k+1)!}. \end{aligned}$$

For each $j = 1, \cdots, n+1$, let $\alpha_j(u) = \frac{1}{\sqrt{t_j - t_{j-1}}} \chi_{(t_{j-1}, t_j]}(u)$. Let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \cdots, \alpha_{n+1}\}$ and let $V^\perp$ denote the orthogonal complement of V. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0,t]$ to V and $V^\perp$, respectively. Let σ be in $\mathcal{M}(L_2[0,t])$. For $\vec{\xi}_n = (\xi_0, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, let $\sigma_{\vec{\xi}_n}$ and $\sigma_{\vec{\xi}_{n+1}}$ be the measures defined by

$$\frac{d\sigma_{\vec{\xi}_n}}{d\sigma}(v) = \exp\left\{ i \sum_{j=1}^{n} (\mathcal{P}v)(t_j)(\xi_j - \xi_{j-1}) \right\} \tag{4.2}$$

and

$$\frac{d\sigma_{\vec{\xi}_{n+1}}}{d\sigma}(v) = \exp\{i(v,[\vec{\xi}_{n+1}])\}, \tag{4.3}$$

respectively, where $v \in L_2[0,t]$. Moreover, let $F_{\vec{\xi}_{n+1}} \in \mathcal{S}_{w_\varphi}$ be defined by

$$F_{\vec{\xi}_{n+1}}(x) = \int_{L_2[0,t]} \exp\{i(v,x)\} d(\sigma_{\vec{\xi}_{n+1}} \circ (\mathcal{P}^\perp)^{-1})(v) \tag{4.4}$$

for w_φ-a.e. $x \in C[0,t]$.

We now have the following theorems [7].

Theorem 4.3. *Let $F \in \mathcal{S}_{w_\varphi}$ and X_{n+1} be given by (2.1) and (3.1), respectively. Then, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ exists for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ and it is given by*

$$\begin{aligned} E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) &= E^{anw_\lambda}[F_{\vec{\xi}_{n+1}}] \\ &= \int_{L_2[0,t]} \exp\left\{-\frac{1}{2\lambda}\|v\|^2\right\} d(\sigma_{\vec{\xi}_{n+1}} \circ (\mathcal{P}^\perp)^{-1})(v), \end{aligned} \tag{4.5}$$

where $\sigma_{\vec{\xi}_{n+1}}$ and $F_{\vec{\xi}_{n+1}}$ are given by (4.3) and (4.4), respectively. Moreover, for a non-zero real q, $E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1})$ is given by (4.5) replacing λ by $-iq$.

Theorem 4.4. *Let $F \in \mathcal{S}_{w_\varphi}$ and X_n be given by (2.1) and (3.2), respectively. Then, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ exists for $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and it is given by*

$$E^{anw_\lambda}[F|X_n](\vec{\xi}_n) = \int_{L_2[0,t]} \exp\left\{-\frac{1}{2\lambda}[\|\mathcal{P}^\perp v\|^2 + (t-t_n)[(\mathcal{P}v)(t)]^2]\right\} d\sigma_{\vec{\xi}_n}(v), \tag{4.6}$$

where $\sigma_{\vec{\xi}_n}$ is given by (4.2). Moreover, for a non-zero real q, $E^{anf_q}[F|X_n](\vec{\xi}_n)$ is given by (4.6) replacing λ by $-iq$.

Corollary 4.5. *Let $F \in \mathcal{S}_{w_\varphi}$ and σ be related by (2.1) and suppose that σ is concentrated on $V^\perp$. Then for a non-zero real q, we have*

$$E^{anf_q}[F|X_n](\vec{\xi}_n) = E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = E^{anf_q}[F] = \int_{L_2[0,t]} \exp\left\{\frac{1}{2qi}\|v\|^2\right\} d\sigma(v)$$

for a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$.

Remark. Because $L_2[0,t] = V \oplus V^\perp$ and V is of finite dimensional, the space $V^\perp$ is non-trivial. For $k = 1,2,\cdots$, let

$$h_k(t) = \frac{(-1)^{j+1}2^{2k-1}}{(t_j - t_{j-1})^{2k-1}}\left(t - \frac{t_{j-1}+t_j}{2}\right)^{2k-1} \quad \text{if} \quad t_{j-1} \le t \le t_j \quad (j = 1,\cdots,n+1).$$

Then we have $\mathcal{P}h_k = 0$ for $k = 1,2,\cdots$. Indeed, every finite linear combination of the h_ks are in $V^\perp$ so that $V^\perp$ is at least uncountable and of infinite dimensional since $L_2[0,t]$ is of infinite dimensional.

Throughout this paper, let $\{v_1, v_2, \cdots, v_l\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^{\perp}v_1, \cdots, \mathcal{P}^{\perp}v_l\}$ are independent. Note that such an orthonormal set can be obtained from the h_ks as given in the above remark. Let $\{e_1, \cdots, e_l\}$ be the orthonormal set obtained from $\{\mathcal{P}^{\perp}v_1, \cdots, \mathcal{P}^{\perp}v_l\}$ by the Gram-Schmidt orthonormalization process. Now, for $k = 1, \cdots, l$, let

$$\mathcal{P}^{\perp}v_k = \sum_{j=1}^{l} \alpha_{kj} e_j$$

be the linear combinations of the e_js and let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1l} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{l1} & \alpha_{l2} & \cdots & \alpha_{ll} \end{bmatrix} \tag{4.7}$$

be the coefficient matrix of the combinations. We can also regard A as the linear transformation $T_A : \mathbb{R}^l \to \mathbb{R}^l$ given by

$$T_A(\vec{z}) = \vec{z}A, \tag{4.8}$$

where $\vec{z}$ is arbitrary row-vector in $\mathbb{R}^l$. Note that A is invertible so that T_A is an isomorphism. For $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ and $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, let

$$\vec{v}_{\vec{\xi}_n} = \left(\sum_{j=1}^{n} (\mathcal{P}v_1)(t_j)(\xi_j - \xi_{j-1}), \cdots, \sum_{j=1}^{n} (\mathcal{P}v_l)(t_j)(\xi_j - \xi_{j-1}) \right)$$

and

$$\vec{v}_{\vec{\xi}_{n+1}} = ((v_1, [\vec{\xi}_{n+1}]), \cdots, (v_l, [\vec{\xi}_{n+1}])).$$

Further, for $s \in [0,t]$ let

$$(\mathcal{P}\vec{v})(s) = ((\mathcal{P}v_1)(s), \cdots, (\mathcal{P}v_l)(s)). \tag{4.9}$$

For $\rho \in \mathcal{M}(\mathbb{R}^l)$, let $\rho_{\vec{\xi}_{n+1}}$ and $\rho_{\vec{\xi}_n}$ be the complex Borel measures on $\mathbb{R}^l$ defined by

$$\frac{d\rho_{\vec{\xi}_{n+1}}}{d\rho}(\vec{z}) = \exp\{i\langle \vec{v}_{\vec{\xi}_{n+1}}, \vec{z}\rangle\}$$

and

$$\frac{d\rho_{\vec{\xi}_n}}{d\rho}(\vec{z}) = \exp\{i\langle \vec{v}_{\vec{\xi}_n}, \vec{z}\rangle\}. \tag{4.10}$$

Now we define $\phi_{\vec{\xi}_{n+1}} \in \hat{\mathrm{M}}(\mathbb{R}^l)$ by

$$\phi_{\vec{\xi}_{n+1}}(\vec{u}) = \int_{\mathbb{R}^l} \exp\{i\langle \vec{u}, \vec{z}\rangle\} d(\rho_{\vec{\xi}_{n+1}} \circ T_A^{-1})(\vec{z}),$$

where T_A^{-1} is the inverse transformation of T_A, and define

$$\Phi_{\vec{\xi}_{n+1}}(x) = \phi_{\vec{\xi}_{n+1}}((e_1,x),\cdots,(e_l,x)) \tag{4.11}$$

for w_φ-a.e. $x \in C[0,t]$. We now have the following theorems.

Theorem 4.6. *Let $\phi \in \hat{\mathrm{M}}(\mathbb{R}^l)$ be given by (2.3) and let $\Phi(x) = \phi((v_1,x),\cdots,(v_l,x))$ for w_φ-a.e. $x \in C[0,t]$. Then we have for $\lambda \in \mathbb{C}_+$*

$$E^{anw_\lambda}[\Phi|X_{n+1}](\vec{\xi}_{n+1}) = E^{anw_\lambda}[\Phi_{\vec{\xi}_{n+1}}] = \int_{\mathbb{R}^l} \exp\left\{-\frac{1}{2\lambda}\|\vec{z}\|^2\right\} d(\rho_{\vec{\xi}_{n+1}} \circ T_A^{-1})(\vec{z}) \tag{4.12}$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where T_A and $\Phi_{\vec{\xi}_{n+1}}$ are given by (4.8) and (4.11), respectively. Moreover, for a non-zero real q, $E^{anf_q}[\Phi|X_{n+1}](\vec{\xi}_{n+1})$ is given by (4.12) replacing λ by $-iq$.

Theorem 4.7. *Let the assumptions and notations be as given in Theorem 4.6. Then we have for $\lambda \in \mathbb{C}_+$*

$$E^{anw_\lambda}[\Phi|X_n](\vec{\xi}_n) = \int_{\mathbb{R}^l} \exp\left\{-\frac{1}{2\lambda}[\|T_A(\vec{z})\|^2 + (t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{z}\rangle^2]\right\} d\rho_{\vec{\xi}_n}(\vec{z}) \tag{4.13}$$

for $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where $\mathcal{P}\vec{v}$ and $\rho_{\vec{\xi}_n}$ are given by (4.9) and (4.10), respectively. Moreover, for a non-zero real q, $E^{anf_q}[\Phi|X_n](\vec{\xi}_n)$ is given by (4.13) replacing λ by $-iq$.

Corollary 4.8. *Let the assumptions and notations be as given in Theorem 4.7 and suppose that $v_k \in V^\perp$ for $k = 1,\cdots,l$. Then for a non-zero real q, we have*

$$E^{anf_q}[\Phi|X_n](\vec{\xi}_n) = E^{anf_q}[\Phi|X_{n+1}](\vec{\xi}_{n+1}) = E^{anf_q}[\Phi] = \int_{\mathbb{R}^l} \exp\left\{\frac{1}{2qi}\|\vec{z}\|^2\right\} d\rho(\vec{z})$$

for a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$.

For convenience, we introduce useful notations from the Gram-Schmidt orthonormalization process. For $v \in L_2[0,t]$, we obtain an orthonormal set $\{e_1,\cdots,e_l,e_{l+1}\}$ as follows; let

$$c_j(v) = \begin{cases} \langle v, e_j\rangle & \text{for } j = 1,\cdots,l \\ \sqrt{\|v\|^2 - \sum_{k=1}^{l}\langle v, e_k\rangle^2} & \text{for } j = l+1 \end{cases} \tag{4.14}$$

and

$$e_{l+1} = \frac{1}{c_{l+1}(v)}\left[v - \sum_{j=1}^{l} c_j(v) e_j\right]$$

if $c_{l+1}(v) \neq 0$. Then we have

$$(4.15) \qquad v = \sum_{j=1}^{l+1} c_j(v) e_j \quad \text{and} \quad \|v\|^2 = \sum_{j=1}^{l+1} [c_j(v)]^2.$$

Note that the equalities in (4.15) hold trivially for the case $c_{l+1}(v) = 0$. Using the above notations we have the following theorem.

Theorem 4.9. *Let the assumptions and notations be as given in Theorems* 4.3 *and* 4.6. *Further, let* $\Psi(x) = F(x)\Phi(x)$ *and for* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ *let* $\Psi_{\vec{\xi}_{n+1}}(x) = F_{\vec{\xi}_{n+1}}(x)\Phi_{\vec{\xi}_{n+1}}(x)$ *for* w_φ*-a.e.* $x \in C[0,t]$, *where* $F_{\vec{\xi}_{n+1}}$ *and* $\Phi_{\vec{\xi}_{n+1}}$ *are given by* (4.4) *and* (4.11), *respectively. Then for* $\lambda \in \mathbb{C}_+$ *we have*

$$(4.16) \qquad E^{anw_\lambda}[\Psi|X_{n+1}](\vec{\xi}_{n+1}) = E^{anw_\lambda}[\Psi_{\vec{\xi}_{n+1}}] = \int_{L_2[0,t]} \int_{\mathbb{R}^l} \exp\Big\{ -\frac{1}{2\lambda}[\|v\|^2 + 2\langle \vec{c}(v), \vec{z}\rangle + \|\vec{z}\|^2] \Big\} d(\rho_{\vec{\xi}_{n+1}} \circ T_A^{-1})(\vec{z}) d(\sigma_{\vec{\xi}_{n+1}} \circ (\mathcal{P}^\perp)^{-1})(v)$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, *where* $\vec{c}(v) = (c_1(v), \cdots, c_l(v))$ *and the* c_j*s are given by* (4.14). *Further, for a non-zero real* q, $E^{anf_q}[\Psi|X_{n+1}](\vec{\xi}_{n+1})$ *is given by* (4.16) *replacing* λ *by* $-iq$.

Theorem 4.10. *Let the assumptions and notations be as given in Theorems* 4.4, 4.7 *and* 4.9. *Then for* $\lambda \in \mathbb{C}_+$ *we have*

$$(4.17) \qquad E^{anw_\lambda}[\Psi|X_n](\vec{\xi}_n) = \int_{L_2[0,t]} \int_{\mathbb{R}^l} \exp\Big\{ -\frac{1}{2\lambda}[\|\mathcal{P}^\perp v\|^2 + 2\langle \vec{c}(\mathcal{P}^\perp v), T_A(\vec{z})\rangle + \|T_A(\vec{z})\|^2 + (t-t_n)[(\mathcal{P}v)(t) + \langle (\mathcal{P}\vec{v})(t), \vec{z}\rangle]^2] \Big\} d\rho_{\vec{\xi}_n}(\vec{z}) d\sigma_{\vec{\xi}_n}(v)$$

for $\vec{\xi}_n \in \mathbb{R}^{n+1}$, *where* $\vec{c}(\mathcal{P}^\perp v) = (c_1(\mathcal{P}^\perp v), \cdots, c_l(\mathcal{P}^\perp v))$ *and the* c_j*s are given by* (4.14). *Further, for a non-zero real* q, $E^{anf_q}[\Psi|X_n](\vec{\xi}_n)$ *is given by* (4.17) *replacing* λ *by* $-iq$.

Let $1 \leq p \leq \infty$ and F_l be given by

$$(4.18) \qquad F_l(x) = f((v_1, x), \cdots, (v_l, x))$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R}^l)$. For $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ let

$$(4.19) \qquad f_{\vec{\xi}_n}(\vec{u}) = f(\mathcal{T}_A(\vec{u}) + \vec{v}_{\vec{\xi}_n})$$

and

$$(4.20) \qquad f_{\vec{\xi}_{n+1}}(\vec{u}) = f(\mathcal{T}_A(\vec{u}) + \vec{v}_{\vec{\xi}_{n+1}}),$$

where $\mathcal{T}_A$ is the linear transformation given by $\mathcal{T}_A(\vec{u}) = \vec{u}A^T$. Note that $\vec{u}$ means any row-vector in $\mathbb{R}^l$ and A^T is the transpose of the matrix A given by (4.7). Moreover let

$$F_{l,\vec{\xi}_{n+1}}(x) = f_{\vec{\xi}_{n+1}}((e_1,x),\cdots,(e_l,x)) \tag{4.21}$$

for w_φ-a.e. $x \in C[0,t]$. Using the above notations, we have the following theorems [7].

Theorem 4.11. *Let $1 \le p \le \infty$ and F_l be given by (4.18). Then for $\lambda \in \mathbb{C}_+$ we have*

$$\begin{aligned} E^{anw_\lambda}[F_l|X_{n+1}](\vec{\xi}_{n+1}) &= E^{anw_\lambda}[F_{l,\vec{\xi}_{n+1}}] \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{\frac{l}{2}} \int_{\mathbb{R}^l} f_{\vec{\xi}_{n+1}}(\vec{u}) \exp\Big\{-\frac{\lambda}{2}\|\vec{u}\|^2\Big\} dm_L^l(\vec{u}) \end{aligned} \tag{4.22}$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $f_{\vec{\xi}_{n+1}}$ and $F_{l,\vec{\xi}_{n+1}}$ are given by (4.20) and (4.21), respectively. Further, if $p = 1$, then for a non-zero real q, $E^{anf_q}[F_l|X_{n+1}](\vec{\xi}_{n+1})$ is given by (4.22) replacing λ by $-iq$.

Theorem 4.12. *Let the assumptions and notations be as given in Theorem 4.11. Then for $\lambda \in \mathbb{C}_+$ we have*

$$\begin{aligned} E^{anw_\lambda}[F_l|X_n](\vec{\xi}_n) &= \Big(\frac{\lambda}{2\pi}\Big)^{\frac{l}{2}} \Big[\frac{1}{1+(t-t_n)\|\mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t))\|^2}\Big]^{\frac{1}{2}} \int_{\mathbb{R}^l} f_{\vec{\xi}_n}(\vec{u}) \\ &\times \exp\Big\{-\frac{\lambda}{2}\Big[\|\vec{u}\|^2 - \frac{(t-t_n)\langle\mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t)),\vec{u}\rangle^2}{1+(t-t_n)\|\mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t))\|^2}\Big]\Big\} dm_L^l(\vec{u}) \end{aligned} \tag{4.23}$$

for $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where $\mathcal{P}\vec{v}$ and $f_{\vec{\xi}_n}$ are given by (4.9) and (4.19), respectively. Further, if $p = 1$, then for a non-zero real q, $E^{anf_q}[F_l|X_n](\vec{\xi}_n)$ is given by (4.23) replacing λ by $-iq$.

Corollary 4.13. *Let F_l be given by (4.18) and suppose that $v_k \in V^\perp$ for $k = 1,\cdots,l$. Then we have for $\lambda \in \mathbb{C}_+$*

$$\begin{aligned} E^{anw_\lambda}[F_l|X_n](\vec{\xi}_n) &= E^{anw_\lambda}[F_l|X_{n+1}](\vec{\xi}_{n+1}) \\ &= E^{anw_\lambda}[F_l] = \Big(\frac{\lambda}{2\pi}\Big)^{\frac{l}{2}} \int_{\mathbb{R}^l} f(\vec{u}) \exp\Big\{-\frac{\lambda}{2}\|\vec{u}\|^2\Big\} dm_L^l(\vec{u}) \end{aligned}$$

for a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Further, if $p = 1$, then for a non-zero real we have

$$\begin{aligned} E^{anf_q}[F_l|X_n](\vec{\xi}_n) &= E^{anf_q}[F_l|X_{n+1}](\vec{\xi}_{n+1}) \\ &= E^{anf_q}[F_l] = \Big(\frac{q}{2\pi i}\Big)^{\frac{l}{2}} \int_{\mathbb{R}^l} f(\vec{u}) \exp\Big\{\frac{qi}{2}\|\vec{u}\|^2\Big\} dm_L^l(\vec{u}). \end{aligned}$$

Theorem 4.14. *Let $G_l = FF_l$, where F and F_l are given by (2.1) and (4.18), respectively. Moreover, for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ let $G_{l,\vec{\xi}_{n+1}} = F_{\vec{\xi}_{n+1}} F_{l,\vec{\xi}_{n+1}}$, where $F_{\vec{\xi}_{n+1}}$ and $F_{l,\vec{\xi}_{n+1}}$ are given by (4.4) and (4.21), respectively. Then we have for $\lambda \in \mathbb{C}_+$*

$$
\begin{aligned}
(4.24) \qquad E^{anw_\lambda}[G_l|X_{n+1}](\vec{\xi}_{n+1}) &= E^{anw_\lambda}[G_{l,\vec{\xi}_{n+1}}] \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{l}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^l} f_{\vec{\xi}_{n+1}}(\vec{u}) \exp\left\{ \frac{1}{2\lambda} \left[\sum_{j=1}^{l} [\lambda i u_j \right.\right. \\
&\quad \left.\left. + c_j(v)]^2 - \|v\|^2 \right] \right\} dm_L^l(\vec{u}) d(\sigma_{\vec{\xi}_{n+1}} \circ (\mathcal{P}^\perp)^{-1})(v),
\end{aligned}
$$

where $\vec{u} = (u_1, \cdots, u_l)$, and $\sigma_{\vec{\xi}_{n+1}}$, $c_j(v)$ and $f_{\vec{\xi}_{n+1}}$ are given by (4.3), (4.14) and (4.20), respectively. Furthermore, if $p = 1$, then for any non-zero real q, $E^{anf_q}[G_l|X_{n+1}]$ $(\vec{\xi}_{n+1})$ is given by (4.24) replacing λ by $-iq$.

Theorem 4.15. *Let the assumptions and notations be given as in Theorems 4.12 and 4.14. Then for $\lambda \in \mathbb{C}_+$ we have*

$$
\begin{aligned}
(4.25) \qquad & E^{anw_\lambda}[G_l|X_n](\vec{\xi}_n) \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{l}{2}} \left[\frac{1}{1 + (t - t_n)\|\mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t))\|^2} \right]^{\frac{1}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^l} f_{\vec{\xi}_n}(\vec{u}) \exp\left\{ \frac{1}{2\lambda} \right. \\
&\quad \times \left[\sum_{j=1}^{l} [\lambda i u_j + c_j(\mathcal{P}^\perp v)]^2 - \|\mathcal{P}^\perp v\|^2 + \frac{t - t_n}{1 + (t - t_n)\|\mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t))\|^2} \right. \\
&\quad \left.\left. \times [i[(\mathcal{P}v)(t) - \langle \vec{c}(\mathcal{P}^\perp v), \mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t))\rangle] + \lambda \langle \mathcal{T}_A^{-1}((\mathcal{P}\vec{v})(t)), \vec{u}\rangle]^2 \right] \right\} dm_L^l \vec{u}) d\sigma_{\vec{\xi}_n}(v)
\end{aligned}
$$

for $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where $\vec{c}(\mathcal{P}^\perp v) = (c_1(\mathcal{P}^\perp v), \cdots, c_l(\mathcal{P}^\perp v))$ and $\sigma_{\vec{\xi}_n}$ is given by (4.2). Further, if $p = 1$, then for a non-zero real q, $E^{anf_q}[G_l|X_n](\vec{\xi}_n)$ is given by (4.25) replacing λ by $-iq$.

Remark. Since $\mathcal{P}^\perp v = v$ and $\mathcal{P}v = 0$ for $v \in V^\perp$, if $\sigma \in \mathcal{M}(L_2[0,t])$ is concentrated on $V^\perp$, then we have $\sigma_{\vec{\xi}_{n+1}} = \sigma_{\vec{\xi}_n} = \sigma$ so that we also have $F_{\vec{\xi}_{n+1}} = F$. If $v_k \in V^\perp$ for $k = 1, \cdots, l$, then $\mathcal{P}v_k = 0$ and $\mathcal{P}^\perp v_k = v_k$ so that $(\mathcal{P}\vec{v})(t) = \vec{0}$. Further, the coefficient matrix A given by (4.7) is the identity matrix and hence $f_{\vec{\xi}_{n+1}} = f_{\vec{\xi}_n} = f$ which implies $F_{l,\vec{\xi}_{n+1}} = F_l$. In each case, we can obtain more simple equations in each theorem of this section.

§ 5. Time-dependent and time-independent conditional Feynman integrals

Throughout this section, let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ be a partition of $[0,t]$, let $X_t(x) = (x(t_0), \cdots, x(t_{n-1}), x(t_n))$ and $Y_t(x) = (x(t_0), \cdots, x(t_{n-1}))$ for $x \in C^r[0,t]$. Our first theorem in this section is a time-dependent version of Theorem 2.7 above; its proof is included in [9].

Theorem 5.1. *Let F_k and $\eta = \mu + \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j}\delta_{p_{l,j}}$ be the function and the measure as described in Theorem 2.7, where $w_{l,j} \in \mathbb{C}$ for all (l,j) and $0 = t_0 < p_{1,1} < p_{1,2} < \cdots < p_{1,r_1} < t_1 < p_{2,1} < \cdots < p_{2,r_2} < t_2 < \cdots < t_{n-1} < p_{n,1} < \cdots < p_{n,r_n} < t_n = t$. Then for $\lambda \in \mathbb{C}_+$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+1)r}$, $E^{anw_\lambda}[F_k|X_t](\vec{\xi}_{n+1})$ is given by*

$$E^{anw_\lambda}[F_k|X_t](\vec{\xi}_{n+1}) = k! \sum_{q_1+\cdots+q_n=k} \prod_{l=1}^{n} A(l,\lambda,\vec{\xi}_{n+1},q_l) \tag{5.1}$$

where

$$\begin{aligned}
&A(l,\lambda,\vec{\xi}_{n+1},q_l) \\
&= \sum_{m_{l,0}+m_{l,1}+\cdots+m_{l,r_l}=q_l} \left(\prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!}\right) \sum_{j_0+j_1+\cdots+j_{r_l}=m_{l,0}} \int_{\Delta_{m_{l,0};j_0,\cdots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} \\
&\exp\Bigg\{ i \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \langle [\vec{\xi}_{n+1}](s_{l,u,v}), \vec{v}_{l,u,v}\rangle - \frac{1}{2\lambda} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} (s_{l,u,v} - s_{l,u,v-1}) \Bigg\| \sum_{\beta=u+1}^{r_l} \\
&\sum_{\gamma=1}^{j_\beta+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} + \sum_{\gamma=v}^{j_u+1} \frac{t_l - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\beta=0}^{u-1} \\
&\sum_{\gamma=1}^{j_\beta+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} \Bigg\|^2 \Bigg\} d\left(\prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}}\right)(\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l)
\end{aligned}$$

with

$$\begin{aligned}
s_{l,0,0} &= t_{l-1}, \quad s_{l,u,0} = p_{l,u} = s_{l,u-1,j_{u-1}+1} \quad \text{for } u = 1,\cdots,r_l, \quad s_{l,r_l,j_{r_l}+1} = t_l, \\
\vec{s}_l &= (s_{l,0,1},\ldots,s_{l,0,j_0},\ldots,s_{l,r_l,1},\ldots,s_{l,r_l,j_{r_l}}), \\
\Delta_{m_{l,0};j_0,\cdots,j_{r_l}} &= \{\vec{s}_l : t_{l-1} < s_{l,0,1} < \cdots < s_{l,0,j_0} < p_{l,1} < s_{l,1,1} < \cdots < s_{l,1,j_1} < p_{l,2} \\
&\qquad < \cdots < p_{l,r_l} < s_{l,r_l,1} < \cdots < s_{l,r_l,j_{r_l}} < t_l\}, \\
\vec{v}_l &= (\vec{v}_{l,0,1},\ldots,\vec{v}_{l,0,j_0},\vec{v}_{l,1,1},\cdots,\vec{v}_{l,1,j_1},\ldots,\vec{v}_{l,r_l,1},\ldots,\vec{v}_{l,r_l,j_{r_l}}), \quad \vec{v}_{l,r_l,j_{r_l}+1} = \vec{0} \in \mathbb{R}^r, \\
\vec{h}_l &= (\vec{h}_{l,1,1},\ldots,\vec{h}_{l,1,m_{l,1}},\vec{h}_{l,2,1},\ldots,\vec{h}_{l,2,m_{l,2}},\ldots,\vec{h}_{l,r_l,1},\ldots,\vec{h}_{l,r_l,m_{l,r_l}}); \\
\vec{v}_{l,u-1,j_{u-1}+1} &= \sum_{v=1}^{m_{l,u}} \vec{h}_{l,u,v} \quad \text{for } u = 1,\ldots,r_l.
\end{aligned}$$

Furthermore, for nonzero real q, $E^{anf_q}[F_k|X_t](\vec{\xi}_{n+1})$ is given by the right hand side of (5.1) replacing λ by $-iq$.

By the same method as used in the proof of Theorem 2.11, we can prove the following theorem.

Theorem 5.2. *Let the assumptions and notations be as given in Theorem 5.1, and let F be as given in Theorem 2.11. Then for nonzero real q and $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+1)r}$, $E^{anf_q}[F|X_t](\vec{\xi}_{n+1})$ exists and it is given by*

$$E^{anf_q}[F|X_t](\vec{\xi}_{n+1}) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[F_k|X_t](\vec{\xi}_{n+1})$$

where $E^{anf_q}[F_k|X_t](\vec{\xi}_{n+1})$ is as given in Theorem 5.1.

Let $\psi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle\vec{u},\vec{v}\rangle\} d\nu(\vec{v})$ for $\nu \in \mathcal{M}(\mathbb{R}^r)$. Then we have for $\vec{\xi}_{n+1} = (\vec{\xi}_0, \vec{\xi}_1, \cdots, \vec{\xi}_n) \in \mathbb{R}^{(n+1)r}$ and $\lambda > 0$

$$\psi(\lambda^{-\frac{1}{2}}(x(t) - [x](t)) + [\vec{\xi}_{n+1}](t)) = \psi(\vec{\xi}_n). \tag{5.2}$$

By (5.2) and Theorem 5.2, we have the following theorem.

Theorem 5.3. *Let G_k and G be as given in Theorem 2.12. Then, under the assumptions and notations as given in Theorem 5.2, we have for nonzero real q and $\vec{\xi}_{n+1} = (\vec{\xi}_0, \vec{\xi}_1, \cdots, \vec{\xi}_n) \in \mathbb{R}^{(n+1)r}$*

$$E^{anf_q}[G_k|X_t](\vec{\xi}_{n+1}) = \psi(\vec{\xi}_n) E^{anf_q}[F_k|X_t](\vec{\xi}_{n+1})$$

and

$$E^{anf_q}[G|X_t](\vec{\xi}_{n+1}) = \psi(\vec{\xi}_n) E^{anf_q}[F|X_t](\vec{\xi}_{n+1}) = \psi(\vec{\xi}_n) + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[G_k|X_t](\vec{\xi}_{n+1}).$$

Now we obtain a time-independent version of Theorem 2.7. To do his, we need the following lemma [9].

Lemma 5.4. *For $\lambda > 0$, $\vec{v} \in \mathbb{R}^r$ and $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1, \cdots, \vec{\xi}_{n-1}) \in \mathbb{R}^{nr}$ let*

$$\Psi(n,\lambda,\vec{v},\vec{\xi}) = \left[\frac{\lambda}{2\pi(t - t_{n-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\Big\{ i \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \langle [(\vec{\xi}, \vec{\xi}_n)](s_{n,u,v}), \vec{v}_{n,u,v}\rangle + i\langle \vec{\xi}_n, \vec{v}\rangle - \frac{\lambda}{2} \frac{\|\vec{\xi}_n - \vec{\xi}_{n-1}\|^2}{t - t_{n-1}} \Big\} dm_L^r(\vec{\xi}_n)$$

where $(\vec{\xi}, \vec{\xi}_n) = (\vec{\xi}_0, \vec{\xi}_1, \cdots, \vec{\xi}_{n-1}, \vec{\xi}_n) \in \mathbb{R}^{(n+1)r}$, $t_{n-1} \le s_{n,u,v} \le t_n = t$ and $\vec{v}_{n,u,v} \in \mathbb{R}^r$ for $u = 0, 1, \cdots, r_n; v = 1, 2, \cdots, j_u + 1$. Then we have

$$\Psi(n,\lambda,\vec{v},\vec{\xi}) = \exp\Big\{ i\Big\langle \vec{\xi}_{n-1}, \vec{v} + \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \vec{v}_{n,u,v} \Big\rangle - \frac{t - t_{n-1}}{2\lambda} \Big\| \vec{v} + \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \frac{s_{n,u,v} - t_{n-1}}{t - t_{n-1}} \vec{v}_{n,u,v} \Big\|^2 \Big\}. \tag{5.3}$$

Theorem 5.5. *Let the assumptions and notations be as given in Theorem* 5.1. *Then for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1, \cdots, \vec{\xi}_{n-1}) \in \mathbb{R}^{nr}$, $E^{anw_\lambda}[F_k|Y_t](\vec{\xi})$ *is given by*

$$E^{anw_\lambda}[F_k|Y_t](\vec{\xi}) = k! \sum_{q_1+\cdots+q_n=k} \left[\prod_{l=1}^{n-1} A(l,\lambda,\vec{\xi},q_l)\right] B(n,\lambda,\vec{0},\vec{\xi},q_n)$$

where for $\vec{v} \in \mathbb{R}^r$, $B(n,\lambda,\vec{v},\vec{\xi},q_n)$ *is given by the expression of* $A(n,\lambda,\vec{\xi},q_n)$ *replacing*

$$\exp\{i \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \langle [\vec{\xi}](s_{n,u,v}), \vec{v}_{n,u,v} \rangle\}$$

by $\Psi(n,\lambda,\vec{v},\vec{\xi})$ *which is given by (5.3). Furthermore, for a nonzero real q,* $E^{anf_q}[F_k|Y_t](\vec{\xi})$ *is given by the above expression replacing* λ *by* $-iq$.

Theorem 5.6. *Let the assumptions and notations be as given in Theorem* 5.5 *and let* $F(x) = \exp\{\int_0^t \theta(s,x(s))d\eta(s)\}$ *for* $x \in C^r[0,t]$. *Then for nonzero real q and* $\vec{\xi} \in \mathbb{R}^{nr}$, $E^{anf_q}[F|Y_t](\vec{\xi})$ *exists and it is given by*

$$E^{anf_q}[F|Y_t](\vec{\xi}) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[F_k|Y_t](\vec{\xi})$$

where $E^{anf_q}[F_k|Y_t](\vec{\xi})$ *is as given in Theorem* 5.5.

Theorem 5.7. *Under the assumptions and notations as given in Theorems* 5.3, 5.5 *and* 5.6, *we have for nonzero real q and* $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1, \cdots, \vec{\xi}_{n-1}) \in \mathbb{R}^{nr}$

$$E^{anf_q}[G_k|Y_t](\vec{\xi}) = k! \sum_{q_1+\cdots+q_n=k} \left[\prod_{l=1}^{n-1} A(l,-iq,\vec{\xi},q_l)\right] \int_{\mathbb{R}^r} B(n,-iq,\vec{v},\vec{\xi},q_n) d\nu(\vec{v})$$

and

$$E^{anf_q}[G|Y_t](\vec{\xi}) = \int_{\mathbb{R}^r} \exp\left\{ i\langle \vec{\xi}_{n-1}, \vec{v} \rangle + \frac{t-t_{n-1}}{2qi} \|\vec{v}\|^2 \right\} d\nu(\vec{v}) + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[G_k|Y_t](\vec{\xi}).$$

Remark.
- If F_k, G_k, F and G are defined on the r-dimensional Wiener space, then we can obtain the same results in Theorems 5.1, 5.2, 5.3, 5.5, 5.6 and 5.7 with $\vec{\xi}_0 = \vec{0} \in \mathbb{R}^r$ in the expressions of $\vec{\xi}_{n+1}$ and $\vec{\xi}$.
- If $\eta = \mu$ or $\eta = \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$, we can obtain more simple expressions in Theorems 5.1, 5.2, 5.3, 5.5, 5.6 and 5.7.
- If some of the $p_{l,j}$s are in the set $\{t_0, t_1, \cdots, t_n\}$, we can obtain all the results including η in Theorem 5.1 with minor modifications.
- If $\eta = \mu + \sum_{l=1}^{n} \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$ and some of the r_ls are ∞, then, using (2.5), we can show that $E^{anf_q}[G|X_t]$ in Theorem 5.3 and $E^{anf_q}[G|Y_t]$ in Theorem 5.7 exist.

§ 6. Application to an integral equation

In this section, we present a solution of an integral equation including the integral equation which is formally equivalent to the Schrödinger partial differential equation.

Let $t \in (0,\infty)$ and $\theta : [0,\infty) \times \mathbb{R}^r \to \mathbb{C}$ be the function given by

$$\theta(s,\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u},\vec{v}\rangle\} d\sigma_s(\vec{v}) \tag{6.1}$$

where $\{\sigma_s : s \in [0,\infty)\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the following conditions;

1. For each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,t]$,
2. $\|\sigma_s\| \in L_1[0,t]$.

Furthermore, for a function $F : C^r[0,t] \to \mathbb{C}$ and $\lambda > 0$, let

$$I_F^\lambda(\vec{\xi}) = E[F(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])], \tag{6.2}$$

where $\vec{\xi} \in \mathbb{R}^{2r}$ and the polygonal functions are taken over the partition $0 < t$. In the following theorem, if we replace F by F_s with $0 < s \le t$, then we will assume that the expectation is taken over $C^r[0,s]$ and the polygonal functions are taken over the partition $0 < s$. We now have the following theorem [5].

Theorem 6.1. *For $t \in (0,\infty)$ let*

$$F_t(x) = \exp\left\{\int_0^t \theta(u,x(u))dm_L(u)\right\}$$

for $x \in C^r[0,t]$, where θ is given by (6.1). Further, for $(t,\vec{\xi}_1,\lambda) \in (0,\infty)\times\mathbb{R}^r\times(0,\infty)$, let

$$H(t,\vec{\xi}_1,\lambda) = \left(\frac{\lambda}{2\pi t}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\} I_{F_t}^\lambda(\vec{\xi}_0,\vec{\xi}_1)d\varphi^r(\vec{\xi}_0),$$

where $I_{F_t}^\lambda$ is given by (6.2) in the sense of the conditioning function X_t being given by $X_t(x) = (x(0),x(t))$ on $C^r[0,t]$. Then $F_t \in S_{w_\varphi}^r$ and H satisfies the following integral equation:

$$\begin{aligned} H(t,\vec{\xi}_1,\lambda) = {} & \left(\frac{\lambda}{2\pi t}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\} d\varphi^r(\vec{\xi}_0) + \int_0^t \left[\frac{\lambda}{2\pi(t-s)}\right]^{\frac{r}{2}} \\ & \times \int_{\mathbb{R}^r} \theta(s,\vec{u})H(s,\vec{u},\lambda)\exp\left\{-\frac{\lambda}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|_2^2\right\} dm_L^r(\vec{u})dm_L(s) \end{aligned} \tag{6.3}$$

for $(t,\vec{\xi}_1,\lambda) \in (0,\infty)\times\mathbb{R}^r\times(0,\infty)$.

Theorem 6.2. *Let the assumptions and notations be as given in Theorem 6.1. Moreover, for $(t,\vec{\xi}_1,\lambda) \in (0,\infty)\times\mathbb{R}^r\times\mathbb{C}_+$ let*

$$H(t,\vec{\xi}_1,\lambda) = \left(\frac{\lambda}{2\pi t}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\} E^{anw_\lambda}[F_t|X_t](\vec{\xi}_0,\vec{\xi}_1)d\varphi^r(\vec{\xi}_0).$$

Then $H(t,\vec{\xi}_1,\lambda)$ satisfies the integral equation (6.3) as given in Theorem 6.1.

For a function f defined on $\mathbb{R}^r$, we adopt the following notation:

$$\overline{\int}_{\mathbb{R}^r} f(\vec{z})dm_L^r(\vec{z}) = \lim_{A\to\infty}\int_{\mathbb{R}^r} f(\vec{z})\exp\left\{-\frac{1}{2A}\|\vec{z}\|^2\right\}dm_L^r(\vec{z})$$

if the limit exists. Using this notation, we have the following theorem.

Theorem 6.3. *Let the assumptions and notations be as given Theorem 6.2. Moreover, for $(t,\vec{\xi}_1,q)\in(0,\infty)\times\mathbb{R}^r\times(\mathbb{R}-\{0\})$ let*

$$H(t,\vec{\xi}_1,-iq) = \left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}E^{anf_q}[F_t|X_t](\vec{\xi}_0,\vec{\xi}_1)d\varphi^r(\vec{\xi}_0).$$

Then $H(t,\vec{\xi}_1,-iq)$ satisfies the following integral equation:

$$\begin{aligned}H(t,\vec{\xi}_1,-iq) &= \left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}d\varphi^r(\vec{\xi}_0)+\int_0^t\left[\frac{q}{2\pi i(t-s)}\right]^{\frac{r}{2}}\\ &\quad\times\overline{\int}_{\mathbb{R}^r}\theta(s,\vec{u})H(s,\vec{u},-iq)\exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|^2\right\}dm_L^r(\vec{u})dm_L(s).\end{aligned}$$

Corollary 6.4. *Under the assumptions and notations as given in Theorem 6.3, if φ^r is the Dirac measure concentrated at $\vec{0}$, then w_φ^r is exactly the r-dimensional Wiener measure on the Borel class of $C_0^r[0,t]$, and we have*

$$H(t,\vec{\xi}_1,-iq) = \left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1\|^2\right\}E^{anf_q}[F_t|X_t](\vec{0},\vec{\xi}_1)$$

so that $H(t,\vec{\xi}_1,-iq)$ satisfies the following integral equation which is formally equivalent to the Schrödinger partial differential equation:

$$\begin{aligned}H(t,\vec{\xi}_1,-iq) &= \left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1\|^2\right\}+\int_0^t\left[\frac{q}{2\pi i(t-s)}\right]^{\frac{r}{2}}\\ &\quad\times\overline{\int}_{\mathbb{R}^r}\theta(s,\vec{u})H(s,\vec{u},-iq)\exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|^2\right\}dm_L^r(\vec{u})dm_L(s)\end{aligned}$$

which is the integral equation as given in Theorem 6 of [10].

Theorem 6.5. *Let the assumptions and notations be as given Theorem 6.3. Suppose that $\varphi^r \ll m_L$, that is, φ^r has the probability density ψ on $\mathbb{R}^r$. Moreover, for $(t,\vec{\xi}_1,q)\in(0,\infty)\times\mathbb{R}^r\times(\mathbb{R}-\{0\})$ let*

$$H(t,\vec{\xi}_1,-iq) = \left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}E^{anf_q}[F_t|X_t](\vec{\xi}_0,\vec{\xi}_1)\psi(\vec{\xi}_0)dm_L^r(\vec{\xi}_0).$$

Then $H(t,\vec{\xi}_1,-iq)$ satisfies the following integral equation:

$$H(t,\vec{\xi}_1,-iq)=\left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r}\psi(\vec{\xi}_0)\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}dm_L^r(\vec{\xi}_0)+\int_0^t\left[\frac{q}{2\pi i(t-s)}\right]^{\frac{r}{2}}$$
$$\times\overline{\int}_{\mathbb{R}^r}\theta(s,\vec{u})H(s,\vec{u},-iq)\exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|^2\right\}dm_L^r(\vec{u})dm_L(s)$$

which is formally equivalent to the Schrödinger partial differential equation.

Remark. If ψ is Lebesgue measurable, then we can take a Borel measurable function ψ_1 with $\psi(\vec{u})=\psi_1(\vec{u})$ for m_L^r-a.e. $\vec{u}\in\mathbb{R}^r$, so that we can assume that ψ is Borel measurable. Furthermore, since $\psi\in L_1(\mathbb{R}^r)$, we have by the dominated convergence theorem

$$\overline{\int}_{\mathbb{R}^r}\psi(\vec{\xi}_0)\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}dm_L^r(\vec{\xi}_0)=\int_{\mathbb{R}^r}\psi(\vec{\xi}_0)\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}dm_L^r(\vec{\xi}_0)$$

so that $H(t,\vec{\xi}_1,-iq)$ satisfies the following integral equation:

$$H(t,\vec{\xi}_1,-iq)=\left(\frac{q}{2\pi it}\right)^{\frac{r}{2}}\overline{\int}_{\mathbb{R}^r}\psi(\vec{\xi}_0)\exp\left\{\frac{iq}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|^2\right\}dm_L^r(\vec{\xi}_0)+\int_0^t\left[\frac{q}{2\pi i(t-s)}\right]^{\frac{r}{2}}$$
$$\times\overline{\int}_{\mathbb{R}^r}\theta(s,\vec{u})H(s,\vec{u},-iq)\exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|^2\right\}dm_L^r(\vec{u})dm_L(s)$$

which is formally equivalent to the Schrödinger partial differential equation [1, 3, 12].

§ 7. Operator-valued function space integral

In this section we investigate that the conditional analytic Feynman integrals of functionals on $C^r[0,t]$ can be applied to express the operator-valued Feynman integrals in terms the conditional Feynman integrals.

Throughout this section, let $t>0$ be fixed, $X_t: C^r[0,t]\to\mathbb{R}^{2r}$ be given by $X_t(x)=(x(0),x(t))$. Further, for $\lambda>0$ and $\vec{\xi}\in\mathbb{R}^r$, let $X_t^{\lambda,\vec{\xi}}(x)=X_t(\lambda^{-\frac{1}{2}}x+\vec{\xi})$ and for a function $F: C^r[0,t]\to\mathbb{C}$, let $F^{\lambda,\vec{\xi}}(x)=F(\lambda^{-\frac{1}{2}}x+\vec{\xi})$. Now we define the analytic operator-valued function space integral.

Definition 7.1. Let $F: C^r[0,t]\to\mathbb{C}$ be a function. For any $\lambda>0$, ψ in $L_1(\mathbb{R}^r)$ and $\vec{\xi}$ in $\mathbb{R}^r$, let $\psi_t^{\lambda,\vec{\xi}}(x)=\psi(\lambda^{-\frac{1}{2}}x(t)+\vec{\xi})$ and

$$(I_\lambda(F)\psi)(\vec{\xi})=\int_{C^r}F^{\lambda,\vec{\xi}}(x)\psi_t^{\lambda,\vec{\xi}}(x)dw_\varphi^r(x).$$

If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{R}^r)$ as a function of $\vec{\xi}$ and if the correspondence $\psi\to I_\lambda(F)\psi$ gives an element of $\mathcal{L}\equiv\mathcal{L}(L_1(\mathbb{R}^r),L_\infty(\mathbb{R}^r))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists an $\mathcal{L}$-valued function which is weakly analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0,\infty)$. Then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{an}(F)$ and is called the analytic operator-valued Wiener integral of F associated with parameter λ. Finally, for a nonzero real q suppose that there exists an operator $J_q^{an}(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{R}^r)$, $I_\lambda^{an}(F)\psi$ converges weakly to $J_q^{an}(F)\psi$, as λ approaches to $-iq$ through $\mathbb{C}_+$. Then $J_q^{an}(F)$ is called the analytic operator-valued Feynman integral of F with parameter q.

Note that in Definition 7.1, the weak limit and the weak analyticity are based on the weak* topology on $L_\infty(\mathbb{R}^r)$ induced by its pre-dual $L_1(\mathbb{R}^r)$ [2, 8, 13].

Lemma 7.2. *Let $\lambda > 0$ and $\vec{\xi} \in \mathbb{R}^r$. Suppose that φ^r is absolutely continuous with respect to m_L^r. Then $P_{X_t^{\lambda,\vec{\xi}}} \ll m_L^{2r}$ and*

$$\frac{dP_{X_t^{\lambda,\vec{\xi}}}}{dm_L^{2r}}(\vec{\eta}_1,\vec{\eta}_2) = \Big(\frac{\lambda}{\sqrt{2\pi t}}\Big)^r \exp\Big\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|^2}{2t}\Big\}\frac{d\varphi^r}{dm_L^r}(\lambda^{\frac{1}{2}}(\vec{\eta}_1-\vec{\xi}))$$

for m_L^{2r}-a.e. $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$.

Theorem 7.3. *Let the assumptions and notations be as in Lemma 7.2. For $F : C^r[0,t] \to \mathbb{C}$, suppose that $E^{anw_\lambda}[F|X_t](\vec{\eta}_1,\vec{\eta}_2)$ exists on $\mathbb{C}_+ \times \mathbb{R}^{2r}$, and for each bounded subset Ω of $\mathbb{C}_+$, there exists $M_\Omega \geq 0$ such that $|E^{anw_\lambda}[F|X_t](\vec{\eta}_1,\vec{\eta}_2)| \leq M_\Omega$ for all $\lambda \in \Omega$ and all $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$. Further, suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying the following conditions;*

(i) *for each $\lambda > 0$ and $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta}) = \frac{d\varphi^r}{dm_L^r}(\lambda^{\frac{1}{2}}\vec{\eta})$,*
(ii) *for each $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta})$ is analytic on $\mathbb{C}_+$ as a function of λ, and*
(iii) *for each bounded subset Ω of $\mathbb{C}_+$, Ψ is bounded on $\Omega \times \mathbb{R}^r$.*

Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener integral $I_\lambda^{an}(F)$ exists as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ and is given by

$$(7.1)\qquad (I_\lambda^{an}(F)\psi)(\vec{\xi}) = \Big(\frac{\lambda}{\sqrt{2\pi t}}\Big)^r \int_{\mathbb{R}^{2r}} E^{anw_\lambda}[F|X_t](\vec{\eta}_1,\vec{\eta}_2)\psi(\vec{\eta}_2) \times\Psi(\lambda,\vec{\eta}_1-\vec{\xi})\exp\Big\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|^2}{2t}\Big\}dm_L^{2r}(\vec{\eta}_1,\vec{\eta}_2)$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\vec{\xi} \in \mathbb{R}^r$. In addition, suppose that for a nonzero real q and $\vec{\xi} \in \mathbb{R}^r$, $E^{anf_q}[F|X_t](\vec{\eta}_1,\vec{\eta}_2)$ exists for $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$. Moreover, suppose that Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ with the following two additional conditions;

(ii)$'$ *for each $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta})$ is continuous at $\lambda = -iq$ as a function of λ, and*
(iii)$'$ *there exists a function $\Phi_q \in L_1(\mathbb{R}^r)$ satisfying*

$$|\Psi(\lambda,\vec{\eta})| \leq |\Phi_q(\vec{\eta})| \text{ for all } (\lambda,\vec{\eta}) \in \Omega_\epsilon \times \mathbb{R}^r,$$

where $\Omega_\epsilon = \{\lambda \in \mathbb{C}_+ : |\lambda + iq| < \epsilon\}$ for some real $\epsilon > 0$.

Then as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ the analytic operator-valued Feynman integral $J_q^{an}(F)$ exists and it is given by (7.1) *replacing λ and E^{anw_λ} by $-iq$ and E^{anf_q}, respectively.*

Example 7.4. For $\vec{\eta} = (\eta_1, \cdots, \eta_r) \in \mathbb{R}^r$ and $\lambda \in \mathbb{C}_+^\sim$, let

$$\frac{d\varphi^r}{dm_L^r}(\vec{\eta}) = \left(\frac{1}{\pi}\right)^r \prod_{j=1}^r \frac{1}{1+\eta_j^2} \text{ and } \Psi(\lambda, \vec{\eta}) = \left(\frac{1}{\pi}\right)^r \prod_{j=1}^r \frac{1}{1+\lambda\eta_j^2}.$$

Then φ^r is a probability measure on the Borel class of $\mathbb{R}^r$ and the condition (i) of Theorem 7.3 is satisfied. Further, for $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda, \vec{\eta})$ is analytic on $\mathbb{C}_+$ and continuous on $\mathbb{C}_+^\sim$ because $1 + \lambda\eta_j^2 \neq 0$ for $\lambda \in \mathbb{C}_+^\sim$, satisfying conditions (ii) and (ii)$'$ of Theorem 7.3. Now we have for $(\lambda, \eta_j) \in \mathbb{C}_+^\sim \times \mathbb{R}$

$$\left|\frac{1}{1+\lambda\eta_j^2}\right|^2 = \frac{1}{(1+\eta_j^2\mathrm{Re}\lambda)^2 + (\eta_j^2\mathrm{Im}\lambda)^2} \leq \min\left\{1, \frac{1}{1+\eta_j^4(\mathrm{Im}\lambda)^2}\right\}$$

which satisfies conditions (iii) and (iii)$'$ of Theorem 7.3. Let F be a function satisfying the assumptions of Theorem 7.3. Applying Theorem 7.3 to F, for any nonzero real q, the analytic operator-valued Feynman integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ and it is given by

$$\begin{aligned}(J_q^{an}(F)\psi)(\vec{\xi}) = \left(\frac{q}{\pi i\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} & E^{anf_q}[F|X_t](\vec{\eta}_1, \vec{\eta}_2)\psi(\vec{\eta}_2) \\ & \times \left[\prod_{j=1}^r \frac{1}{1 - iq(\eta_{1,j} - \xi_j)^2}\right] \exp\left\{\frac{iq\|\vec{\eta}_2 - \vec{\eta}_1\|^2}{2t}\right\} dm_L^{2r}(\vec{\eta}_1, \vec{\eta}_2)\end{aligned}$$

for $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} = (\xi_1, \cdots, \xi_r) \in \mathbb{R}^r$, where $\vec{\eta}_1 = (\eta_{1,1}, \cdots, \eta_{1,r})$. Note that the probability distribution φ having the above density for $r = 1$ is known as the Cauchy distribution.

Using the same method as used in the proof of Theorem 7.3, we can easily prove the following theorem.

Theorem 7.5. *If the conditions (iii) and (iii)$'$ in Theorem* 7.3 *are replaced by the one condition that for each bounded subset Ω of $\mathbb{C}_+$ there exists a function $\Phi_\Omega \in L_1(\mathbb{R}^r)$ satisfying*

$$|\Psi(\lambda, \vec{\eta})| \leq |\Phi_\Omega(\vec{\eta})| \textit{ for all } (\lambda, \vec{\eta}) \in \Omega \times \mathbb{R}^r, \tag{7.2}$$

then we have all the results of Theorem 7.3.

Theorem 7.6. *Let $F \in S_{w_\varphi}^r$ be given by* (2.1). *Then we have for $\lambda \in \mathbb{C}_+$*

$$E^{anw_\lambda}[F|X_t](\vec{\eta}_1, \vec{\eta}_2) = \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}[t\|\vec{v}\| - \|\vec{V}_t\|^2] + \frac{i}{t}\langle\vec{\eta}_2 - \vec{\eta}_1, \vec{V}_t\rangle\right\} d\sigma(\vec{v}) \tag{7.3}$$

for $(\vec{\eta}_1, \vec{\eta}_2) \in \mathbb{R}^{2r}$, where $\vec{V}_t = (\int_0^t v_1(s)ds, \cdots, \int_0^t v_r(s)ds)$. Moreover, for any nonzero real q, $E^{anf_q}[F|X_t]$ is given by the right hand side of (7.3) *replacing λ by $-iq$.*

For $F \in \mathcal{S}^r_{w_\varphi}$ given by (2.1), we know from Theorem 7.6 that $E^{anw}[F|X_t]$ is bounded by $\|\sigma\|$, where $\|\sigma\|$ denotes the total variation of σ. Combining Lemma 7.2, Theorems 7.3, 7.5 and 7.6, we have the following theorem [8].

Theorem 7.7. *Let the assumptions be as given in Lemma 7.2 and let $F \in \mathcal{S}^r_{w_\varphi}$. Suppose that for a nonzero real q there exists a function Ψ on $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ satisfying the conditions (i), (ii), (ii)$'$ of Theorem 7.3, and either (iii) and (iii)$'$ of Theorem 7.3 or (7.2) of Theorem 7.5. Then the analytic operator-valued Feynman integral $J^{an}_q(F)$ exists as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ and it is given by (7.1) replacing λ and E^{anw_λ} by $-iq$ and E^{anf_q}, respectively, where $E^{anf_q}[F|X_t]$ is given by Theorem 7.6.*

Theorem 7.8. *Let the assumptions and notations be as given in Theorem 7.6. Furthermore, suppose that φ^r is normally distributed with the mean vector $\vec{0}$ and the variance-covariance matrix $\alpha^2 I_r$, where I_r is the r-dimensional identity matrix. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener integral $I^{an}_\lambda(F)$ exists as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ and it is given by*

$$(7.4)\qquad (I^{an}_\lambda(F)\psi)(\vec{\xi}) = \left[\frac{\lambda}{2\pi(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}[t\|\vec{v}\|^2 - \|\vec{V}_t\|^2]\right\} \int_{\mathbb{R}^r} \psi(\vec{\eta}) H\left(\lambda, \vec{\xi}, \vec{\eta}, \frac{1}{t}\vec{V}_t\right) dm^r_L(\vec{\eta}) d\sigma(\vec{v})$$

for $\psi \in L_1(\mathbb{R}^r)$ and m^r_L-a.e. $\vec{\xi} \in \mathbb{R}^r$, where

$$H(\lambda, \vec{\xi}, \vec{\eta}, \vec{\zeta}) = \exp\left\{-\frac{\lambda}{2\alpha^2}\|\vec{\xi} - \vec{\eta}\|^2 - \frac{t\alpha^2}{2\lambda(t+\alpha^2)}\left\|\vec{\zeta} + \frac{\lambda i}{\alpha^2}(\vec{\xi} - \vec{\eta})\right\|^2\right\}$$

for $\lambda \in \mathbb{C}_+^\sim$ and $\vec{\zeta} \in \mathbb{R}^r$. Furthermore, for any nonzero real q the analytic operator-valued Feynman integral $J^{an}_q(F)$ exists as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ and it is given by (7.4) replacing λ by $-iq$.

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