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MINIMAX THEOREMS FOR SET-VALUED MAPS*
(集合値写像に対するミニマックス定理)

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Abstract

In the paper, we introduce several types of definitions for minimax and maxmini values of set-valued maps and show minimax theorems for set-valued maps by using properties of nonconvex scalarizing functions for sets.

1 Introduction

Recently, some researchers have been investigated minimax theorems including saddle point problems for set-valued maps based on vector optimization (see [1, 2, 4, 5]). They show minimax theorems by using scalarization methods for vectors. In [8], we propose new concepts for minimax and maxmini values of set-valued maps, and show some minimax theorems by using several properties of scalarizing functions for sets introduced in [6].

The aim of the paper is to introduce several vector-valued set-valued minimax theorems.

The organization of the paper is as follows. In Section 2, we introduce some preliminary results. In Section 3, we recall some properties of scalarizing functions for sets introduced in [6, 7]. Moreover, by using these results, we define concepts of efficient solutions for set-valued optimization problems. In Section 4, we introduce several types of vector-valued set-valued minimax theorems.

2 Mathematical Preliminaries

Let A, B be nonempty subsets of a real topological vector space. We denote the topological interior and complement of A by $\mathrm{int}A$ and A^c, respectively; the algebraic sum, algebraic difference of A and B by $A+B := \{a+b \mid a \in A,\ b \in B\}$, $A-B := \{a-b \mid a \in A,\ b \in B\}$, respectively; the composite function of two functions f and g by $g \circ f$. Moreover, we denote the algebraic sum and algebraic difference of a set A and a family of nonempty sets $\mathcal{V} \subset \wp(Z)$ by $A+\mathcal{V} := \{A+B : B \in \mathcal{V}\}$ and $A-\mathcal{V} := \{A-B : B \in \mathcal{V}\}$, respectively.

Throughout the paper, (X, τ_X), (Y, τ_Y) are real Hausdorff topological vector spaces (X, Y for short, respectively), (Z, τ_Z) is a finite dimensional Euclidean space, $\wp(Z)$ is the family of all nonempty subsets of Z, C is a nontrivial closed convex cone in Z (that is,

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$C \neq \{\theta_Z\}$, $C \neq Z$, $C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with nonempty topological interior. We define a partial ordering $\leq_C$ as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x, y \in Z.$$

When $x \leq_C y$ for $x, y \in Z$, we define the order interval between x and y by $[x, y] := \{z \in Z | x \leq_C z \,\text{and}\, z \leq_C y\}$. We say that C is normal for the topology τ_Z (normal, for short) if there is a base of neighborhoods of $\{\theta_Z\}$ consisting of sets S with the property $S = (S + C) \cap (S - C)$.

Now we define some C-property. A subset A of Z is said to be C-closed if $A + C$ is closed; C-proper if $A + C \neq Z$. Moreover, we say that a map $F : X \to \wp(Z)$ is C-property valued on X if $F(x)$ has the C-property for every $x \in X$.

Let $\mathcal{V} \subset \wp(Z)$. Then, A_0 is said to be

(i) minimal element of $\mathcal{V}$ if for any $A \in \mathcal{V}$

$$A_0 \subset A + C \quad \text{implies} \quad A \subset A_0 + C;$$

(ii) maximal element of $\mathcal{V}$ if for any $A \in \mathcal{V}$

$$A_0 \subset A - C \quad \text{implies} \quad A \subset A_0 - C.$$

If C is replaced by int$C$ then we call it weak minimal element (resp., weak maximal element) of $\mathcal{V}$. We denote the family of minimal element (resp., maximal element) by Min$\mathcal{V}$ (resp., Max$\mathcal{V}$). Also, we denote the family of weak minimal element (resp., weak maximal element) by $\text{Min}_{\text{w}}\mathcal{V}$ (resp., $\text{Max}_{\text{w}}\mathcal{V}$).

Next, we define some convexity and continuity notions for set-valued maps.

Definition 2.1 ([6]). Let $F : X \to \wp(Z)$. Then F is called

(i) natural quasi C-convex if for any $x, y \in X$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$\mu F(x) + (1 - \mu)F(y) \subset F(\lambda x + (1 - \lambda)y) + C;$$

(ii) natural quasi C-concave if for any $x, y \in X$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$\mu F(x) + (1 - \mu)F(y) \subset F(\lambda x + (1 - \lambda)y) - C.$$

If C is replaced by int$C$ we call it strictly natural quasi C-convex (resp., strict natural quasi C-concave).

Definition 2.2 ([3]). Let $F : X \to \wp(Z)$ and $x \in X$. Then,

(i) F is called lower continuous at x if for every open set V with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap V \neq \emptyset$ for all $y \in U$. We shall say that F is lower continuous on X if it is lower continuous at every point $x \in X$.

(ii) F is called upper continuous at x if for every open set V with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V$ for all $y \in U$. We shall say that F is upper continuous on X if it is upper continuous at every point $x \in X$.

(iii) F is called continuous on X if F is both lower and upper continuous on X.

3 Nonlinear scalarization for sets

To show main results, we consider the following two types of nonlinear scalarizing functions for sets which are special cases of unified types of scalarizing functions introduced in [6]:

Let $A \in \wp(Z)$, direction $k \in \mathrm{int}C$ and $v \in Z$. $I_k^v : \wp(Z) \to \mathbb{R} \cup \{\pm\infty\}$ and $S_k^v : \wp(Z) \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I_k^v(A) := \inf \{t \in \mathbb{R} \mid tk + v \in A + C\},$$

$$S_k^v(A) := \sup \{t \in \mathbb{R} \mid tk + v \in A - C\},$$

respectively.

In this section, we introduce some properties of these functions.

Proposition 3.1 ([6]). *Let $A, B \in \wp(Z)$, $k \in \mathrm{int}C$ and $v \in Z$. Then, the following statements hold:*

(i) *For any $\alpha \in \mathbb{R}$,*

$$I_k^v(A + \alpha k) = I_k^v(A) + \alpha,$$

$$S_k^v(A + \alpha k) = S_k^v(A) + \alpha.$$

(ii) *If $B \subset A + C$ then $I_k^v(A) \leq I_k^v(B)$.*

(iii) *If $A \subset B - C$ then $S_k^v(A) \leq S_k^v(B)$.*

Proposition 3.2 ([7]). *Let $A \in \wp(Z)$, $k \in \mathrm{int}C$ and $v \in Z$. Then, the following statements hold:*

(i) *$I_k^v(A) < \infty$ and $S_k^v(A) > -\infty$.*

(ii) *A is C-proper if and only if $I_k^v(A) > -\infty$.*

(iii) *A is $(-C)$-proper if and only if $S_k^v(A) < \infty$.*

Proposition 3.3 ([7]). *Let $A \in \wp(Z)$, $k \in \mathrm{int}C$ and $v \in Z$. Then the following statements hold:*

(i) *If A is C-closed then $(I_k^v(A)k + v) \in A + C$,*

(ii) *If A is $(-C)$-closed then $(S_k^v(A)k + v) \in A - C$.*

Proposition 3.4 ([8]). *Let $A, B \in \wp(Z)$, $k \in \mathrm{int}C$ and $v \in Z$. Assume that I_k^v and S_k^v are finite. Then, the following statements hold:*

(i) *If B is C-closed and $B \subset A + \mathrm{int}C$, then*

$$I_k^v(A) < I_k^v(B).$$

(ii) *If A is $(-C)$-closed and $A \subset B - \mathrm{int}C$, then*

$$S_k^v(A) < S_k^v(B).$$

Let $F : X \to \wp(Z)$, $k \in \mathrm{int}C$ and $v \in Z$. We consider the following set-valued optimization problems:

$$(\mathrm{SP}) \begin{cases} \text{Optimize} & F(x) \\ \text{Subject to} & x \in X. \end{cases}$$

We say that x_0 is a minimal efficient solution (resp., maximal, weak minimal, weak maximal efficient solution) of (SP) if $F(x_0)$ is a minimal element (resp., maximal, weak minimal, weak maximal element) of $F(X)$. Let us consider the following two composite functions:

$$(I_k^v \circ F)(x) := I_k^v(F(x)), \quad x \in X,$$

$$(S_k^v \circ F)(x) := S_k^v(F(x)), \quad x \in X.$$

Then we show sufficient conditions for the existence of these solutions by using properties of $I_k^v \circ F$ and $S_k^v \circ F$.

Lemma 3.1 ([8]). *Let $F : X \to \wp(Z)$. Assume that F is C-closed valued on X. Then the following statements hold:*

(i) *For each $k \in \text{int}C$ and $v \in Z$, there exists $x(k;v) \in X$ such that $x(k;v)$ is a solution of $\inf_{x \in X}(I_k^v \circ F)(x)$, then $x(k;v)$ is a weak minimal efficient solution of* (SP).

(ii) *For each $k \in \text{int}C$ and $v \in Z$, there exists $x(k;v) \in X$ such that $x(k;v)$ is a unique solution of $\inf_{x \in X}(I_k^v \circ F)(x)$, then $x(k;v)$ is a minimal efficient solution of* (SP).

Similarly, we obtain the following result:

Lemma 3.2 ([8]). *Let $F : X \to \wp(Z)$. Assume that F is $(-C)$-closed valued on X. Then the following statements hold:*

(i) *For each $k \in \text{int}C$ and $v \in Z$, there exists $x(k;v) \in X$ such that $x(k;v)$ is a solution of $\sup_{x \in X}(S_k^v \circ F)(x)$, then $x(k;v)$ is a weak maximal efficient solution of* (SP).

(ii) *For each $k \in \text{int}C$ and $v \in Z$, there exists $x(k;v) \in X$ such that $x(k;v)$ is a unique solution of $\sup_{x \in X}(S_k^v \circ F)(x)$, then $x(k;v)$ is a maximal efficient solution of* (SP).

In [8], we define another solution concepts for (SP) based on results in Lemmas 3.1 and 3.2. Let $F : X \to \wp(Z)$. Then, $x_0 \in X$ is said to be proper minimal efficient solution (resp., proper maximal efficient solution) of (SP) if there exists $k \in \text{int}C$ and $v \in Z$ such that $x(k;v)$ is a unique solution of $\inf_{x \in X}(I_k^v \circ F)(x)$ (resp., $\sup_{x \in X}(S_k^v \circ F)(x)$); proper weak minimal efficient solution (resp., proper weak maximal efficient solution) of (SP) if there exists $k \in \text{int}C$ and $v \in Z$ such that $x(k;v)$ is a solution of $\inf_{x \in X}(I_k^v \circ F)(x)$ (resp., $\sup_{x \in X}(S_k^v \circ F)(x)$). We denote the family of sets as the image of proper minimal efficient solution (resp., proper maximal efficient solution) and proper weak minimal efficient solution (resp., proper weak maximal efficient solution) of (SP) by $\text{Min}^{\text{P}} F(X)$ (resp., $\text{Max}^{\text{P}} F(X)$) and $\text{Min}_{\text{w}}^{\text{P}} F(x)$ (resp., $\text{Max}_{\text{w}}^{\text{P}} F(X)$), respectively.

4 Minimax theorems for set-valued maps

Let $F : X \times Y \to \wp(Z) \setminus \{\emptyset\}$. Based on several solution concepts of (SP) introduced in Section 3, we consider the following two types of minimax and maxmini values of F:

$$\text{MinMax}^{\text{P}} F(x,y) := \text{Min}\{F(x,y) | F(x,y) \in \text{Max}^{\text{P}} F(x,Y),\ x \in X\},$$

$$\text{MaxMin}^{\text{P}} F(x,y) := \text{Max}\{F(x,y) | F(x,y) \in \text{Min}^{\text{P}} F(X,y),\ y \in Y\},$$

$$\mathrm{MinMax}^{\mathrm{p}}_{\mathrm{w}}F(x,y) := \mathrm{Min}\{F(x,y)|F(x,y) \in \mathrm{Max}^{\mathrm{p}}_{\mathrm{w}}F(x,Y),\ x \in X\},$$

$$\mathrm{MaxMin}^{\mathrm{p}}_{\mathrm{w}}F(x,y) := \mathrm{Max}\{F(x,y)|F(x,y) \in \mathrm{Min}^{\mathrm{p}}_{\mathrm{w}}F(X,y),\ y \in Y\}.$$

In this section, we introduce some minimax theorems for set-valued maps.

Theorem 4.1 ([8]). *Let A and B be nonempty compact convex subsets of X and Y, respectively. Assume that C is normal. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is continuous and compact valued on $A \times B$,*
(ii) *for any $y \in B$, $F(\cdot, y)$ is natural quasi C-convex on A,*
(iii) *for any $x \in A$, $F(x, \cdot)$ is natural quasi C-concave on B,*

then,

$$(\mathrm{MaxMin}^{\mathrm{p}}_{\mathrm{w}}F(x,y) - C) \cap (\mathrm{MinMax}^{\mathrm{p}}_{\mathrm{w}}F(x,y) + C) \neq \emptyset.$$

As a special case of Theorem 4.1, we obtain the following vector-valued minimax theorem.

Corollary 4.1 ([8]). *Let A and B be nonempty compact convex subsets of X and Y, respectively. Assume that C is normal and $D \subset \mathrm{int}C \cup \{\theta_Z\}$ a closed convex cone with $\mathrm{int}D \neq \emptyset$. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is singleton on $A \times B$,*
(ii) *F is continuous on $A \times B$,*
(iii) *for any $y \in B$, $F(\cdot, y)$ is natural quasi C-convex on A,*
(iv) *for any $x \in A$, $F(x, \cdot)$ is natural quasi C-concave on B,*

then,

$$(\mathrm{MaxMin}_{\mathrm{w}}F(x,y) - C) \cap (\mathrm{MinMax}_{\mathrm{w}}F(x,y) + C) \neq \emptyset.$$

By using Corollary 4.1 and Theorem 3.2 in [10], we obtain another type of minimax theorem for vector-valued functions.

Corollary 4.2. *Let A and B be nonempty compact convex subsets of X and Y, respectively. Assume that C is normal. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *$F(x,y) := f_1(x) + f_2(y)$ where $f_1 : A \to Z$ and $f_2 : B \to Z$,*
(ii) *F is continuous on $A \times B$,*

then,

$$(\mathrm{MaxMin}_{\mathrm{w}}F(x,y) - C) \cap (\mathrm{MinMax}_{\mathrm{w}}F(x,y) + C) \neq \emptyset.$$

Remark 4.1. Let $f : X \times Y \to Z$. In Corollaries 4.1 and 4.2, we present vector-valued minimax theorems based on $\mathrm{MinMax}_{\mathrm{w}}f(x,y)$ and $\mathrm{MaxMin}_{\mathrm{w}}f(x,y)$. In general, minimax and maxmini values of f are defined as follows (see [9, 10, 11]):

$$\min \bigcup_{x \in X} \mathrm{Max}_{\mathrm{w}}f(x,y) \quad \text{and} \quad \max \bigcup_{y \in Y} \mathrm{Min}_{\mathrm{w}}f(x,y)$$

where $\min A := \{a \in A | (a - C) \cap A = \{a\}\}$ and $\max A := \{a \in A | (a + C) \cap A = \{a\}\}$. The following simple example shows that $\mathrm{MaxMin}_{\mathrm{w}}f(x,y) \neq \max \bigcup_{x \in X} \mathrm{Min}_{\mathrm{w}}f(x,y)$. Let $X := [0,1]$, $Y := [-2,2]$, $C := \{(x,y)^t \in \mathbb{R}^2 | 0 \leq x,\ 0 \leq y \leq x\}$ where $(\cdot,\cdot)^t$ is the transpose of $(\cdot,\cdot)$. Then we define a vector-valued function $f : X \times Y \to \mathbb{R}^2$ by

$$f(x,y) := (0,x)^t + (y,y^2)^t.$$

Then $\mathrm{MaxMin_w} f(x,y) = \{\mathrm{Min_w} f(x,y) | y \geq 1\}$ and

$$\max \bigcup_{x\in X} \mathrm{Min_w} f(x,y) = \{f(x,y)|y=-2\} \cup \{f(x,y)|1 \leq y,\ x=0\}.$$

Hence $\mathrm{MaxMin_w} f(x,y) \neq \max \bigcup_{x\in X} \mathrm{Min_w} f(x,y)$. By a similar simple example, we can check that $\mathrm{MinMax_w} f(x,y) \neq \min \bigcup_{y\in Y} \mathrm{Max_w} f(x,y)$.

Next, we introduce a strong minimax theorem for set-valued maps.

Theorem 4.2 ([8]). *Let A and B be nonempty compact convex subsets of X and Y, respectively. Assume that C is normal. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is continuous and compact valued on $A \times B$,*
(ii) *for any $y \in B$, $F(\cdot, y)$ is strictly natural quasi C-convex on A,*
(iii) *for any $x \in A$, $F(x, \cdot)$ is strictly natural quasi C-concave on B,*

then,

$$(\mathrm{MaxMin^P} F(x,y) - C) \cap (\mathrm{MinMax^P} F(x,y) + C) \neq \emptyset.$$

By Corollary 4.1 and Theorem 4.2, we obtain the following corollary.

Corollary 4.3 ([8]). *Let A and B be nonempty compact convex subsets of X and Y, respectively. Assume that C is normal. If $F : A \times B \to \wp(Z)$ satisfies that*

(i) *F is singleton on $A \times B$,*
(ii) *F is continuous on $A \times B$,*
(iii) *for any $y \in B$, $F(\cdot, y)$ is strictly natural quasi C-convex on A,*
(iv) *for any $x \in A$, $F(x, \cdot)$ is strictly natural quasi C-concave on B,*

then,

$$(\mathrm{MaxMin} F(x,y) - C) \cap (\mathrm{MinMax} F(x,y) + C) \neq \emptyset.$$

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