数理解析研究所講究録
第 1806 巻 2012 年 65-78

On the transcendental degrees of the fields generated by special values of power series

金子 元 (Kaneko Hajime) *
Department of Mathematics
College of Science and Technology, Nihon University

Abstract

We study arithmetical properties of the values of power series

$$f(w(n);X)=\sum_{n=0}^{\infty}X^{w(n)}$$

at algebraic points, where $w(n)$ $(n=0,1,\ldots)$ is a sequence of non-negative integers with $w(n+1)>w(n)$ for any sufficiently large n. In [5] the author proved algebraic independence of such numbers in the case where $X=b^{-1}$ with $b\in\mathbb{Z}, b\geq 2$ and

$$w(n)=\beta(y;n):=\left[\exp\left(\log(n)^{1+y}\right)\right]$$

with $y\in\mathbb{R}$, $y\geq 1$. In this paper we study transcendental degrees of the field generated by $\mathbb{Q}$ and $f(\beta(y;n);\alpha^{-1})$ for certain algebraic integers α. In section 3 we give another elementary proofs of certain Fredholm numbers in order to show main ideas for our results.

1 Introduction

In this paper we study arithmetical properties of the values of power series

$$f(w(n);X):=\sum_{n=0}^{\infty}X^{w(n)}$$

*This work is supported by the JSPS fellowship.

at algebraic points $X = \alpha$, where $w(n)$ $(n = 0, 1, \ldots)$ is a sequence of non-negative integers with

$$w(n+1) > w(n) \tag{1.1}$$

for any sufficiently large n. In this section we introduce transcendence and algebraic independence of such values in the case where $w(n)$ $(n = 0, 1, \ldots)$ is lacunary. Recall that $w(n)$ $(n = 0, 1, \ldots)$ is lacunary if

$$\liminf_{n\to\infty} \frac{w(n+1)}{w(n)} > 1.$$

Note that if $w(n)$ $(n = 0, 1, \ldots)$ is lacunary, then there exist a positive constant δ satisfying

$$w(n) > (1+\delta)^n \tag{1.2}$$

for any sufficiently large n. Liouville [8, 9] showed for any integer b greater than 1 that the number

$$f(n!; b^{-1}) = \sum_{n=0}^{\infty} b^{-n!}$$

is transcendental, using Diophantine inequalities. By his method we verify the following: Assume that $w(n)$ $(n = 0, 1, \ldots)$ satisfies

$$\lim \frac{w(n+1)}{w(n)} = \infty. \tag{1.3}$$

Then $f(w(n); b^{-1}) = \sum_{n=0}^{\infty} b^{-w(n)}$ is transcendental for any integer b greater than 1. Note that if $w(n)$ $(n = 0, 1, \ldots)$ satisfies (1.3), then $f(w(n); X)$ is called a gap series.

We consider the case where $f(w(n); X)$ is not a gap series. Let k and b be integers greater than 1. Kempner [6] showed that the Fredholm number

$$f(k^n; b^{-1}) = \sum_{n=0}^{\infty} b^{-k^n}$$

is transcendental. Moreover, Mahler [10] proved for any algebraic number α with $0 < |\alpha| < 1$ that

$$f(k^n; \alpha) = \sum_{n=0}^{\infty} \alpha^{k^n}$$

is transcendental. He verified transcendence of such numbers, using the functional equation

$$f(k^n; X^k) = \sum_{n=0}^{\infty} X^{k^{n+1}} = \sum_{n=1}^{\infty} X^{k^n} = f(k^n; X) - X.$$

Applying Mahler's method, we deduce that $f(w(n); \alpha)$ is transcendental in the case where $w(n)$ $(n = 0, 1, \ldots)$ is a liner recurrence with certain assumptions. We recall that $w(n)$ is linear recurrence if there exist a positive integer l and complex numbers $c_1, \ldots, c_l$ such that

$$w_{n+l} = c_1 w_{n+l-1} + c_2 w_{n+l-2} + \cdots + c_l w_n \tag{1.4}$$

for any nonnegative integer n. For instance, let F_n $(n = 0, 1, \ldots)$ be the sequence of Fibonacci numbers defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for any nonnegative integer n. Then $f(F_n; \alpha)$ is transcendental for any algebraic number α with $0 < |\alpha| < 1$.

Let again α be an algebraic number with $0 < |\alpha| < 1$. Then Corvaja and Zannier [3] showed for an arbitrary lacunary sequence $w(n)$ $(n = 0, 1, \ldots)$ that $f(w(n); \alpha)$ is transcendental, using the Schmidt subspace theorem. Note that if $w(n) = k^n$ $(n = 0, 1, \ldots)$, then the transcendence of $f(k^n; \alpha)$ follows from the Roth-Ridout theorem.

In the rest of this section we study algebraic independence of $f(w(n); \alpha)$ for distinct lacunary sequences $w(n)$ $(n = 0, 1, \ldots)$. First we consider the case where $f(w(n); X)$ is a gap series. Schmidt [13] proved for any integer b greater than 1 that the set

$$\left\{ f((hn)!; b^{-1}) = \sum_{n=0}^{\infty} b^{-(hn)!} \,\middle|\, h = 1, 2, \ldots \right\}$$

is algebraically independent. We recall that a nonempty set S of complex numbers is algebraically independent if arbitrary numbers of distinct elements $\xi_1, \ldots, \xi_r$ in S are algebraically independent. Next, let γ be a positive algebraic number less than 1. Then Durand [4] verified that the continuous set

$$\left\{ f([x(n)!]; \gamma) = \sum_{n=0}^{\infty} \gamma^{[x(n)!]} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\}$$

is algebraically independent, where $[y]$ is the integral part of a real number y. Shiokawa [14] verified criteria for algebraic independence of the values of gap series at algebraic points, which give generalizations of the results by

Schmidt and Durand above. For instance, using his criteria, we deduce the following: For any positive real number x we take an algebraic number α_x with $0 < |\alpha_x| < 1$. Then the continuous set

$$\left\{ f([x(n)!];\alpha_x) = \sum_{n=0}^{\infty} \alpha_x^{[x(n)!]} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\}$$

is algebraically independent.

We now consider the case where $w(n)$ $(n = 0, 1, \ldots)$ is not lacunary. Nishioka [11] verified for any algebraic number α with $0 < |\alpha| < 1$ that the set

$$\left\{ f(k^n;\alpha) = \sum_{n=0}^{\infty} \alpha^{k^n} \,\middle|\, k = 2, 3, \ldots \right\}$$

is algebraically independent, using Mahler's method. We give elementary proof of algebraic independence of

$$\sum_{n=0}^{\infty} b^{-2^n} \text{ and } \sum_{n=0}^{\infty} b^{-4^n}$$

in Section 3, where b is an integer greater than 1. Let $(v_n^{(1)})_{n=0}^{\infty}$, $(v_n^{(2)})_{n=0}^{\infty}, \ldots$ be distinct linear recurrences satisfying (1.4) with common coefficients $c_1, \ldots, c_l$. Tanaka [15] investigated algebraic independence of $f(v_n^{(1)};\alpha), f(v_n^{(2)};\alpha), \ldots$, applying Mahler's method. For instance, let $(v_n^{(1)})_{n=0}^{\infty}, (v_n^{(2)})_{n=0}^{\infty}$ be sequences of nonnegative integers satisfying

$$v_{n+2}^{(i)} = v_{n+1}^{(i)} + v_n^{(i)} \text{ for } i = 1, 2.$$

Then, for any algebraic number α with $0 < |\alpha| < 1$, two numbers $f(v_n^{(1)};\alpha)$ and $f(v_n^{(2)};\alpha)$ are algebraically dependent if and only if there exists an integer N satisfying

$$v_n^{(2)} = v_{n+N}^{(1)}$$

for any sufficiently large n. For more details on Mahler's method, see [12].

2 Main results

Let b be an integer greater than 1 and $w(n)$ $(n = 0, 1, \ldots)$ a sequence of nonnegative integers with (1.1). Then

$$f(w(n);b^{-1}) = \sum_{n=0}^{\infty} b^{-w(n)}$$

gives the base-b expansion of a positive real number. Borel [2] conjectured that any positive algebraic irrational number ξ is normal in base-b. In particular, if this conjecture holds, then ξ is simply normal in base-b. Namely, let

$$\xi = \sum_{n=0}^{\infty} s_n b^{-n}$$

be the base-b expansion of ξ, where $s_0 = [\xi]$ and $s_n \in \{0, 1, \ldots, b-1\}$ for $n \geq 1$. Let $0 \leq k \leq b-1$. Put

$$\lambda(b, k, \xi; N) := \mathrm{Card}\{n \in \mathbb{Z} \mid s_n = k,\ 1 \leq n \leq N\},$$

where Card denotes the cardinality. Then it is believed that

$$\lim_{N\to\infty} \frac{\lambda(b, k, \xi; N)}{N} = \frac{1}{b}. \tag{2.1}$$

If Borel's conjecture above is true, then we deduce the following: For any $w(n)$ $(n = 0, 1, \ldots)$ with

$$\lim_{n\to\infty} \frac{w(n)}{n} = \infty,$$

then the number $f(w(n); b^{-1})$ is transcendental. In fact, assume that $f(w(n); b^{-1})$ is algebraic. Thus, $f(w(n); b^{-1})$ is an algebraic irrational number because the base-b expansion of $f(w(n); b^{-1})$ is not ultimately periodic. We have

$$\lim_{N\to\infty} \frac{\lambda(b, 0, \xi; N)}{N} = 1$$

with $\xi = f(w(n); b^{-1})$, which contradicts (2.1).

Bailey, Borwein, Crandall, and Pomerance [1] showed the following: Assume that

$$\lim_{n\to\infty} \frac{w(n)}{n^R} = \infty \tag{2.2}$$

for any positive R. Then $f(w(n); b^{-1})$ is transcendental. Note that they proved the results on transcendence above only in the case of $b = 2$. However, transcendence of $f(w(n); b^{-1})$ for an arbitrary b is proved in the same way.

We give applications of the transcendental results above. First we consider the case where $w(n)$ $(n = 0, 1, \ldots)$ is lacunary. Then (1.2) implies that $w(n)$ satisfies (2.2). Hence, $f(w(n); b^{-1})$ is transcendental, which gives special cases of the transcendental results by Corvaja and Zannier [3]. Next we consider the case where $w(n)$ $(n = 0, 1, \ldots)$ is not lacunary. For a positive real number y and a positive integer n, we put

$$\beta(y; n) := \left[\exp\left((\log n)^{1+y}\right)\right],$$

where $[x]$ denotes the integral part of a real number x. Let

$$\mu(y; X) := f(\beta(y; n); X) = \sum_{n=1}^{\infty} \beta(y; n) X^n.$$

Then $\beta(y; n)$ satisfies (2.2) because

$$n^R = \exp(R \log n).$$

Hence, $\mu(y; b^{-1})$ is transcendental for any integer b greater than 1. Note that $\beta(y; n)$ $(n = 0, 1, \ldots)$ is not lacunary because $\beta(y; n)$ does not satisfy (1.2). Thus, we cannot prove transcendence of $\mu(y; b^{-1})$ by the criteria for transcnedence by Corvaja and Zannier.

Let again b be an integer greater than 1. The author [5] showed that the continuous set

$$\{\mu(y; b^{-1}) \mid y \geq 1, \ y \in \mathbb{R}\}$$

is algebraically independent. Moreover, in the same paper the author verified for any positive distinct real numbers x and y that $\mu(x; b^{-1})$ and $\mu(y; b^{-1})$ are algebraically independent. On the other hand, it is unknown whether $\mu(y; -b^{-1})$ is transcendental for a positive real number y.

In what follows we consider arithmetical properties of $\mu(y; \alpha^{-1})$, where α is an algebraic integer with certain assumptions.

Let α be an algebraic integer of degree d. We write the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. We say that α is represented by an expanding nonnegative matrix if α satisfies the following two assumptions:

1.
$$|\alpha_i| > 1 \text{ for } i = 1, \ldots, d. \tag{2.3}$$

2. There exists a square matrix A of order d whose entries are nonnegative integers such that the eigenvalues of A are $\alpha_1, \ldots, \alpha_d$.

For instance, $\alpha = 3 + \sqrt{2}$ is an algebraic integer represented by an expanding nonnegative matrix. In fact, $\alpha_1 = \alpha$ and $\alpha_2 = 3 - \sqrt{2}$ satisfy (2.3). Moreover, the eigenvalues of

$$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$$

are α_1 and α_2.

THEOREM 2.1. *Let α be an algebraic integer of degree d. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ be the conjugates of α. Assume that α is represented by an expanding nonnegative matrix.*
(1) *There exists an i with $1 \le i \le d$ such that the continuous set*

$$\{\mu(y;\alpha_i^{-1}) \mid y \ge 1, y \in \mathbb{R}\}$$

is algebraically independent. In particular, let r be a positive integer and $y_1, \ldots, y_r$ be distinct real numbers not less than 1. Then

$$\mathrm{tr.deg}\mathbb{Q}(\{\mu(y_j;\alpha_i^{-1}) \mid 1 \le i \le d \ 1 \le j \le r\}) \ge r.$$

(2) *Let x and y be distinct positive real numbers. Then there exists i with $1 \le i \le d$ such that $\mu(x;\alpha_i^{-1})$ and $\mu(y;\alpha_i^{-1})$ are algebraically independent. In particular,*

$$\mathrm{tr.deg}\mathbb{Q}\big(\mu(x;\alpha_1^{-1}), \ldots, \mu(x;\alpha_d^{-1}), \mu(y;\alpha_1^{-1}), \ldots, \mu(y;\alpha_d^{-1})\big) \ge 2.$$

3 Elementary proof of algebraic independence of certain Fredholm numbers

In this section we denote the set of nonnegative integers by $\mathbb{N}$. Let b be an integer greater than 1. Put

$$\xi_1 := \sum_{n=0}^{\infty} b^{-2^n}, \ \xi_2 := \sum_{n=0}^{\infty} b^{-4^n}.$$

Knight [7] gave a simple proof of transcendence of ξ_1. Namely, he proved transcendence of ξ_1, calculating the base-b expansion of $\xi_1, \xi_1^2, \xi_1^3, \ldots$. In this section, applying his method, we show that ξ_1 and ξ_2 are algebraically independent in order to show main ideas for our results of algebraic independence. That is, we verify the following: for any nonzero polynomial

$$P(X,Y) = \sum_{\mathbf{k}=(k,l)\in\Lambda} A_{\mathbf{k}} X^k Y^l$$

with integral coefficients, we have $P(\xi_1, \xi_2) \neq 0$. Here, Λ is a nonempty finite subset of $\mathbb{N}^2$ and $A_{\mathbf{k}}$ is a nonzero integer for any $\mathbf{k} \in \Lambda$. We introduce the graded lexicographic order $\succ$ in $\mathbb{N}^2$ satisfying $(1,0) \succ (0,1)$. Namely, $(k,l) \succ (k',l')$ if $k+l > k'+l'$, or if $k+l = k'+l'$ and simultaneously $k > k'$. Let $\mathbf{g} = (g,h)$ be the maximal element of Λ with respect to $\succ$. Without loss of generality, we may assume that XY divides $P(X,Y)$ and that $A_{\mathbf{g}} \geq 1$. Since XY divides $P(X,Y)$, we have $k, l \geq 1$ for any $(k,l) \in \Lambda$. Set

$$\begin{aligned}\Lambda_+ &:= \{\mathbf{k} \in \Lambda \mid A_{\mathbf{k}} > 0\},\\ \Lambda_- &:= \{\mathbf{k} \in \Lambda \mid A_{\mathbf{k}} < 0\}.\end{aligned}$$

Moreover, put

$$\eta_1 := \sum_{\mathbf{k}=(k,l)\in\Lambda_+} A_{\mathbf{k}}\xi_1^k\xi_2^l,\ \eta_2 := \sum_{\mathbf{k}=(k,l)\in\Lambda_-} |A_{\mathbf{k}}|\xi_1^k\xi_2^l.$$

For the proof of algebraic independence of ξ_1 and ξ_2, it suffices to show that $\eta_1 \neq \eta_2$. The proof relies on the calculation of the base-b expansions of η_1 and η_2. Let $(k,l) \in \Lambda$. We calculate $\xi_1^k\xi_2^l$. Put

$$\begin{aligned}S_1 &:= \{2^n \mid n \in \mathbb{N}\} = \{1,2,4,8,\ldots\},\\ S_2 &:= \{4^n \mid n \in \mathbb{N}\} = \{1,4,16,64,\ldots\}.\end{aligned}$$

Then ξ_1 and ξ_2 are written as

$$\xi_1 = \sum_{i\in S_1} b^{-i},\ \xi_2 = \sum_{j\in S_2} b^{-j}.$$

Let

$$kS_1 + lS_2 := \left\{ \sum_{i=1}^k x_i + \sum_{j=1}^l y_j \,\middle|\, x_1, x_2, \ldots, x_k \in S_1,\ y_1, y_2, \ldots, y_l \in S_2 \right\}$$

and

$$\rho(k,l;n) := \mathrm{Card}\left\{ (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l) \in S_1^k \times S_2^l \,\middle|\, \sum_{i=1}^k x_i + \sum_{j=1}^l y_j = n \right\}.$$

It is easily seen that

$$\rho(k,l;n) \leq n^{k+l} \tag{3.1}$$

because $1 \leq x_i, y_j \leq n$ for any $1 \leq i \leq k$ and $1 \leq j \leq l$. We obtain

$$\begin{aligned}\xi_1^k\xi_2^l &= \left(\sum_{x\in S_1} b^{-x}\right)^k \left(\sum_{y\in S_2} b^{-y}\right)^l\\ &= \sum_{x_1,\ldots,x_k\in S_1} b^{-(x_1+\cdots+x_k)} \sum_{y_1,\ldots,y_l\in S_2} b^{-(y_1+\cdots+y_l)}\\ &= \sum_{n\in kS_1+lS_2} \rho(k,l;n) b^{-n}. \qquad (3.2)\end{aligned}$$

In particular, we deduce that

$$\eta_1 = \sum_{\mathbf{k}=(k,l)\in\Lambda_+} A_{\mathbf{k}} \sum_{n\in kS_1+lS_2} \rho(k,l;n)b^{-n} \tag{3.3}$$

and that

$$\eta_2 = \sum_{\mathbf{k}=(k,l)\in\Lambda_-} |A_{\mathbf{k}}| \sum_{n\in kS_1+lS_2} \rho(k,l;n)b^{-n}. \tag{3.4}$$

Let m be a nonnegative integer. Put

$$B := 2^0 + 2^2 + \cdots + 2^{2h-2} + 2^{2h-1} + 2^{2h+1} + \cdots + 2^{2h+2g-3}$$

and

$$N(m) := 2^{2m}B.$$

LEMMA 3.1. *Let m be an integer greater than 1 and $(k,l) \in \Lambda$. Then*

$$(kS_1 + lS_2) \cap \left[N(m) - 2^{2m-2}, N(m) + 2^{2m-2}\right] = \begin{cases} \emptyset & ((k,l) \neq (g,h)), \\ \{N(m)\} & ((k,l) = (g,h)). \end{cases}$$

Proof. Put

$$\Gamma(k,l;m) := (kS_1 + lS_2) \cap \left[N(m) - 2^{2m-2}, N(m) + 2^{2m-2}\right].$$

For each $n \in \mathbb{N}$, let us write the sum of digits of the binary expansion of n by $\sigma(n)$. Let $n \in kS_1 + lS_2$. Namely,

$$n = \sum_{i=1}^{k} x_i + \sum_{j=1}^{l} y_j,$$

where $x_i \in S_1$ for $i = 1, \ldots, k$ and $y_j \in S_2$ for $j = 1, \ldots, l$. The right-hand side of the equality above causes carry in the binary expansion of n. Thus,

$$\sigma(n) \le k + l \le g + h \tag{3.5}$$

because (g,h) is the maximal element of Λ with respect to $\succ$. Moreover, suppose that $n \in kS_1 + lS_2$ and that $\sigma(n) = k + l$. Then write the binary expansion of n by

$$n = \sum_{i\in\Omega} 2^i,$$

where Ω is a finite subset of $\mathbb{N}$ with Card $\Omega = k + l$. Then we get

$$n = \sum_{i\in\Omega_1,\ 2^i\in S_1} 2^i + \sum_{j\in\Omega_2,\ 2^j\in S_2} 2^j, \tag{3.6}$$

where Ω_1 and Ω_2 are disjoint subsets of Ω with Card $\Omega_1 = k$, Card $\Omega_2 = l$.

On the other hand, let $(t_l \cdots t_1 t_0)_2$ be the binary expansion of B, where $t_0 = 1$. Then the binary expansions of $N(m)$, $N(m) - 2^{2m-2}$, $N(m) + 2^{2m-2}$ are represented as

$$\begin{aligned}
N(m) &= (t_l \ldots t_1 100 \underbrace{0\ldots 0}_{2m-2>0})_2,\\
N(m) - 2^{2m-2} &= (t_l \ldots t_1 011 \underbrace{0\ldots 0}_{2m-2>0})_2,\\
N(m) + 2^{2m-2} &= (t_l \ldots t_1 101 \underbrace{0\ldots 0}_{2m-2>0})_2.
\end{aligned}$$

Thus, let $n' \in [N(m) - 2^{2m-2}, N(m))$. Then

$$\begin{aligned}
\sigma(n') &\geq \sigma\big(N(m) - 2^{2m-2}\big) = \sigma(B) - 1 + 2\\
&> \sigma(B) = g + h.
\end{aligned} \tag{3.7}$$

Combining (3.5) and (3.7), we obtain that $n' \notin kS_1 + lS_2$.

Similarly, let $n'' \in (N(m), N(m) + 2^{2m-2}]$. Then

$$\sigma(n'') > \sigma(B) = g + h$$

implies that $n'' \notin kS_1 + lS_2$. Hence, we deduce that

$$\Gamma(k, l; m) \subset \{N(m)\}.$$

Suppose that $(k, l) = (g, h)$. We have

$$\begin{aligned}
N(m) &= 2^{2m} + 2^{2m+2} + \cdots + 2^{2m+2h-2}\\
&\quad + 2^{2m+2h-1} + 2^{2m+2h+1} + \cdots + 2^{2m+2h+2g-3}\\
&\in gS_1 + hS_2,
\end{aligned} \tag{3.8}$$

because

$$2^{2m+2i} \in S_2 \text{ for } i = 0, 1, \ldots h - 1$$

and

$$2^{2m+2h-1+2j} \in S_1 \text{ for } j = 0, 1, \ldots, g - 1.$$

Thus,

$$\Gamma(g, h; m) = \{N(m)\}$$

We now consider the case where $(k, l) \neq (g, h)$. Suppose that

$$\Gamma(k, l; m) = \{N(m)\}.$$

Using (3.5) and $N(m) \in kS_1 + lS_2$, we get

$$\sigma(N(m)) = g + h \leq k + l.$$

The maximality of (g, h) implies that $k + l = g + h$. We apply (3.6) to (3.8). Using

$$2^{2m+2h+2j-3} \notin S_2,$$

we get

$$2^{2m+2h+2j-3} \in \Omega_1 \subset S_1$$

for any $j = 1, 2, \ldots, g$. Hence, we obtain $k = \operatorname{Card} \Omega_1 \geq g$, which contradicts the maximality of (g, h) with respect to $\succ$. Therefore, we deduce that $\Gamma(k, l; m) = \emptyset$. □

For any positive real number ξ with base-b expansion

$$\xi = \sum_{n=-R}^{\infty} t_n b^{-n},$$

where we do not use the infinite word $(b-1)(b-1)\ldots$ for the base-b expansion of ξ. We set

$$\xi(m) = \sum_{n=N(m)-2^{2m-2}}^{N(m)} t_n b^{-n}.$$

Put

$$\psi_{1,m} := \sum_{\mathbf{k}=(k,l)\in\Lambda_+} A_{\mathbf{k}} \sum_{\substack{n\in kS_1+lS_2 \\ n\leq N(m)}} \rho(k, l; n) b^{-n}$$

and

$$\psi_{2,m} := \sum_{\mathbf{k}=(k,l)\in\Lambda_-} |A_{\mathbf{k}}| \sum_{\substack{n\in kS_1+lS_2 \\ n\leq N(m)}} \rho(k, l; n) b^{-n}.$$

LEMMA 3.2. *Let m be any sufficiently large integer. Then*

$$\eta_i(m) = \psi_{i,m}(m)$$

for $i = 1, 2$.

Proof. Let $i \in \{1, 2\}$. We only have to show that

$$\sum_{\mathbf{k}=(k,l)\in\Theta(i)} |A_{\mathbf{k}}| \sum_{\substack{n\in kS_1+lS_2 \\ n>N(m)}} \rho(k,l;n)b^{-n} < b^{-N(m)}$$

for any sufficiently large m, where $\Theta(i) = \Lambda_+$ if $i = 1$ and $\Theta(i) = \Lambda_-$ if $i = 2$. Put

$$N(m) + 2^{2m-2} = \left(1 + \frac{1}{4B}\right) N(m) =: (1+\tau)N(m),$$

where τ is a positive constant independent of m. Combining (3.1) and Lemma 3.1, we get

$$\begin{aligned} \sum_{\mathbf{k}=(k,l)\in\Theta(i)} |A_{\mathbf{k}}| \sum_{\substack{n\in kS_1+lS_2 \\ n>N(m)}} \rho(k,l;n)b^{-n} &= \sum_{\mathbf{k}=(k,l)\in\Theta(i)} |A_{\mathbf{k}}| \sum_{\substack{n\in kS_1+lS_2 \\ n>(1+\tau)N(m)}} \rho(k,l;n)b^{-n} \\ &\le \sum_{\mathbf{k}=(k,l)\in\Theta(i)} |A_{\mathbf{k}}| \sum_{n>(1+\tau)N(m)} n^{k+l}b^{-n} \\ &\le C\big((1+\tau)N(m)\big)^{k+l} b^{-(1+\tau)N(m)} \\ &< b^{-N(m)} \end{aligned}$$

for all sufficiently large m, where C is a positive constant independent of m. □

Using Lemmas 3.1 and 3.2, we calculate $\eta_1(m)$ and $\eta_2(m)$. Note that $A_{\mathbf{g}}2^{-N(m)}$ causes carry in the binary expansion of η_1. Thus, for any sufficiently large m, we obtain

$$\eta_1(m) = \psi_{1,m}(m) = A_{\mathbf{g}}2^{-N(m)}$$

and

$$\eta_2(m) = \psi_{2,m}(m) = 0.$$

In particular, we deduce that $\eta_1 \neq \eta_2$ and that ξ_1, ξ_2 are algebraically independent. □

References

[1] D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, On the binary expansions of algebraic numbers, J. Théor. Nombres Bordeaux **16** (2004), 487-518.

[2] É. Borel, Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaîne, C. R. Acad. Sci. Paris **230** (1950), 591-593.

[3] P. Corvaja and U. Zannier, Some new applications of the subspace theorem. Compositio Math. **131** (2002), 319-340.

[4] A. Durand, Indépendance algébrique de nombres complexes et critère de transcendance. Compositio Math. **35** (1977), 259–267.

[5] H. Kaneko, Algebraic independence of real numbers with low density of nonzero digits, to appear in Acta Arith.

[6] A. J. Kempner, On Transcendental Numbers, Trans. Amer. Math. Soc. **17** (1916), 476—482.

[7] M. J. Knight, An "ocean of zeros" proof that a certain non-Liouville number is transcendental, Amer. Math. Monthly **98** (1991), 947-949.

[8] J. Liouville, Remarques relatives 1° à des classes très-étendues de quantités dont la valeur n'est ni rationnelle ni même réducible à des irrationnelles algébri-ques; 2° à un passage du livre des Principes où Newton calcule l'action exercée par une sphère sur un point extérieur, C. R. Acad. Sci. Paris **18** (1844), 883–885.

[9] J. Liouville, Nouvelle démonstration d'un théorème sur les irrationnelles algébriques, C. R. Acad. Sci. Paris **18** (1844), 910–911.

[10] K. Mahler, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, Math. Ann. **101** (1929), 342-366.

[11] K. Nishioka, Algebraic independence by Mahler's method and S-unit equations, Compositio Math. **92** (1994), 87-110.

[12] K. Nishioka, Mahler Functions and Transcendence, in: Lecture Notes in Mathematics, Vol. 1631, Springer, Berlin, 1996.

[13] W. M. Schmidt, Simultaneous approximation and algebraic independence of numbers, Bull. Amer. Math. Soc. **68** (1962), 475-478.

[14] I. Shiokawa, Algebraic independence of certain gap series, Arch. Math. **38** (1982), 438-442.

[15] T. Tanaka, Algebraic independence of the values of power series generated by linear recurrences, Acta Arith. **74** (1996), 177-190.