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DECOMPOSITIONS OF POLYHEDRAL PRODUCTS

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1. Introduction

This is an announcement of a result in the paper [IK].

Put $[n]=\{1,\ldots,n\}$. Let K be an abstract simplicial complex on the vertex set $[n]$, and let $(\underline{X},\underline{A})$ be a collection of pairs of spaces $\{(X_i,A_i)\}_{i\in[n]}$ labelled by $[n]$. For a simplex σ of K which is a (possible empty) subset of $[n]$, we put

$$(\underline{X},\underline{A})^{\sigma}=Y_1\times\cdots\times Y_n,\quad\text{where}\quad Y_i=\begin{cases}X_i & i\in\sigma\\ A_i & i\notin\sigma.\end{cases}$$

The polyhedral product, or the generalized moment angle complex, $\mathcal{Z}_K(\underline{X},\underline{A})$ is defined as the union of all $(\underline{X},\underline{A})^{\sigma}$ for all simplices σ of K. Polyhedral products are classical objects including many interesting classes of spaces. For example, (higher order) Whitehead products can be described by the natural maps between polyhedral products over boundaries of simplices. See [P]. After a work by Davis and Januszkiewicz [DJ], the special polyhedral products $\mathcal{Z}_K(\underline{D}^2,\underline{S}^1)$, called the moment angle complex of K, was focused as a source producing manifolds with torus actions which have properties analogous to toric varieties. So the homotopy types of polyhedral products appeal now to a variety of areas in mathematics.

Among other things, let us consider decompositions of polyhedral products. In [BBCG], Bahri, Bendersky, Cohen and Gitler gave a wedge decomposition of a suspension of the polyhedral complex $\mathcal{Z}_K(\underline{X},\underline{A})$, which is a simple generalization of the standard wedge decomposition

$$\Sigma(X\times Y)\simeq\Sigma X\vee\Sigma Y\vee\Sigma(X\wedge Y).$$

As a special case, they proved the following. For a simplicial complex K on the vertex set V and a subset $I\subset V$, the induced subcomplex K_I is the maximum subcomplex of K whose vertex set is I. We denote the geometric realization of K by $|K|$ and the smash product $\bigwedge_{j\in J}X_j$ by $\widehat{X}^J$.

Theorem 1.1 (Bahri, Bendersky, Cohen and Gitler [BBCG]). *Let K be a simplicial complex on the index set $[n]$, and let $(C\underline{X},\underline{X})=\{(CX_i,X_i)\}_{i\in[n]}$. Then there is a homotopy equivalence*

$$\Sigma\mathcal{Z}_K(C\underline{X},\underline{X})\simeq\Sigma\bigvee_{\emptyset\neq I\subset[n]}|K_I|*\widehat{X}^I.$$

On the other hand, Grbić and Theriault [GT] formerly proved that the decomposition in Theorem 1.1 holds without a suspension for moment angle complexes of special simplicial complexes. To state their result, let us define shifted complexes.

Definition 1.2. A simplicial complex K on a finite chain V is called shifted if for any $v < w \in V$ and any simplex $\sigma \in K$ with $v \in \sigma$, $(\sigma - v) \cup w$ is also a simplex of K.

Remark 1.3. While shifted complexes may not be familiar to topologists, they have been studied extensively in combinatorics.

Theorem 1.4 (Grbić and Theriault [GT]). *The moment angle complex of a shifted complex has the homotopy type of a wedge of spheres.*

Remark 1.5. The proof of Theorem 1.4 heavily depends on the fact that S^1 has the classifying space, and then their proof does not work for a polyhedral product of general pairs $\{(CX_i, X_i)\}_{i\in[n]}$.

Supported by Theorem 1.1 and 1.4, Bahri, Bendersky, Cohen and Gitler [BBCG] posed the following conjecture.

Conjecture 1.6 (Bahri, Bendersky, Cohen and Gitler [BBCG]). Let K be a shifted complex on the index set $[n]$, and let $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i\in[n]}$. Then there is a homotopy equivalence

$$\mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset\neq I\subset[n]} |K_I| * \widehat{X}^I.$$

In [IK], the authors gave a resolution for this conjecture.

Theorem 1.7 (Iriye and Kishimoto [IK]). *The Conjecture 1.6 is true.*

Remark 1.8. Grbić and Theriault announced at the arxiv that they solved conjecture 1.6 but their proofs include several serious mistakes.

In what follows, we will sketch ideas of the proof of Theorem 1.7. The proofs of related lemmas will be quite brief and we refer to [IK] for details.

2. Proof of Theorem 1.7

2.1. **Topology of shifted complexes.** Let us first look at the topology of shifted complexes. The connected components of shifted complexes are quite simple as follows. The proof is an easy verification and then we omit it here.

Lemma 2.1. *Let K be a shifted complex on the index set $[n]$ and let K_0 be the connected component of the maximum vertex n. Then*

$$V(K_0) = V(\mathrm{star}_K(n)) = \{n_K, n_K + 1, \dots, n\}$$

for some $n_K \in [n]$ and $K - K_0$ is discrete, where $V(L)$ means the vertex set of a simplicial complex L.

This lemma shows that we can treat separately the component of the maximum vertex and the rest discrete part.

We now describe the homotopy types of shifted complexes. For a shifted complex K with the maximum vertex v, we put

$$\overline{K} = |K|/|\mathrm{star}_K(v)|.$$

Note that since $|\mathrm{star}_K(v)|$ is contractible, $\overline{K}$ and $|K|$ have the same homotopy type. Let $\mathfrak{m}(K)$ denote the set of all maximal simplices of K which do not contain the maximum vertex v.

Lemma 2.2. *For a shifted complex K, we have*

$$\overline{K} = \bigvee_{\sigma \in \mathfrak{m}(K)} S^{\dim \sigma}.$$

Moreover, if L is a shifted subcomplex of K on the same vertex set, the inclusion $L \to K$ induces

$$1 \vee * : \overline{L} = \left(\bigvee_{\sigma \in \mathfrak{m}(L) \cap \mathfrak{m}(K)} S^{\dim \sigma} \right) \vee \left(\bigvee_{\tau \in \mathfrak{m}(L) - \mathfrak{m}(K)} S^{\dim \tau} \right)$$
$$\to \overline{K} = \left(\bigvee_{\sigma \in \mathfrak{m}(L) \cap \mathfrak{m}(K)} S^{\dim \sigma} \right) \vee \left(\bigvee_{\tau \in \mathfrak{m}(K) - \mathfrak{m}(L)} S^{\dim \tau} \right).$$

Proof. For $\sigma \in K$, it holds that $\partial \sigma \subset \mathrm{star}_K(v)$ for the maximum vertex v. Then the result follows. $\square$

2.2. **Lemmas on pushouts.** We collect two lemmas on pushouts of spaces, one is classical and well-known and the other concerns with the decompositions of pushouts.

Lemma 2.3. *Suppose there is a commutative cube*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "g"] \arrow[dd, "f_1"'] & & B \arrow[dr] \arrow[dd, "f_2" near end] & \\
& C \arrow[rr, crossing over] & & D \arrow[dd, "f_4"] \\
E \arrow[rr] \arrow[dr, "h"'] & & F \arrow[dr] & \\
& G \arrow[rr] \arrow[from=uu, crossing over, "f_3" near start] & & H
\end{tikzcd}$$

in which the top and the bottom faces are pushouts and g, h are cofibrations. If f_1, f_2, f_3 are homotopy equivalences, so is f_4.

For spaces X, Y, we put $X \ltimes Y = X \times Y / X \times *$.

Lemma 2.4. *Define Q as a pushout*

$$\begin{tikzcd} A \times (B \vee C) \arrow[r, "i\times 1"] \arrow[d, "1\times(1\vee *)"] & CA \times (B \vee C) \arrow[d] \\ A \times (B \vee D) \arrow[r] & Q, \end{tikzcd}$$

where $i : A \to CA$ is the inclusion. Then there is a homotopy equivalence $Q \xrightarrow{\simeq} B \vee (A \ltimes D) \vee (\Sigma A \wedge C)$ which is natural with respect to A, B, C, D.

Proof. One can easily get an analogous result by replacing $\times$ in the diagram with $\ltimes$. Then we obtain the result by looking at the relation between two pushouts of the diagram with $\times$ and $\ltimes$. □

2.3. **The proof of Theorem 1.7.** Fix a collection of spaces $\{X_i\}_{i\in[n]}$. For a shifted complex K on $[n]$, we define a space $\mathcal{W}_K$ as

$$\mathcal{W}_K = \bigvee_{\emptyset\neq I\subset[n]} \Sigma\overline{K_I} \wedge \widehat{X}^I.$$

Note that $\mathcal{W}_K$ and the right hand side of the homotopy equivalence in Conjecture 1.6 are homotopy equivalent. For a shifted subcomplex L of K on the same vertex set $[n]$, we also define a map $\lambda : \mathcal{W}_L \to \mathcal{W}_K$ in the obvious manner. The proof of Theorem 1.7 is done by constructing a homotopy equivalence

$$\epsilon_K : \mathcal{Z}_K(C\underline{X}, \underline{X}) \to \mathcal{W}_K$$

inductively on n, the number of vertices of K. More precisely, we construct ϵ_K by comparing, in virtue of Lemma 2.3, pushouts for $\mathcal{Z}_\bullet(C\underline{X}, \underline{X})$ and $\mathcal{W}_\bullet$ induced from the pushout of simplicial complexes

$$\begin{tikzcd} \mathrm{link_K(n)} \arrow[r] \arrow[d] & \mathrm{star}_K(n) \arrow[d] \\ K_{[n-1]} \arrow[r] & K. \end{tikzcd}$$

As for $\mathcal{Z}_\bullet(C\underline{X}, \underline{X})$, we have the following pushout

$$\begin{tikzcd} X_n \times \mathcal{Z}_{\mathrm{link}_K(n)}(C\underline{X}, \underline{X}) \arrow[r, "\mathrm{incl}"] \arrow[d, "\mathrm{incl}"] & \mathcal{Z}_{\mathrm{star}_K(n)}(C\underline{X}, \underline{X}) \arrow[r, equal] \arrow[d, "\mathrm{incl}"] & CX_n \times \mathcal{Z}_{\mathrm{link}_K(n)}(C\underline{X}, \underline{X}) \\ \mathcal{Z}_{K_{[n-1]}}(C\underline{X}, \underline{X}) \arrow[r, "\mathrm{incl}"] & \mathcal{Z}_K(C\underline{X}, \underline{X}). & \end{tikzcd}$$

Then as for connected shifted complexes, our remaining task is to prove the following. For the disconnected case, the proof is similar and we omit it. See [IK] for details.

Theorem 2.5. *For a connected shifted complex on $[n]$, define Q as a pushout*

$$\begin{tikzcd} X_n \times \mathcal{W}_{\mathrm{link}_K(n)} \arrow[r, "\mathrm{incl}"] \arrow[d, "1\times\lambda"] & CX_n \times \mathcal{W}_{\mathrm{link}_K(n)} \arrow[d] \\ X_n \times \mathcal{W}_{K_{[n-1]}} \arrow[r] & Q. \end{tikzcd}$$

Then there is a natural homotopy equivalence

$$Q \xrightarrow{\simeq} \mathcal{W}_K.$$

Proof. Combine Lemma 2.2 and 2.4. □

Remark 2.6. The most crucial point of Theorem 2.5 is the naturality of the homotopy equivalence since we use Lemma 2.3, a commutative cube, to compare pushouts.

Finally, we have obtained the following which includes Theorem 1.7.

Theorem 2.7. *Let K be a shifted complex on $[n]$ and let $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i\in[n]}$. Then there is a homotopy equivalence*

$$\epsilon_K : \mathcal{Z}_K(C\underline{X}, \underline{X}) \xrightarrow{\simeq} \mathcal{W}_K$$

such that for a shifted subcomplex L of K on the same vertex $[n]$, there is a commutative diagram

$$\begin{tikzcd} \mathcal{Z}_L(C\underline{X}, \underline{X}) \arrow[r, "\epsilon_L"] \arrow[d, "\mathrm{incl}"] & \mathcal{W}_L \arrow[d, "\lambda"] \\ \mathcal{Z}_K(C\underline{X}, \underline{X}) \arrow[r, "\epsilon_K"] & \mathcal{W}_K. \end{tikzcd}$$

Remark 2.8. In [IK], naturality of ϵ_K with respect to more general shifted subcomplexes is proved.

References

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