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On Optimality Conditions for Robust Optimization Problems

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Abstract

In this paper, we review our recent works for optimality conditions of robust optimization problems. We give optimality conditions for the robust counterparts(the worst-case counterparts) of uncertain (multiobjective) optimization problems with uncertainty data. We present necessary and sufficient optimality theorems for the robust counterpart of a nondifferentiable convex optimization problem in the face of data uncertainty, a necessary optimality theorem for the robust counterpart of a differentiable nonconvex optimization problem in the face of data uncertainty, and a necessary optimality theorem for the robust counterpart of a differentiable multiobjective problem with uncertainty data.

1. Introduction

Recently, many authors ([1-4], [7-15]) have studied optimization problems in the face of data uncertainty within the framework of robust optimization. In this paper, we review our recent works for optimality conditions of robust optimization problems. We give optimality conditions for the robust counterparts (the worst-case counterparts) of uncertain (multiobjective) optimization problems with uncertainty data. We give a necessary and sufficient optimality theorem for the robust counterpart of a nondifferentiable convex optimization problem in the face of data uncertainty ([15]), a necessary optimality theorem for the robust counterpart of a differentiable nonconvex optimization problem in the face of data uncertainty ([12]), and a necessary optimality theorem for the robust counterpart of a differentiable multiobjective problem with uncertainty data ([13]).

2. A Necessary and Sufficient Optimality Theorem for Robust Convex Optimization Problem

The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. For a set A in $\mathbb{R}^n$, the closure of A is denoted by $\mathrm{cl}A$. We say A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. The indicator function $\delta_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if} \quad x \in A, \\ +\infty, & \text{otherwise}. \end{cases} \tag{1}$$

For an extended real-valued function f on $\mathbb{R}^n$, the effective domain and the epigraph are respectively defined by $\mathrm{dom}f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\mathrm{epi}f := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. We say that f is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\mathrm{dom}f \neq \emptyset$. Moreover, if $\liminf_{x' \to x} f(x') \geq f(x)$ for all $x \in \mathbb{R}^n$, we say f is a lower semicontinuous function. A function

$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\mu \in [0,1]$ $f((1-\mu)x+\mu y) \leq (1-\mu)f(x)+\mu f(y)$ for all $x, y \in \mathbb{R}^n$. Moreover, we say f is concave if $-f$ is convex. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq f(y) - f(x), \forall y \in \mathbb{R}^n\}, \\ \text{if}, x \in \text{dom} f, \\ \emptyset, \qquad\qquad \text{otherwise.} \end{cases} \tag{2}$$

More generally, for any $\epsilon \geq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^m$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^m : \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon \forall y \in \mathbb{R}^m\}, \\ \text{if}, x \in \text{dom} f, \\ \emptyset, \qquad\qquad \text{otherwise.} \end{cases} \tag{3}$$

As usual, for any proper convex function f on $\mathbb{R}^n$, its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^*(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f(x)\} \text{ for all } x^* \in \mathbb{R}^n.$$

For details see [16].

Lemma 2.1. (cf. [6]) Let I be an arbitrary index set and let f_i, $i \in I$, be proper lower semicontinuous convex functions on $\mathbb{R}^n$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i \in I} f_i(x_0) < \infty$. Then

$$\text{epi}(\sup_{i \in I} f_i)^* = \text{cl}(\text{ co} \bigcup_{i \in I} \text{epi} f_i^*),$$

where $\sup_{i\in I} f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i\in I} f_i)(x) = \sup_{i\in I} f_i(x)$ for all $x \in \mathbb{R}^n$.

Consider the following uncertain optimization problem:

$$\begin{aligned} \text{(UP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x, v_i) \leqq 0, \; i = 1, \cdots, m, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \cdots, m$, are functions, $\mathcal{V}_i$, $i = 1, \cdots, m$, are nonempty subsets in $\mathbb{R}^q$ and $v_i \in \mathcal{V}_i$, $i = 1, \cdots, m$. Here we suppose that we do not know the exact values of v_i, $i = 1, \cdots, m$, but know that v_i, $i = 1, \cdots, m$ belongs to some uncertainty sets $\mathcal{V}_i$, $i = 1, \cdots, m$.

The robust counterpart of (UP) is given as follows (see [1,2]);

$$\begin{aligned} \text{(RUP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x, v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i = 1, \cdots, m. \end{aligned}$$

A vector $x \in \mathbb{R}^n$ is said to be a robust feasible solution of (UP) if $g_i(x, v_i) \leqq 0$, $\forall v_i \in \mathcal{V}_i, i = 1, \cdots, m$. Let F be the set of all the robust feasible solutions of (UP), that is,

$$F := \{x \in \mathbb{R}^n \mid g_i(x, v_i) \leq 0, \; \forall v_i \in \mathcal{V}_i, \; i = 1, \cdots, m\}.$$

We say that x^* is a robust global minimizer of (UP) if $x^* \in F$ and $\forall x \in F$,

$$f(x) \geq f(x^*).$$

In this section, using (RUP), we present Lagrange optimality conditions for a robust global solution for (UP). The interesting feature of the Lagrange optimality conditions is that the number of the Lagrangean multipliers coincides with the number of constraint functions.

The following proposition, which describes the relationship between the epigraph of a conjugate function and the ϵ−subdifferential and which plays a key role in deriving the main results, was recently given in [5].

Proposition 2.1. Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function and let $a \in \text{dom}f$. Then

$$\text{epi}h^* = \cup_{\epsilon \geq 0}\{(v, v^T a + \epsilon - h(a)) : \partial_\epsilon h(a)\}.$$

The following theorem, which is the robust version of an alternative theorem, can be obtained from Theorem 2.4 and Proposition 2.3 in [8]. For the sake of completeness, we give a short proof here.

Theorem 2.1. [8] **(Robust Theorem of the Alternative)** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q$, $i = 1, \cdots, m$ be continuous functions such that $g_i(\cdot, v_i)$ is a convex function for each $u_i \in \mathbb{R}^q$. Let $\mathcal{V}_i$ be a nonempty convex subset of $\mathbb{R}^q$, $i = 1, \cdots, m$.

Let $F := \{x \in \mathbb{R}^n \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \cdots, m\} \neq \emptyset$.

Suppose that for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is a concave function. Then exact one of the following two statements holds :

$$\text{(i)} \quad (\exists x \in \mathbb{R}^n)\ f(x) < 0, \quad g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i, \quad i = 1, \cdots, m,$$

$$\text{(ii)} \quad (0, 0) \in \text{epi}f^* + \text{cl}(\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^*).$$

Proof. Suppose that (i) does not hold. Then for any $x \in F$, $f(x) \geqq 0$ and hence $\inf_{x \in \mathbb{R}^n}\{f(x) + \delta_F(x)\} \geqq 0$. By assumptions, $\delta_F(\cdot)$ is proper, lower semicontinuous and convex. So, $(0,0) \in \text{epi}(f + \delta_F)^* = \text{epi}f^* + \text{epi}\delta_F^*$. Since $\delta_F(x) = \sup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \sum_{i=1}^{m} \lambda_i g_i(x, v_i)$, it follows from Lemma 2.1 that

$$\text{epi}\delta_F^* = \text{epi}(\sup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* = \text{cl}(\text{co}(\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*)).$$

Moreover, we can check that the concavity assumption on the functions $g_i(x, \cdot)$ implies the convexity of the set

$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*$ (see the proof of Proposition 2.3 in [8]). Thus (ii) holds.

Conversely, suppose that (ii) holds. Then $(0,0) \in \text{epi}(f + \delta_F)^*$ and hence $\inf_{x \in \mathbb{R}^n}\{f(x) + \delta_F(x)\} \geqq 0$. Thus for any $x \in F$, $f(x) \geqq 0$. Hence (i) does not hold.

¿From Proposition 2.1 and Theorem 2.1(Robust Theorem of the Alternative), we can obtain the following necessary and sufficient optimality theorem for (UP) in [15], which is a robust version of that for convex optimization problem. In [15], we obtained the following theorem as a corollary of a sequential optimality theorem for convex optimization problem.

Theorem 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q$, $i = 1, \cdots, m$ be continuous functions such that $g_i(\cdot, v_i)$ is a convex function for each $u_i \in \mathbb{R}^q$. Let $\mathcal{V}_i$ be a nonempty convex subset of $\mathbb{R}^q$, $i = 1, \cdots, m$. Let $F := \{x \in \mathbb{R}^n \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \cdots, m\} \neq \emptyset$. Suppose that for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is a concave function. Let $\bar{x} \in F$. Suppose that the set $\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*$ is closed.

Then the following statements are equivalent:

(i) $\bar{x}$ is a robust global solution of (UP),

(ii) $(\exists \bar{v}_i \in \mathcal{V}_i,\ \bar{\lambda}_i \geqq 0,\ \ i = 1, \cdots, m)$

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}, \bar{v}_i), \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

Remark 2.1. If $g_i : \mathbb{R}^n \times \mathbb{R}^q$, $i = 1, \cdots, m$ are continuous functions such that $g_i(\cdot, v_i)$ is a convex function for each $v_i \in \mathbb{R}^q$, $\mathcal{V}_i$ is a nonempty convex and compact subset of $\mathbb{R}^q$, $i = 1, \cdots, m$, and the Slater type condition holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0, v_i) < 0$ for all $i = 1, \cdots, m$ and all $v_i \in \mathcal{V}_i$, then the set $\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*$ is closed [8].

3. A Necessary Optimality Theorem for Robust Nonconvex Optimization Problem

Consider the following uncertain optimization problem:

$$\begin{aligned} \text{(UP)} \quad & \min \quad f(x) \\ & \text{s.t.} \quad g_i(x, v_i) \leqq 0,\ i = 1, \cdots, m, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \cdots, m$, are continuously differentiable functions, $\mathcal{V}_i$, $i = 1, \cdots, m$, are nonempty convex compact subsets in $\mathbb{R}^q$ and $v_i \in \mathcal{V}_i$, $i = 1, \cdots, m$.

The robust counterpart of (UP) is given as follows (see [1,2,8]);

$$\begin{aligned}
\text{(RUP)} \quad \min \quad & f(x) \\
\text{s.t.} \quad & g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \cdots, m.
\end{aligned}$$

A vector $x \in \mathbb{R}^n$ is said to be a robust feasible solution of (UP) if $g_i(x, v_i) \leqq 0$, $\forall v_i \in \mathcal{V}_i, i = 1, \cdots, m$. Let F be the set of all the robust feasible solutions of (UP), that is,

$$F := \{x \in \mathbb{R}^n \mid g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \cdots, m\}.$$

We say that x^* is a robust local minimizer of (UP) if $x^* \in F$ and $\exists \epsilon > 0$ such taht $\forall x \in F \cap B_\epsilon(x^*)$, $f(x) \geqq f(x^*)$, where $B_\epsilon(x^*) = \{x \in \mathbb{R}^n \mid ||x - x^*|| < \delta\}$.

Let $x^* \in F$. Let us decompose $I := \{1, \cdots, m\}$ into two index sets $I = I_1(x^*) \cup I_2(x^*)$, where $I_1(x^*) = \{i \in I \ : \ \exists v_i \in \mathcal{V}_i \text{ s.t. } g_i(x^*, v_i) = 0\}$ and $I_2(x^*) = I \setminus I_1(x^*)$. Let $\mathcal{V}_i^0 = \{v_i \in \mathcal{V}_i \mid g_i(x^*, v_i) = 0\}$ for $i \in I_1(x^*)$. Now, we define an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) as follows:

$$(\exists d \in \mathbb{R}^n)(\forall v_i \in \mathcal{V}_i^0) \ \ \nabla_1 g_i(x^*, v_i)^T d < 0, \ i \in I_1(x^*).$$

In this section, we present a robust Karush-Kuhn-Tucker (KKT) necessary optimality condition for (UP) in [12], where f and g_i, $i = 1, \cdots, m$, are continuously differentiable, as follow: As in the classical approach to necessary optimality conditions, the proof of the robust necessary condition employs the robust Gordan's theorem and linearization.

Theorem 3.1. [12] **(Robust KKT necessary optimality condition)** Let x^* be a robust local minimizer of (UP). Suppose that $g_i(x,\cdot)$ is concave on $\mathcal{V}_i$, for each $x \in \mathbb{R}^n$ and for each $i = 1, \ldots, m$. Then, there exist $\lambda_i \geq 0$ with $\sum_{i=0}^{m} \lambda_i = 1$ and $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that

$$\lambda_0 \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x^*, v_i) = 0 \text{ and } \lambda_i g_i(x^*, v_i) = 0, \; i = 1, \ldots, m. \quad (4)$$

Moreover, if we further assume that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds, then

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x^*, v_i) = 0 \text{ and } \lambda_i g_i(x^*, v_i) = 0, \; i = 1, \ldots, m. \quad (5)$$

4. An Extension to Robust Multiobjective Optimization Problem

Consider a uncertain multiobjective optimization problem:

$$\text{(UMP)} \qquad \text{minimize} \quad (f_1(x), \cdots, f_l(x))$$
$$\text{subject to} \quad g_j(x, v_j) \leqq 0, \quad j = 1, \cdots, m,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $\; i = 1, \cdots, l$ and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $\; j = 1, \cdots, m$ are continuous functions and v_j is a uncertain parameter, and $v_j \in \mathcal{V}_j$ for some convex compact set $\mathcal{V}_j$ in $\mathbb{R}^q$.

When $l = 1$, (UMP) becomes a uncertain optimization problem (UP), which has been intensively studied in [1-3,8].

In this section, we treat the robust approach for (UMP), which is the worst-case approach for (UMP). Now we associates with the uncertain multiobjective optimization problem (UMP) its robust counterpart :

$$\text{(RMP)} \qquad \text{minimize} \qquad (f_1(x), \cdots, f_l(x))$$
$$\text{subject to} \qquad \max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leqq 0, \quad j = 1, \cdots, m.$$

A vector $x \in \mathbb{R}^n$ is a robust feasible solution of (UMP) if $\max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leqq 0, \quad j = 1, \cdots, m$.

Let F be the set of all the robust feasible solutions of (UMP).

A robust feasible solution $\bar{x}$ of (UMP) is a weakly robust efficient solution of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$f_i(x) < f_i(\bar{x}), \quad i = 1, \cdots, m.$$

Let $\bar{x} \in F$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$ where $J_1(\bar{x}) = \{j \in J \mid \exists v_j \in \mathcal{V}_j \text{ s.t. } g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = J \setminus J_1(\bar{x})$. Since $\bar{x} \in F$, $J_1(\bar{x}) = \{j \in J \mid \max_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = \{j \in J \mid \max_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) < 0\}$. Let $\mathcal{V}_j^0 = \{v_j \in \mathcal{V}_j \mid g_j(\bar{x}, v_j) = 0\}$ for $j \in J_1(\bar{x})$.

Assume that $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, l$, and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $j = 1, \cdots, m$ are continuously differentiable.

Now we define an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) for (UMP) as follows: there exists $d \in \mathbb{R}^n$ such that for any $j \in J_1(\bar{x})$ and any $v_j \in \mathcal{V}_j^0$,

$$\nabla_1 g_j(\bar{x}, v_j)^T d < 0.$$

Now we present a necessary optimality theorems for weakly robust efficient solution for (UMP), which can be obtained from Theorem 3.3 in [13] and can be regarded as a generalization of Theorem 3.1 in Section 3.

Theorem 4.1. Let $\bar{x} \in F$ be a weakly robust efficient solution of (UMP). Suppose that $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j$, $j = 1, \cdots, m$. Then there exist $\lambda_i \geqq 0$, $i = 1, \cdots, l$, $\mu_j \geqq 0$, $j = 1, \cdots, m$, not all zero, and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \cdots, m$ such that

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0 \tag{6}$$

and $$\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m. \tag{7}$$

Moreover, if we further assume that the Extended Mangasarian-Fromovitz constraint qualification (shortly, EMFCQ) holds, then there exist $\lambda_i \geqq 0$, $i = 1, \cdots, l$, not all zero, and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \cdots, m$ such that (6) and (7) hold.

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