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Stability, Bifurcation and Classification of Minimal Sets in Random Complex Dynamics

Hiroki Sumi
Department of Mathematics, Osaka University,
1-1, Machikaneyama, Toyonaka, Osaka, 560-0043, Japan
E-mail: sumi@math.sci.osaka-u.ac.jp
http://www.math.sci.osaka-u.ac.jp/~sumi/

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Since nature has many random terms, it is natural and important to investigate random dynamical systems. Many physicists are investigating "noise-induced phenomena" (new phenomena caused by noise and randomness, e.g. [1]) in random dynamical systems. Regarding the dynamics of a rational map h with $\deg(h)\geq 2$ on the Riemann sphere $\hat{\mathbb{C}}$, we always have the **chaotic part** in $\hat{\mathbb{C}}$. **However**, we show that in the (i.i.d.) random dynamics of polynomials on $\hat{\mathbb{C}}$, **generically**, (1) **the chaos of the averaged system disappears**, due to the automatic cooperation of many kinds of maps in the system (**cooperation principle**), and (2) the limit states are **stable** under perturbations of the system.

Moreover, we investigate the **bifurcation** of 1-parameter families of random complex dynamical systems.

Definition 1.

(1) *We denote by* $\hat{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ *the Riemann sphere and denote by* d *the spherical distance on* $\hat{\mathbb{C}}$.

(2) *We set* $\mathrm{Rat}:=\{h:\hat{\mathbb{C}}\to\hat{\mathbb{C}}\mid h$ *is a non-const. rational map*$\}$ *endowed with the distance* η *defined by* $\eta(f,g):=\sup_{z\in\hat{\mathbb{C}}}d(f(z),g(z))$.
We set $\mathrm{Rat}_{+}:=\{h\in\mathrm{Rat}\mid\deg(h)\geq 2\}$.

(3) *We set* $\mathcal{P}:=\{h:\hat{\mathbb{C}}\to\hat{\mathbb{C}}\mid h$ *is a polynomial map*$,\deg(h)\geq 2\}$ *endowed with the relative topology from* Rat.

(4) *For a metric space* X, *we denote by* $\mathfrak{M}_1(X)$ *the space of all Borel probability measures on* X. *We set*

$$\mathfrak{M}_{1,c}(X):=\{\tau\in\mathfrak{M}_1(X)\mid \operatorname{supp}\tau \text{ is compact}\},$$

where $\operatorname{supp}\tau$ *denotes the topological support of* τ.

From now on, we take a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *and we consider the* **(i.i.d.) random dynamics** *on* $\hat{\mathbb{C}}$ *such that at every step we choose a map* $h \in \mathrm{Rat}$ *according to* τ. *This determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space* $\hat{\mathbb{C}}$ *such that for each* $x \in \hat{\mathbb{C}}$ *and for each Borel measurable subset* A *of* $\hat{\mathbb{C}}$, *the transition probability* $p(x, A)$ *from* x *to* A *is defined as*

$$p(x, A) = \tau(\{h \in \mathrm{Rat} \mid h(x) \in A\}).$$

(5) *Note that* Rat *and* $\mathcal{P}$ *are* **semigroups** *where the semigroup operation is functional composition. A subsemigroup of* Rat *is called a* **rational semigroup**. *A subsemigroup* G *of* $\mathcal{P}$ *is called a* **polynomial semigroup**.

(6) *For a rational semigroup* G, *we set*

$$F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } G \text{ is equicontinuous on } U\}.$$

This $F(G)$ *is called the* **Fatou set** *of* G. *Moreover, we set*

$$J(G) := \hat{\mathbb{C}} \setminus F(G).$$

This $J(G)$ *is called the* **Julia set** *of* G.

(7) (**Key**) *For a rational semigroup* G, *we set*

$$J_{\mathrm{ker}}(G) := \bigcap_{h \in G} h^{-1}(J(G)).$$

This is called the **kernel Julia set** *of* G.

(8) *For a* $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, *let* G_τ *be the rational semigroup generated by* $\operatorname{supp}\tau$. *Thus* G_τ *is the set of all finite compositions of elements in* $\operatorname{supp}\tau$.

Remark: Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. If there exists an $f_0 \in \mathcal{P}$ and a non-empty open subset U of $\mathbb{C}$ s.t. $\{f_0 + c \mid c \in U\} \subset \operatorname{supp}\tau$, then $J_{\mathrm{ker}}(G_\tau) = \emptyset$. Thus, for **most** $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, $J_{\mathrm{ker}}(G_\tau) = \emptyset$.

Theorem 0.1 (Theorem A, Cooperation Principle and Disappearance of Chaos). *Let* $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$. *Suppose* $J_{\ker}(G_\tau) = \emptyset$. *Then, we have all of the following* (1)(2)(3).

(1) *We say that a non-empty compact subset* L *of* $\hat{\mathbb{C}}$ *is a* **minimal set** *of* G_τ *if* L *is minimal in*

$$\{K \subset \hat{\mathbb{C}} \mid \emptyset \neq K \text{ is compact}, \forall h \in G_\tau, h(K) \subset K\}$$

with respect to the inclusion. Moreover, we set

$$\mathrm{Min}(G_\tau) := \{L \mid L \text{ is a minimal set of } G_\tau\}.$$

Then, $1 \leq \sharp \mathrm{Min}(G_\tau) < \infty$.

(2) *For* **each** $z \in \hat{\mathbb{C}}$, *there exists a Borel subset* $\mathcal{A}_z$ *of* $(\mathrm{Rat})^{\mathbb{N}}$ *with* $(\Pi_{j=1}^{\infty}\tau)(\mathcal{A}_z) = 1$ *such that for each* $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, *the following* (a) *and* (b) *hold.*

   (a) *There exists a* $\delta = \delta(z, \gamma) > 0$ *such that* $\mathrm{diam}\gamma_n \cdots \gamma_1(B(z,\delta)) \to 0$ *as* $n \to \infty$.

   (b) $d(\gamma_n, \cdots \gamma_1(z), \bigcup_{L \in \mathrm{Min}(G_\tau)} L) \to 0$ *as* $n \to \infty$.

(3) *We set* $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi \text{ is conti.}\}$ *endowed with the sup. norm* $\|\cdot\|_\infty$. *Let* $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ *be the operator defined by*

$$M_\tau(\varphi)(z) := \int_{\mathrm{Rat}} \varphi(h(z))\, d\tau(h), \ \forall \varphi \in C(\hat{\mathbb{C}}), \forall z \in \hat{\mathbb{C}}.$$

Let $\mathcal{U}_\tau$ *be the space of all finite linear combinations of unitary eigenvectors of* $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, *where an eigenvector is said to be* **unitary** *if the absolute value of the corresponding eigenvalue is* 1.

Then, $1 \leq \dim_{\mathbb{C}} \mathcal{U}_\tau < \infty$ *and*

$$C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}.$$

Moreover, each $\varphi \in \mathcal{U}_\tau$ *is locally constant on* $F(G_\tau)$ *and is Hölder continuous on* $\hat{\mathbb{C}}$.

Remark: Theorem A describes **new phenomena** which **cannot hold in the usual iteration dynamics** of a single $h \in \mathrm{Rat}$ with $\deg(h) \geq 2$.

When $J_{\ker}(G_\tau) = \emptyset$?

Definition 2. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$. We say that τ is* **mean stable** *if there exist non-empty open subsets U, V of $F(G_\tau)$ and a number $n \in \mathbb{N}$ such that all of the following* (1)(2)(3) *hold.*

(1) $\overline{V} \subset U \subset F(G_\tau)$.

(2) *For all $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\operatorname{supp} \tau)^{\mathbb{N}}$, $(\gamma_n \circ \cdots \circ \gamma_1)(U) \subset V$.*

(3) *For all $z \in \hat{\mathbb{C}}$, there exists an $h \in G_\tau$ such that $h(z) \in U$.*

Remark: If τ is mean stable, then $J_{\mathrm{ker}}(G_\tau) = \emptyset$. Note that **the converse is NOT true** in general.

When is a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ **mean stable**?

Definition 3. *Let $\mathcal{Y}$ be a closed subset of* Rat. *Let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ if and only if*

(1) *$\int \varphi \, d\tau_n \to \int \varphi \, d\tau$ for each bounded continuous function $\varphi : \mathcal{Y} \to \mathbb{R}$, and*

(2) *$\operatorname{supp} \tau_n \to \operatorname{supp} \tau$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $\mathcal{Y}$.*

Theorem 0.2 (Theorem B). (Density of Mean Stable Systems) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau \text{ is mean stable}\}$ is* **open and dense** *in $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.*

Theorem 0.3 (Theorem C, Stability). *Suppose $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ is mean stable. Then there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ such that all of the following* (1)(2)(3) *hold.*

(1) *For each $\nu \in \Omega$, ν is mean stable, $J_{\mathrm{ker}}(G_\nu) = \emptyset$ and thus* Theorem A *for ν holds.*

(2) *The map $\nu \mapsto \mathcal{U}_\nu$ is continuous on Ω.*

(3) *The map $\nu \mapsto \sharp \mathrm{Min}(G_\nu)$ is constant on Ω.*

Theorem 0.4 (Theorem D, Bifurcation). *For each $t \in [0,1]$, let μ_t be an element of $\mathfrak{M}_{1,c}(\mathrm{Rat}_+)$. Suppose that all of the following* (1)–(4) *hold.*

(1) *$t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ is continuous on $[0,1]$.*

(2) *If* $t_1, t_2 \in [0,1]$ *and* $t_1 < t_2$, *then* $\operatorname{supp} \mu_{t_1} \subset \operatorname{int}(\operatorname{supp} \mu_{t_2})$ *with respect to the topology of* Rat_+.

(3) $\operatorname{int}(\operatorname{supp} \mu_0) \neq \emptyset$ *and* $F(G_{\mu_1}) \neq \emptyset$.

(4) $\sharp\mathrm{Min}(G_{\mu_0}) \neq \sharp\mathrm{Min}(G_{\mu_1})$.

Let $B := \{t \in [0,1] \mid \mu_t$ *is not mean stable*$\}$.
Then, we have all of the following (a)(b)(c)(d).

(a) *For each* $t \in [0,1]$, *we have* $J_{\ker}(G_{\mu_t}) = \emptyset$ *and all statements in Theorem A (with* $\tau = \mu_t$*) hold.*

(b) $1 \leq \sharp(B \cap [0,1)) \leq \sharp\mathrm{Min}(G_{\mu_0}) - \sharp\mathrm{Min}(G_{\mu_1}) < \infty$.

(c) *For each* $t \in [0,1] \setminus B$ *and for each* $L \in \mathrm{Min}(G_{\mu_t})$, L *is* **attracting** *for* G_{μ_t}, *i.e. there exist non-empty open subsets* U, V *of* $F(G_{\mu_t})$ *and a number* $n \in \mathbb{N}$ *such that*

   (i) $L \subset V \subset \overline{V} \subset U \subset F(G_{\mu_t})$, *and*

   (ii) *for each* $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\operatorname{supp} \mu_t)^{\mathbb{N}}$, $\gamma_n \cdots \gamma_1(U) \subset V$.

(d) *For each* $t \in B$, *there exists an* $L \in \mathrm{Min}(G_{\mu_t})$ *such that either*

   (i) L *is* J-**touching** *for* G_{μ_t}, *i.e.,* $L \cap J(G_{\mu_t}) \neq \emptyset$, *or*

   (ii) L *is* **sub-rotative** *for* G_{μ_t}, *i.e.,* $L \subset F(G_{\mu_t})$ *and* L *meets a Siegel disc or a Hermann ring of some element of* G_{μ_t}.

Idea of proofs of results.

Lemma 1 (Classification of Minimal Sets). *Let* $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$. *Let* $L \in \mathrm{Min}(G_\tau)$. *Then, exactly one of the following holds.*

(a) L *is attracting for* G_τ.

(b) L *is* J*-touching for* G_τ.

(c) L *is sub-rotative for* G_τ.

(For the definitions of the terms "attracting", "J*-touching" and "sub-rotative", see Theorem D with* μ_t *replaced by* τ*.)*

Outline of the proof of Theorem B: Take any $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Enlarge $\operatorname{supp}\tau$ just a little bit. Then any J-touching or sub-rotative minimal set of G_τ collapses. Now we observe that each minimal set of G_ν is attracting if and only if ν is mean stable.

Summary and Remarks: (1) Regarding the random dynamics of polynomials, **generically**, the chaos of the averaged system **disappears** and the limit states are **stable** under perturbations of the system. (2) In order to prove the above result, we need the **classification of minimal sets**. (3) We can investigate the **bifurcation** of the 1-parameter family of random complex dynamical systems. (4) There exist a lot of examples of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\ker}(G_\tau) = \emptyset$ (thus the chaos disappears) but τ is **not mean stable**. At such a τ, a kind of **bifurcation** occurs. (5) There exists an example of means stable $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp \operatorname{supp}\tau < \infty$ such that there exists a $\varphi \in \mathcal{U}_\tau$ whose Hölder exponent is strictly less than 1 ("**Devil's Coliseum**", which is the function of probability of tending to ∞. To prove this result, we use ergodic theory and potential theory). Therefore, even if the chaos disappears in the "C^0" sense, the chaos may remain in the "C^1" sense (or in the space of Hölder continuous functions with some exponent $\alpha_0 < 1$). Thus, in random dynamics, we have a kind of gradation between non-chaos and chaos. It is interesting to investigate the pointwise Hölder exponent of the above φ. The above φ is a continuous function on $\hat{\mathbb{C}}$ which varies precisely on the Julia set $J(G_\tau)$, which is a thin fractal set. Thus it is important to estimate the Hausdorff dimension $\dim_H(J(G_\tau))$ of $J(G_\tau)$. By using the thermodynamical formalisms, we can show that $\dim_H(J(G_\tau))$ is equal to the zero of the pressure function, under certain conditions. Also, in order to investigate the pointwise Hölder exponent of this function φ in detail, we can sometimes apply the thermodynamical formalisms. We are very interested in studying the pointwise-Hölder-exponent spectrum of this function $\varphi \in \mathcal{U}_\tau$.

References

[1] K. Matsumoto and I. Tsuda, *Noise-induced order*, J. Statist. Phys. 31 (1983) 87-106.

[2] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (2011) 102(1), pp 50–112.

Also available from http://arxiv.org/abs/0812.4483. (**Some of the contents of this presentation are included in this paper**.)

[3] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, preprint 2010, http://arxiv.org/abs/1008.3995. (**Some of the contents of this presentation are included in this paper**.)

[4] H. Sumi, *Random complex dynamics and devil's coliseums*, preprint 2011, http://arxiv.org/abs/1104.3640.