数理解析研究所講究録
第 1806 巻 2012 年 210-218

A transformation formula for Maass-type Eisenstein series of two variables

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Abstract

Let $\boldsymbol{s} = (s_1, s_2)$ be complex variables, and $\boldsymbol{z} = (z_1, z_2)$ complex parameters with $(z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, where $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) denotes the upper (resp. lower) half-plane. The main object of this report is the double Eisenstein series (of two variables) $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ defined by (2.1) below, which includes (as a particular case) the non-holomorphic Eisenstein series $\zeta_{\mathbb{Z}^2}(s; z)$ (defined by (1.1)) attached to $SL(2, \mathbb{Z})$.

We first show a Fourier-type series expansion for a two variable extension of the bilateral Hurwitz zeta-function (Theorem 1), which further allows us to obtain a similar type of series expansion for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ (Theorem 2) by means of Mellin-Barnes type integrals. This eventually leads us to establish complete asymptotic expansions for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ in the descending order of $z = z_1 - z_2$ as $z \to \infty$ through the (upper-half) sector $0 < \arg z < \pi$ (Theorem 3). It can be shown from Theorem 3 certain functional properties of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ (Corollaries 1–2), as well as several closed form evaluations for specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ at some integer lattice arguments (Corollary 3).

1 Introduction

Let $s = \sigma + it$ be a complex variable and let $\mathcal{H}^+ = \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi\}$ and $\mathcal{H}^- = \{z \in \mathbb{C} \mid -\pi < \arg(z) < 0\}$ be the complex half-planes. For an arbitrary even integer k and the complex parameter $z \in \mathcal{H}^+$, the non-holomorphic Eisenstein series $E_k(s; z)$ of weight k attached to $SL(2, \mathbb{Z})$ is defined by the meromorphic continuation of the series

$$E_k(s, z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \mathrm{GCD}(c,d)=1}} (cz + d)^{-k} |cz + d|^{-2s} \tag{1.1}$$

to the whole s-plane (see [10, Chap.4, Sect.3]). It is readily seen when $k = 0$ that the relation

$$E_0(s; z) = \zeta_{\mathbb{Z}^2}(s; z)/2\zeta(2s) \tag{1.2}$$

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holds with the Riemann zeta-function $\zeta(s)$, and the Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s;z)$ defined by

$$\zeta_{\mathbb{Z}^2}(s;z)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}|m+nz|^{-2s}\qquad(\mathrm{Re}\,s>1),\tag{1.3}$$

which can be continued to a meromorphic function over the whole s-plane (cf. [1][2]).

The principal aim of this report is to introduce a class of double Eisenstein series (2.1) below, which includes (as a particular case) the Epstein zeta-function (1.3) and so the non-holomorphic (or real analytic) Eisenstein series (1.1), and further to present a Fourier-type series expansion for the double Eisenstein series (of two variables) $\widetilde{\zeta_{\mathbb{Z}^2}}(s;\boldsymbol{z})$ defined by (2.1) below (Theorem 2). This eventually leads us to obtain complete asymptotic expansions of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;\boldsymbol{z})$ in the descending order of $z=z_1-z_2$ as $z\to\infty$ through the (upper-half) sector $0<\arg z<\pi$ (Theorem 3). Certain functional properties of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;\boldsymbol{z})$ (Corollaries 1–2), as well as several closed form evaluations for specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;\boldsymbol{z})$ at some integer lattice arguments will be presented (Corollary 3).

We give in what follows a brief overview of several results relevant to the present direction of research. As to *holomorphic* Eisenstein series, complete asymptotic expansions (with respect to the parameter z) were obtained by Matsumoto in [11, Corollary 1], while Noda [12] studied an asymptotic formula for the *non-holomorphic* Eisenstein series $E_0(s;z)$ as $t\to+\infty$ on the critical line $\sigma=1/2$. Katsurada [4] derived complete asymptotic expansions for (1.3) in the descending order of $y=\mathrm{Im}\,z$ as $y\to+\infty$, where key rôles in the proofs were played by Mellin-Barnes type integrals. This result further allows him to yield for $\zeta_{\mathbb{Z}^2}(s;z)$ a new proof of its functional equation and its Kronecker limit formula when $s\to1$, as well as its closed form evaluations of certain specific values at integer arguments. The main formula in [4, Theorem 1] is readily switched to an asymptotic expansion of $E_0(s;z)$ as $y\to+\infty$ by the relation (1.2). It is in fact possible to transfer from $E_0(s;z)$ to $E_k(s;z)$ by using Maass' weight change operators (see [10, Chap.4,(12),(13)]); this leads the authors to establish in [5, Theorem 1] complete asymptotic expansions for (1.1) as $y\to+\infty$ with any even weight k. The main formula in [5, Theorem 1] yields various consequences similar to those in the case of (1.2).

The classical Lipschitz formula (see (2.4) below) was recently extended in [6][7] into a form of two variables, where the pair of parameters (z_1,z_z) belongs to either $(\mathcal{H}^+)^2$ or $(\mathcal{H}^-)^2$. As an application, they further derived a transformation formula for a class of double Eisenstein series, which can be regarded as a two variable analogue of the Fourier series expansion of the *holomorphic* Eisenstein series attached to $SL(2,\mathbb{Z})$.

The class of double Eisenstein series such as in (2.1) can be regarded as one of those of double *bilateral* Dirichlet series. Double and further multiple Dirichlet series have been the subject of recent extensive research, where its major portion covers multiple *unilateral* Dirichlet series. We mention here several results relevant to (2.1). The functional equations of some Euler-type double Eisenstein series were recently given in [8], while Pasles and Pribitkin [13] have shown a kind of triple and quadruple analogues of the Lipschitz formula, which give generalizations of the Maass-type formula. They also applied their results to study a class of generalized non-analytic automorphic forms.

2 Statement of results

Let $\boldsymbol{s} = (s_1, s_2)$ be complex variables, and $\boldsymbol{z} = (z_1, z_2)$ be complex parameters with $\boldsymbol{z} = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$. Throughout this report, the notation $\langle \boldsymbol{s} \rangle = s_1 + s_2$ will be used. We define the double Eisenstein series of Maass-type by

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z}) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m + nz_1)^{-s_1}(m + nz_2)^{-s_2}. \tag{2.1}$$

Here the argument of $(m + nz)$ for $z \in \mathcal{H}^+ \cup \mathcal{H}^-$ is chosen so as $|\arg(m + nz)| \le \pi$. More precisely, for negative integer n in $(m + nz)$, we let $\arg(n) = -\pi$ for $z \in \mathcal{H}^+$ and $\arg(n) = \pi$ for $z \in \mathcal{H}^-$. Similarly, for negative integer m in $(m + nz)$ when $n = 0$, we define $\arg(m) = -\pi$ for $z \in \mathcal{H}^+$ and $\arg(m) = \pi$ for $z \in \mathcal{H}^-$. The argument of $m \in \mathbb{Z}$ in $(m + w)$ for $w \in \mathcal{H}^+ \cup \mathcal{H}^-$ is defined as follows:

$$\arg(m) = \lim_{\delta\to+0} \arg(m + \delta w) = \begin{cases} 0 & \text{if } m > 0, \\ \pi & \text{if } m < 0 \text{ and } w \in \mathcal{H}^+, \\ -\pi & \text{if } m < 0 \text{ and } w \in \mathcal{H}^-. \end{cases} \tag{2.2}$$

The complex power is defined by $w^s = \exp\{(\sigma + it)(\log |w| + i \arg(w))\}$. The right-hand side of (2.1) converges absolutely and locally uniformly for $\mathrm{Re}\langle \boldsymbol{s} \rangle > 2$.

As usual, $\Gamma(s)$ and $\zeta(s)$ denote the gamma and the Riemann zeta function respectively. We write $\sigma_s(l) = \sum_{0<d|l} d^s$ and use the notation $e(z) = e^{2\pi i z}$. Let $U(\alpha, \gamma; Z)$ be the confluent hypergeometric function of the second kind, defined by

$$U(\alpha, \gamma; Z) = \frac{1}{\Gamma(s_1)} \int_0^\infty e^{-Zu} u^{\alpha-1}(1 + u)^{\gamma-\alpha-1} du$$

for $\mathrm{Re}(\alpha) > 0$ and $|\arg(Z)| < \pi/2$ (see [3, 6.5.(2)]). Further, we let $\zeta_{\mathbb{Z}}(s; z)$ be the bilateral Hurwitz zeta-function defined by

$$\zeta_{\mathbb{Z}}(s; z) = \sum_{m=-\infty}^{\infty} (m + z)^{-s} \qquad (\mathrm{Re}\, s > 1), \tag{2.3}$$

for $z \in \mathcal{H}^+$ or $z \in \mathcal{H}^-$. Then the Lipschitz formula (cf. [6, Proposition 2])

$$\zeta_{\mathbb{Z}}(s; z) = \begin{cases} \dfrac{(-2\pi i)^s}{\Gamma(s)} \displaystyle\sum_{l=1}^{\infty} l^{s-1} e(lz) & \text{if } z \in \mathcal{H}^+, \\ \dfrac{(2\pi i)^s}{\Gamma(s)} \displaystyle\sum_{l=1}^{\infty} l^{s-1} e(-lz) & \text{if } z \in \mathcal{H}^-, \end{cases} \tag{2.4}$$

holds. Here the l-sum on the right side converges absolutely for all complex s, and hence (2.4) provides the holomorphic continuation of $\zeta_{\mathbb{Z}}(s; z)$ to the whole s-plane. This includes the classical Lipschitz formula, which asserts that

$$\sum_{m=-\infty}^{\infty} (z + m)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} \exp(2\pi i n z) \qquad (z \in \mathcal{H}^+),$$

for any integer $k \geq 2$ (cf. [9]). We next define the bilateral Hurwitz zeta-function of two variables, $\widetilde{\zeta_{\mathbb{Z}}}(\boldsymbol{s};\boldsymbol{z})$ for complex variables $\boldsymbol{s} = (s_1, s_2)$ and complex parameters $\boldsymbol{z} = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, by

$$\widetilde{\zeta_{\mathbb{Z}}}(\boldsymbol{s};\boldsymbol{z}) = \sum_{m=-\infty}^{\infty} (m+z_1)^{-s_1}(m+z_2)^{-s_2} \qquad (\mathrm{Re}\langle \boldsymbol{s}\rangle > 1). \tag{2.5}$$

In this paper, we first extend the Lipschitz formula (2.4) into a form of two variables.

Theorem 1. *Let* $\boldsymbol{z} = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, *and define*

$$a_l(\boldsymbol{s};\boldsymbol{z}) = \begin{cases} \Gamma(s_2)e(lz_1)U(s_2, \langle \boldsymbol{s}\rangle; 2\pi i l(z_2 - z_1)) & \text{if } l > 0, \\ \Gamma(s_1)e(lz_2)U(s_1, \langle \boldsymbol{s}\rangle; 2\pi i|l|(z_2 - z_1)) & \text{if } l < 0. \end{cases} \tag{2.6}$$

Then the formula

$$\begin{aligned}\widetilde{\zeta_{\mathbb{Z}}}(\boldsymbol{s};\boldsymbol{z}) &= 2\pi i^{2s_2-1}\frac{\Gamma(\langle \boldsymbol{s}\rangle - 1)}{\Gamma(s_1)\Gamma(s_2)}(z_1 - z_2)^{1-\langle \boldsymbol{s}\rangle} \\ &\quad + \frac{(2\pi)^{\langle \boldsymbol{s}\rangle} i^{s_2-s_1}}{\Gamma(s_1)\Gamma(s_2)}\sum_{l\neq 0} |l|^{\langle \boldsymbol{s}\rangle-1} a_l(\boldsymbol{s};\boldsymbol{z})\end{aligned} \tag{2.7}$$

holds for $\mathrm{Re}\langle \boldsymbol{s}\rangle > 1$. *Here the l-sum on the right side converges absolutely for all* $\boldsymbol{s} \in \mathbb{C}^2$, *and hence (2.7) provides the meromorphic continuation of* $\widetilde{\zeta_{\mathbb{Z}}}(\boldsymbol{s};\boldsymbol{z})$ *to the whole s-space* $\mathbb{C}^2$.

Theorem 1 yields the following transformation formula (Fourier-type expansion) for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$.

Theorem 2. *Let* $\boldsymbol{z} = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, *define*

$$\begin{aligned}\mathcal{E}_0(\boldsymbol{s};\boldsymbol{z}) = \{1 + e^{\pi i(s_1 - s_2)}\}\Big\{&\zeta(\langle \boldsymbol{s}\rangle) + 2\pi i^{2s_2-1}\frac{\Gamma(\langle \boldsymbol{s}\rangle - 1)}{\Gamma(s_1)\Gamma(s_2)} \\ &\times (z_1 - z_2)^{1-\langle \boldsymbol{s}\rangle}\zeta(\langle \boldsymbol{s}\rangle - 1)\Big\},\end{aligned} \tag{2.8}$$

and let $a_l(\boldsymbol{s};\boldsymbol{z})$ *be as in Theorem 1. Then the formula*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z}) = \mathcal{E}_0(\boldsymbol{s};\boldsymbol{z}) + \frac{(2\pi)^{\langle \boldsymbol{s}\rangle} i^{s_2-s_1}}{\Gamma(s_1)\Gamma(s_2)}\{1 + e^{\pi i(s_1-s_2)}\}\sum_{l\neq 0}\sigma_{\langle \boldsymbol{s}\rangle-1}(l)a_l(\boldsymbol{s};\boldsymbol{z}) \tag{2.9}$$

holds. Here the l-sum on the right side converges for all $\boldsymbol{s} \in \mathbb{C}^2$, *and hence (2.9) provides the meromorphic continuation of* $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ *to the whole s-space* $\mathbb{C}^2$.

For $s_j \in \mathbb{C}$, we write $s_j = \sigma_j + it_j$ $(j = 1, 2)$. Further let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n be the shifted factorial of s, and

$$\Phi_{r,s}(e(w)) = \sum_{h,k=1}^{\infty} h^r k^s e(hkw) = \sum_{l=1}^{\infty} \sigma_{r-s}(l) l^s e(lw), \tag{2.10}$$

the function first introduced by Ramanujan [14]; this converges absolutely for all $(r, s) \in \mathbb{C}^2$ when $w \in \mathcal{H}^+$, and defines there an entire function.

Theorem 3. *Let $\boldsymbol{z} = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$ and $z_1 - z_2 = z \in \mathcal{H}^+$. Then for any integer $N_1 \geq 1$ and $N_2 \geq 1$, the asymptotic expansion*

$$\begin{aligned}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z}) = \mathcal{E}_0(\boldsymbol{s}; \boldsymbol{z}) &+ \frac{2(2\pi)^{\langle \boldsymbol{s} \rangle}}{\Gamma(s_1)} \cos\{\pi(s_1 - s_2)/2\} \left\{S_{1,N_1}(\boldsymbol{s}; \boldsymbol{z}) + R_{1,N_1}(\boldsymbol{s}; \boldsymbol{z})\right\} \\ &+ \frac{2(2\pi)^{\langle \boldsymbol{s} \rangle}}{\Gamma(s_2)} \cos\{\pi(s_1 - s_2)/2\} \left\{S_{2,N_2}(\boldsymbol{s}; \boldsymbol{z}) + R_{2,N_2}(\boldsymbol{s}; \boldsymbol{z})\right\},\end{aligned} \tag{2.11}$$

holds in the region of $\boldsymbol{s}$ with $-N_2 < \sigma_1 < 1 + N_1$ and $-N_1 < \sigma_2 < 1 + N_2$. Here $\mathcal{E}_0(\boldsymbol{s}, \boldsymbol{z})$ is defined as in Theorem 2,

$$\begin{aligned}&S_{1,N_1}(\boldsymbol{s}; \boldsymbol{z}) \\ &= \sum_{n=0}^{N_1-1} \frac{(-1)^n (s_2)_n (1-s_1)_n}{n!} \Phi_{s_1-n-1,-s_2-n}(e(z_1)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-s_2-n},\end{aligned} \tag{2.12}$$

$$\begin{aligned}&S_{2,N_2}(\boldsymbol{s}; \boldsymbol{z}) \\ &= \sum_{n=0}^{N_2-1} \frac{(-1)^n (s_1)_n (1-s_2)_n}{n!} \Phi_{s_2-n-1,-s_1-n}(e(-z_2)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-s_1-n},\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}R_{1,N_1}(\boldsymbol{s}; \boldsymbol{z}) &= \frac{(-1)^{N_1} (s_2)_{N_1} (1-s_1)_{N_1}}{(N_1 - 1)!} \sum_{h,k=1}^{\infty} e(hkz_1) h^{\langle \boldsymbol{s} \rangle - 1} \\ &\times \int_0^1 \xi^{-s_2-N_1} (1-\xi)^{N_1-1} U(s_2 + N_1, \langle \boldsymbol{s} \rangle; 2\pi hke^{-\pi i/2} z/\xi) d\xi,\end{aligned} \tag{2.14}$$

$$\begin{aligned}R_{2,N_2}(\boldsymbol{s}; \boldsymbol{z}) &= \frac{(-1)^{N_2} (s_1)_{N_2} (1-s_2)_{N_2}}{(N_2 - 1)!} \sum_{h,k=1}^{\infty} e(-hkz_2) h^{\langle \boldsymbol{s} \rangle - 1} \\ &\times \int_0^1 \xi^{-s_1-N_2} (1-\xi)^{N_2-1} U(s_1 + N_2, \langle \boldsymbol{s} \rangle; 2\pi hke^{-\pi i/2} z/\xi) d\xi.\end{aligned} \tag{2.15}$$

Let $\theta = \arg(-iz)$, then the expansion above gives the asymptotic series in the descending order of z, and R_{j,N_j} $(j = 1, 2)$ is the remainder terms satisfying the estimates

$$R_{1,N_1}(\boldsymbol{s}; \boldsymbol{z}) = O\left(e^{-2\pi \mathrm{Im}(z_1)} |z|^{-\sigma_2 - N_1}\right), \tag{2.16}$$

$$R_{2,N_2}(\boldsymbol{s}; \boldsymbol{z}) = O\left(e^{2\pi \mathrm{Im}(z_2)} |z|^{-\sigma_1 - N_2}\right), \tag{2.17}$$

as $z \to \infty$ through the sector $\delta \le \arg z \le \pi - \delta$ with any small $\delta > 0$. Here the O-constants depend on N_j, t_j $(j = 1, 2)$ and θ.

From Theorems 2 and 3, we can determine the singularities of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$, and to derive the functional equations under some natural conditions. In order to describe our results, we put for any $m \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{M}_m &= \{\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2;\ s_1 - s_2 = m\}, \\ \mathcal{N}_m &= \{\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2;\ \langle \boldsymbol{s} \rangle = s_1 + s_2 = m\}, \\ \mathcal{P} &= \{\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2;\ \langle \boldsymbol{s} \rangle \in \{2, 1, 0, -2, -4, \cdots\}\}, \\ \mathcal{Q} &= \{\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2; s_1 \in \{0, -1, -2, \cdots\} \text{ or } s_2 \in \{0, -1, -2, \cdots\}\}. \end{aligned}$$

Generally, it is possible to explain that the singularities (and a part of zeros) and the functional equation come from $\mathcal{E}_0(\boldsymbol{s}; z)$, which is called "the constant term" of the Eisenstein series. In our case, this fact is stated as follows. :

Corollary 1. *(i) For any $l \in \mathbb{Z}$ and any $\boldsymbol{s}^* = (s_1^*, s_2^*) \in \mathcal{M}_{2l+1}$ with $\boldsymbol{s}^* \notin \mathcal{P}$,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}^*; \boldsymbol{z}) = 0,$$

namely, $\mathcal{M}_{2l+1} \setminus \mathcal{P}$ is the set of the zeros of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$.

(ii) $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ has singularities on $\boldsymbol{s} \in \mathcal{P}$. In particular, $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ has indeterminacy singularities on $\boldsymbol{s} \in \mathcal{P} \cap \mathcal{Q}$ or on $\boldsymbol{s} \in \mathcal{P} \cap \mathcal{M}_{2l+1}$ for any $l \in \mathbb{Z}$.

The functional equation of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ come into existence under the condition $\boldsymbol{s} \in \mathcal{M}_{2k}$ $(k \in \mathbb{Z})$, which seems to be a natural generalization of the functional equation of the non-holomorphic Eisenstein series $E_k(s, z)$ attached to $SL(2, \mathbb{Z})$ (see Corollary 5 below). It is also possible to obtain a functional equation under a different kind of the condition that $\boldsymbol{s} \in \mathcal{N}_{2l+1}$ $(l \in \mathbb{Z})$ upon subtracting the constant term $\mathcal{E}_0(\boldsymbol{s}; z)$ of the double Eisenstein series. In the following, we use the notation $\widehat{\boldsymbol{1} - \boldsymbol{s}} = (1 - s_2, 1 - s_1)$ for $\boldsymbol{s} = (s_1, s_2)$.

Corollary 2. *(i) The functional equation*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\widehat{\boldsymbol{1} - \boldsymbol{s}}; \boldsymbol{z}) = \left(\frac{2\pi i}{z}\right)^{1 - \langle \boldsymbol{s} \rangle} \frac{\Gamma(s_1)}{\Gamma(1 - s_2)} \widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$$

holds on the hyper-plane $\mathcal{M}_{2k}$ $(k \in \mathbb{Z})$ except on the poles.

(ii) The functional equation

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\widehat{\boldsymbol{1} - \boldsymbol{s}}; \boldsymbol{z}) - \mathcal{E}_0(\widehat{\boldsymbol{1} - \boldsymbol{s}}; \boldsymbol{z}) = \left(\frac{2\pi i}{z}\right)^{1 - \langle \boldsymbol{s} \rangle} \frac{\Gamma(s_1)}{\Gamma(1 - s_2)} \left\{\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z}) - \mathcal{E}_0(\boldsymbol{s}; \boldsymbol{z})\right\}$$

holds on the hyper-planes $\mathcal{M}_{2k}$ $(k \in \mathbb{Z})$ or $\mathcal{N}_{2l+1}$ $(l \in \mathbb{Z})$ except on the poles.

From Theorem 3, we obtain the following closed form expressions for specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}, \boldsymbol{z})$ at positive integer lattice arguments:

Corollary 3. *For any* $\boldsymbol{m} = (m_1, m_2) \in \mathbb{N}^2$ $(\mathbb{N} = \{1, 2, \cdots\})$ *with* $\langle \boldsymbol{m} \rangle > 2$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m}; \boldsymbol{z}) = \mathcal{E}_0(\boldsymbol{m}; \boldsymbol{z}) + 2(2\pi)^{\langle \boldsymbol{m} \rangle} \cos\{\pi(m_1 - m_2)/2\} \left\{ \frac{T_1(\boldsymbol{m}; \boldsymbol{z})}{(m_1 - 1)!} + \frac{T_2(\boldsymbol{m}; \boldsymbol{z})}{(m_2 - 1)!} \right\}. \tag{2.18}$$

Here

$$\begin{aligned} \mathcal{E}_0(\boldsymbol{m}; \boldsymbol{z}) &= -\frac{(2\pi)^{\langle \boldsymbol{m} \rangle} i^{m_1 - m_2} \cos\{\pi(m_1 - m_2)/2\} B_{\langle \boldsymbol{m} \rangle}}{\cos(\pi \langle \boldsymbol{m} \rangle / 2) \langle \boldsymbol{m} \rangle !} \\ &+ \frac{4\pi i^{\langle \boldsymbol{m} \rangle - 1}(\langle \boldsymbol{m} \rangle - 2)! \cos\{\pi(m_1 - m_2)/2\} \zeta(\langle \boldsymbol{m} \rangle - 1)}{(m_1 - 1)!(m_2 - 1)!(z_1 - z_2)^{\langle \boldsymbol{m} \rangle - 1}}, \end{aligned} \tag{2.19}$$

where B_m *is the* m*-th Bernoulli number and*

$$T_1(\boldsymbol{m}; \boldsymbol{z}) = \sum_{n=0}^{m_1 - 1} \binom{m_1 - 1}{n} (m_2)_n \Phi_{m_1 - n - 1, -m_2 - n}(e(z_1)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-m_2 - n}, \tag{2.20}$$

$$T_2(\boldsymbol{m}; \boldsymbol{z}) = \sum_{n=0}^{m_2 - 1} \binom{m_2 - 1}{n} (m_1)_n \Phi_{m_2 - n - 1, -m_1 - n}(e(-z_2)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-m_1 - n}. \tag{2.21}$$

From Theorems 1 and 2, the following two corollaries are readily derived.

Corollary 4 (Maass [10]). *Let* $z = x + iy \in \mathcal{H}^+$, *and define*

$$a_n(y; s_1, s_2) = i^{s_2 - s_1} (2\pi)^{\langle \boldsymbol{s} \rangle} \times \begin{cases} n^{\langle \boldsymbol{s} \rangle - 1} \Gamma(s_1)^{-1} U(s_2, \langle \boldsymbol{s} \rangle; 4\pi n y) & \text{if } n > 0, \\ |n|^{\langle \boldsymbol{s} \rangle - 1} \Gamma(s_2)^{-1} U(s_1, \langle \boldsymbol{s} \rangle; 4\pi |n| y) & \text{if } n < 0, \\ \Gamma(s_1)^{-1} \Gamma(s_2)^{-1} \Gamma(\langle \boldsymbol{s} \rangle - 1)(4\pi y)^{1 - \langle \boldsymbol{s} \rangle} & \text{if } n = 0. \end{cases}$$

Then the formula

$$\sum_{m=-\infty}^{\infty} (z + m)^{-s_1} (\overline{z} + m)^{-s_2} = \sum_{n=-\infty}^{\infty} a_n(y; s_1, s_2) e(nx + i|n|y)$$

holds, where the n*-sum (with* $n \neq 0$*) on the right side converges for all* $(s_1, s_2) \in \mathbb{C}^2$, *and this formula provides the meromorphic continuation of the left side to the whole* (s_1, s_2)*-space* $\mathbb{C}^2$.

Corollary 5 (Maass [10]). *Under the same notation as in Corollary 4, we have*

$$\begin{aligned} E_k(s, z) &= 1 + \frac{\zeta(k + 2s - 1)}{\zeta(k + 2s)} a_0(y; k + s, s) \\ &+ \frac{1}{\zeta(k + 2s)} \sum_{l \neq 0} \sigma_{1-k-2s}(l) a_l(y; k + s, s) e(lx + i|l|y), \end{aligned}$$

and the functional equation

$$\pi^{-s}\Gamma(s)\zeta(2s)y^s E_k(s,z)$$
$$= \pi^{-1+s+k}\Gamma(1-s-k)\zeta(2-2s-2k)y^{1-s-k}E_k(1-s-k,z).$$

Remark. The double Lipschitz formula and the Fourier series expansion of non-holomorphic Eisenstein series were shown by Maass [10], in which he described his results in terms of Whittaker functions. Equivalent statements described in terms of confluent hypergeometric functions can be found, for e.g., in [15, p. 132].

3 Acknowledgment

The author was supported by Grants-in-Aid for Scientific Research (No. 23540032), Japan Society for the Promotion of Science (JSPS) and the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

[1] S. Chowla and A. Selberg, *On Epstein's zeta-function* (I), Proc Nat. Acad. Sci. USA **35** (1949), 371–374.

[2] S. Chowla and A. Selberg, *On Epstein's zeta-function*, J. Reine Angew. Math. **227** (1967), 86–110.

[3] A. Erdélyi, et al., *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[4] M. Katsurada, *Complete asymptotic expansions associated with Epstein zeta-functions*, Ramanujan J. (2007), 14, 249-275

[5] M. Katsurada and T. Noda, *Differential actions on the asymptotic expansions of non-holomorphic Eisenstein series*, International Journal of Number Theory, , 6, Vol. 5 1061-1088, (2009).

[6] M. Katsurada and T. Noda, *On generalized Lipschitz formulae and applications*, Proceedings of Diophantine Analysis and Related Fields 2010, AIP Conference Proceedings 1264, A.I.P. Press, September 129–138 (2010).

[7] M. Katsurada and T. Noda, *Transformation formulas for a double Eisenstein series and applications*, preprint

[8] Y. Komori, K. Matsumoto and H. Tsumura *Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type*, Publ. Math. Debrecen 77/1-2 (2010), 15-31.

[9] R. Lipschitz, *Untersuchung der Eigenschaften einer Gattung von unendlichen*, J. Reine und Angew. Math. CV, 127–156 (1889).

[10] H. Maass, *Lectures on Modular Functions of One Complex Variable*, Tata Institute of Fundamental Research, Bombay, 1964.

[11] K. Matsumoto, *Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series*, Nagoya Math. J. 172, 2003, 59-102

[12] T. Noda, *Asymptotic expansions of the non-holomorphic Eisenstein series*, R.I.M.S. Kôkyûroku, No. 1319 (2003), 29–32.

[13] P. C. Pasles and W. A. Pribitkin, *A generalization of the Lipschitz summation formula and some applications*, Proc. A.M.S. Vol. 129, 3177–3184 (2001).

[14] S. Ramanujan *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. Vol. 22, 159–184 (1916)

[15] G. Shimura, *Elementary Dirichlet series and modular forms*, Springer, 2007.