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Extension theorems concerned with results by Ponnusamy and Karunakaran

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1 Introduction

Let $\mathcal{A}(n,k)$ be the class of functions $f(z)$ of the form

$$f(z)=z^{n}+\sum_{m=n+k}^{\infty}a_{m}z^{m}\qquad(n\geqq 1,k\geqq 1) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For two functions $f(z)$ and $g(z)$ belonging to the class $\mathcal{A}(1,1)$, Sakaguchi [5] has proved the following result.

Theorem A *Let $f(z)\in\mathcal{A}(1,1)$ and $g(z)\in\mathcal{A}(1,1)$ be starlike in* $\mathbb{U}$. *If $f(z)$ and $g(z)$ satisfy*

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right)>0\qquad(z\in\mathbb{U}), \tag{1.2}$$

then

$$\mathrm{Re}\left(\frac{f(z)}{g(z)}\right)>0\qquad(z\in\mathbb{U}). \tag{1.3}$$

After Theorem A, many mathematicians studying this field have applied this theorem to get some results. In 1989, Ponnusamy and Karunakaran [4] have improved Theorem A as following.

Theorem B *Let α be a complex number with $\mathrm{Re}\,\alpha>0$ and $\beta<1$. Further, let $f(z)\in\mathcal{A}(n,k)$ and $g(z)\in\mathcal{A}(n,j)$ $(j\geqq 1)$ satisfies*

$$\mathrm{Re}\left(\frac{\alpha g(z)}{zg'(z)}\right)>\delta\qquad(z\in\mathbb{U}) \tag{1.4}$$

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with $0 \leqq \delta < \dfrac{\operatorname{Re}\alpha}{n}$. *If* $f(z)$ *and* $g(z)$ *satisfy*

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}\right\}>\beta \qquad (z\in\mathbb{U}), \tag{1.5}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)>\frac{2\beta+\delta k}{2+\delta k} \qquad (z\in\mathbb{U}). \tag{1.6}$$

It is the purpose of the present paper is to discuss Theorem B applying the lemma due to Fukui and Sakaguchi [1]. To discuss our problems, we need the following lemmas.

Lemma 1 *Let* $w(z)=\sum_{n=k}^{\infty}a_n z^n$ $(a_k\neq 0, k\geqq 1)$ *be analytic in* $\mathbb{U}$. *If the maximum value of* $|w(z)|$ *on the circle* $|z|=r<1$ *is attained at* $z=z_0$, *then we have*

$$\frac{z_0 w'(z_0)}{w(z_0)}=\ell\geqq k, \tag{1.7}$$

which shows that $\dfrac{z_0 w'(z_0)}{w(z_0)}$ *is a positive real number.*

The proof of Lemma 1 can be found in [1] and we see that Lemma 1 is a generalization of Jack's lemma given by Jack [2]. Applying Lemma 1, we derive

Lemma 2 *Let* $p(z)=1+\sum_{n=k}^{\infty}c_n z^n$ $(c_k\neq 0, k\geqq 1)$ *be analytic in* $\mathbb{U}$ *with* $p(z)\neq 0$ $(z\in\mathbb{U})$. *If there exists a point* $z_0\in\mathbb{U}$ *such that*

$$\operatorname{Re}p(z)>0 \qquad (|z|<|z_0|)$$

and

$$\operatorname{Re}p(z_0)=0,$$

then we have

$$-z_0 p'(z_0)\geqq\frac{\ell}{2}(1+|p(z_0)|^2) \tag{1.8}$$

and so

$$\frac{z_0 p'(z_0)}{p(z_0)}=i\ell, \tag{1.9}$$

where

$$k\leqq\frac{k}{2}\left(a+\frac{1}{a}\right)\leqq\ell \qquad \left(\arg p(z_0)=\frac{\pi}{2}\right) \tag{1.10}$$

and

$$-k\geqq-\frac{k}{2}\left(a+\frac{1}{a}\right)\geqq\ell \qquad \left(\arg p(z_0)=-\frac{\pi}{2}\right) \tag{1.11}$$

with $p(z_0)=\pm ia$ $(a>0)$.

Proof. Let us consider

$$\phi(z)=\frac{1-p(z)}{1+p(z)}=\frac{c_k}{2}z^k+\cdots \tag{1.12}$$

for $p(z)$. Then, it follows that $\phi(0)=\phi'(0)=\cdots=\phi^{(k-1)}(0)=0$, $|\phi(z)|<1 \quad (|z|<|z_0|)$ and $|\phi(z_0)|=1$. Therefore, applying Lemma 1, we have that

$$\frac{z_0\phi'(z_0)}{\phi(z_0)}=\frac{-2z_0p'(z_0)}{1-\left(p(z_0)\right)^2}=\frac{-2z_0p'(z_0)}{1+|p(z_0)|^2}=\ell\geqq k. \tag{1.13}$$

This implies that $z_0p'(z_0)$ is a negative real number and

$$-z_0p'(z_0)\geqq\frac{k}{2}(1+|p(z_0)|^2). \tag{1.14}$$

Let us use the same method by Nunokawa [3]. If $\arg p(z_0)=\frac{\pi}{2}$, then we write $p(z_0)=ia$ $(a>0)$. This gives us that

$$\operatorname{Im}\left(\frac{z_0p'(z_0)}{p(z_0)}\right)=\operatorname{Im}\left(-\frac{iz_0p'(z_0)}{a}\right)\geqq\frac{k}{2}\left(a+\frac{1}{a}\right).$$

If $\arg p(z_0)=-\frac{\pi}{2}$, then we write $p(z_0)=-ia \quad (a>0)$. Thus we have that

$$\operatorname{Im}\left(\frac{z_0p'(z_0)}{p(z_0)}\right)=\operatorname{Im}\left(\frac{iz_0p'(z_0)}{a}\right)\leqq-\frac{k}{2}\left(a+\frac{1}{a}\right).$$

This completes the proof of Lemma 2. □

2 Main results

With the help of Lemma 2, we derive the following theorem.

Theorem 1 *Let α be a complex number with $\operatorname{Re}\alpha>0$ and $\beta<1$. Further, let $f(z)\in\mathcal{A}(n,k)$ and $g(z)\in\mathcal{A}(n,j)$ $(j\geqq 1)$ satisfies*

$$\operatorname{Re}\left(\frac{\alpha g(z)}{zg'(z)}\right)>\delta \qquad (z\in\mathbb{U}) \tag{2.1}$$

with $0\leqq\delta<\frac{\operatorname{Re}\alpha}{n}$. If $f(z)$ and $g(z)$ satisfy

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}\right\}+\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z)}{g(z)}-\beta_1\right|^2>\beta \qquad (z\in\mathbb{U}) \tag{2.2}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)>\beta_1 \qquad (z\in\mathbb{U}), \tag{2.3}$$

where $\beta_1=\frac{2\beta+\delta k}{2+\delta k}$.

Proof. Defining the function $p(z)$ by

(2.4) $$p(z)=\frac{\dfrac{f(z)}{g(z)}-\beta_1}{1-\beta_1},$$

we see that $p(0)=1$ and

(2.5) $$(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}-\beta$$

$$=(\beta_1-\beta)+(1-\beta_1)\left(p(z)+\frac{\alpha g(z)}{zg'(z)}zp'(z)\right)$$

$$>-\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z)}{g(z)}-\beta_1\right|^2$$

for all $z\in\mathbb{U}$. Let us suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$|\arg p(z_0)|<\frac{\pi}{2}\qquad(|z|<|z_0|)$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}.$$

Then, by means of Lemma 2, we have that

(2.6) $$-z_0p'(z_0)\geqq\frac{k}{2}(1+|p(z_0)|^2).$$

If follows from the above that

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)}+\alpha\frac{f'(z_0)}{g'(z_0)}-\beta\right\}$$

$$=(\beta_1-\beta)+(1-\beta_1)\mathrm{Re}\left\{p(z_0)+\frac{\alpha g(z_0)}{z_0g'(z_0)}z_0p'(z_0)\right\}$$

$$=(\beta_1-\beta)-(1-\beta_1)\mathrm{Re}\left\{\frac{\alpha g(z_0)}{z_0g'(z_0)}(-z_0p'(z_0))\right\}$$

$$\leqq(\beta_1-\beta)-(1-\beta_1)\frac{\delta k}{2}(1+|p(z_0)|^2)$$

$$=-\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z_0)}{g(z_0)}-\beta_1\right|^2$$

which contradicts (2.5). This completes the proof of the theorem. □

Remark 1 If $f(z)$ and $g(z)$ satisfy $f(z_0) = \beta_1 g(z_0)$ in Theorem 1, then Theorem 1 becomes Theorem B given by Ponnusamy and Karunakaran [4]. We also have

Theorem 2 *Let α be a complex number with $\operatorname{Re}\alpha > 0$ and $\beta < 1$. Further, let $f(z) \in \mathcal{A}(n,k)$ and $g(z) \in \mathcal{A}(n,j)$ $(j \geqq 1)$ satisfies the condition (2.1) with $0 \leqq \delta < \dfrac{\operatorname{Re}\alpha}{n}$. If $f(z)$ and $g(z)$ satisfy*

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta\right\}\right| < \frac{\pi}{2} + \operatorname{Tan}^{-1}\left(\frac{\delta k|p(z)|}{2\left(\frac{2r}{1-r^2} + \frac{|\operatorname{Im}\alpha|}{n} + 1\right)}\right) \tag{2.7}$$

for $|z| = r < 1$, then

$$\left|\arg\left(\frac{f(z)}{g(z)} - \beta_1\right)\right| < \frac{\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.8}$$

or

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > \beta_1 \qquad (z \in \mathbb{U}), \tag{2.9}$$

where $\beta_1 = \dfrac{2\beta + \delta k}{2 + \delta k}$ and

$$p(z) = \frac{\frac{f(z)}{g(z)} - \beta_1}{1 - \beta_1}.$$

Proof. Note that the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. It follows that

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta\right\}\right| = \left|\arg\left\{(\beta_1 - \beta) + (1-\beta_1)\left(p(z) + \frac{\alpha g(z)}{zg'(z)} zp'(z)\right)\right\}\right|$$

$$< \frac{\pi}{2} + \operatorname{Tan}^{-1}\left(\frac{\delta k|p(z)|}{2\left(\frac{2r}{1-r^2} + \frac{|\operatorname{Im}\alpha|}{n} + 1\right)}\right)$$

for $|z| = r < 1$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z_0)| < \frac{\pi}{2} \qquad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2},$$

then, by Lemma 2, we have that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell,$$

where

$$\frac{k}{2}\left(a + \frac{1}{a}\right) \leqq \ell \qquad \left(\arg p(z_0) = \frac{\pi}{2}\right)$$

and

$$-\frac{k}{2}\left(a + \frac{1}{a}\right) \geqq \ell \qquad \left(\arg p(z_0) = -\frac{\pi}{2}\right)$$

with $p(z_0) = \pm ia \quad (a > 0)$. If $\arg p(z_0) = \dfrac{\pi}{2}$, then it follows that

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\}$$

$$= \arg p(z_0)\left(\frac{\beta_1 - \beta}{p(z_0)} + (1-\beta_1)\left(1 + \frac{\alpha g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right)\right\}$$

$$= \frac{\pi}{2} + \arg\left\{-\left(\frac{\beta_1 - \beta}{a}\right)i + (1-\beta_1)\left(1 + i\ell\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right\}$$

$$= \frac{\pi}{2} + \arg I(z_0),$$

where

(2.9) $$I(z_0) = -\left(\frac{\beta_1 - \beta}{a}\right)i + (1-\beta_1)\left(1 + i\ell\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right).$$

Note that

(2.10) $$\operatorname{Im} I(z_0) = \frac{\beta - \beta_1}{a} + (1-\beta_1)\ell \operatorname{Re}\frac{\alpha g(z_0)}{z_0 g'(z_0)}$$

$$> (1-\beta_1)\delta\ell + \frac{\beta - \beta_1}{a}$$

$$\geqq \frac{\delta k}{2}(1-\beta_1)\left(a + \frac{1}{a}\right) + \frac{\beta - \beta_1}{a}$$

$$= \frac{\delta k}{2}(1-\beta_1)a > 0$$

and

(2.11) $$\operatorname{Re} I(z_0) = (1-\beta_1)\left(1 - \ell \operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right) \leqq (1-\beta_1)\left(1 + \ell\left|\operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right|\right).$$

Letting

$$q(z)=\frac{\alpha g(z)}{zg'(z)}+1-\frac{\alpha}{n}, \tag{2.12}$$

we know that $q(z)$ is analytic in $\mathbb{U}$ with $q(0)=1$. This gives us that

$$|\operatorname{Im} q(z)|=\left|\operatorname{Im}\left(\frac{\alpha g(z)}{zg'(z)}+1-\frac{\alpha}{n}\right)\right| \leqq \frac{2r}{1-r^2} \tag{2.13}$$

for $|z|=r<1$. Thus we have that

$$\left|\operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right| \leqq \frac{2r}{1-r^2}+\frac{|\operatorname{Im}\alpha|}{n} \qquad (|z|=r<1). \tag{2.14}$$

Using (2.11) and (2.14), we obtain that

$$\arg I(z_0)=\operatorname{Tan}^{-1}\left(\frac{\operatorname{Im} I(z_0)}{\operatorname{Re} I(z_0)}\right) \geqq \operatorname{Tan}^{-1}\left(\frac{\delta k a}{2\left(\frac{2r}{1-r^2}+\frac{|\operatorname{Im}\alpha|}{n}+1\right)}\right),$$

which contradicts our condition (2.7).
If $\arg p(z_0)=-\frac{\pi}{2}$, using the same way, we also have that

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)}+\alpha\frac{f'(z_0)}{g'(z_0)}-\beta\right\} \leqq -\left\{\frac{\pi}{2}+\operatorname{Tan}^{-1}\left(\frac{\delta k a}{2\left(\frac{2r}{1-r^2}+\frac{|\operatorname{Im}\alpha|}{n}+1\right)}\right)\right\},$$

which contradicts (2.7). □

Remark 2 If $f(z)$ satisfies the conditions in Theorem B, then $f(z)$ satisfies the conditions of Theorem 2. In this case, we see that Theorem 2 becomes Theorem B.

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