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ON THE INVISCID LIMIT PROBLEM FOR VISCOUS INCOMPRESSIBLE FLOWS IN THE HALF PLANE

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1. Introduction

This article is a resume of the author's recent work [21]. We are concerned with the Navier-Stokes equations for viscous incompressible flows in the half plane under the no-slip boundary conditions:

(NS_ν)

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, & t>0,\ x\in\mathbb{R}^2_+, \\ u = 0 & t\geq 0,\ x\in\partial\mathbb{R}^2_+, \\ u|_{t=0} = a & x\in\mathbb{R}^2_+. \end{cases}$$

Here $\mathbb{R}^2_+ = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_2>0\}$ and ν is the kinematic viscosity which is assumed to be a positive constant, and $u = u(t,x) = (u_1(t,x), u_2(t,x))$, $p = p(t,x)$ denote the velocity field, the pressure field, respectively. We will use the standard notations for derivatives; $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} u = \sum_{j=1}^2 \partial_j u_j$, and $u\cdot\nabla u = \sum_{j=1}^2 u_j\partial_j u$.

The behavior of viscous incompressible flows at the inviscid limit is a classical issue in the fluid dynamics. However, in the presence of nontrivial boundary one is faced with a serious difficulty in this problem even in the two-dimensional case if the no-slip boundary condition is imposed on the velocity field. This is due to the appearance of the boundary layer, whose formation is formally explained by Prandtl's theory. But because of its strong instability mechanism so far the rigorous description of the formation of the boundary layer and the outer flow was achieved only for some limited cases. For example, it is proved in [3, 33, 34] that for analytic initial data the solution of (NS_ν) converges to the one of the Euler equations outside the boundary layer and to the one of the Prandtl equations in the boundary layer. When the domain and the initial data possess a circular symmetry the significant cancellation occurs in the nonlinear term, and hence the convergence is affirmatively justified; see [24, 5, 18, 19, 14, 26]. On the other hand, the necessary and sufficient condition for the L^2 convergence of the Navier-Stokes flows to the Euler flows was given by [12], which was extended by several authors [36, 38, 13, 14].

Since the appearance of the boundary layer is considered as the formation of a vortex sheet (or line in the two dimension) along the boundary, it is natural to investigate the behavior of vorticity fields at the inviscid limit. However, under the no-slip boundary condition on the velocity field the vorticity field has to be subject to a nonlocal and nonlinear boundary condition, from which it is still not easy to derive useful informations. This is contrasting with the case of the whole plane (i.e., no nontrivial boundary), where the detailed analysis has been established [22, 8]. In the case of the half plane the situation is somewhat relaxed, since the solution formula is available for the linearized problem, which enables us to estimate the behavior of vorticity near the boundary in details at least in the linear level; see [20].

In [21] the inviscid limit of (NS_ν) is studied by using the vorticity formulation in [20] when the initial vorticity is located away from the boundary. This class of initial data includes a dipole-type localized vortex, which is often used in numerical works to investigate the interaction between the vorticity created on the boundary and the original vorticity away from the boundary; cf. [31, 15, 29]. For such a localized initial vorticity [21] proved the following asymptotic expansion at the inviscid limit for a short time $T > 0$ (but T is independent of the viscosity):

$$(1.1)\quad \omega^{(\nu)}(t,x) = \omega_E(t,x) + \frac{1}{\nu^{\frac12}} w_P(t,x_1,\frac{x_2}{\nu^{\frac12}}) + \frac{1}{\nu^{\frac12}} w_{IP}^{(\nu)}(t,x_1,\frac{x_2}{\nu^{\frac12}}) + w_{II}^{(\nu)}(t,x).$$

Here $\omega^{(\nu)}$ is the vorticity field of the Navier-Stokes flows (NS_ν), ω_E is the vorticity field of the Euler flows (see (E) below), w_P is the vorticity field of the Prandtl flows (see (P) below), and the remainder parts $w_{IP}^{(\nu)}$, $w_{II}^{(\nu)}$ are of the order $\mathcal{O}(\nu^{1/2})$ in suitable norms. It should be noted here that, even if there is no vorticity near the boundary at the initial time, the vorticity is immediately created there and forms a vortex line along the boundary in positive time. From the Biot-Savart law the asymptotic expansion for the velocity field can be also obtained as follows.

Theorem 1.1 ([21, Theorem 1.1]). *Assume that the initial velocity $a = (a_1, a_2)$ belongs to $\dot{W}^{1,p}_{0,\sigma}(\mathbb{R}^2_+)$ for some $1 < p < 2$ and the initial vorticity $b = \partial_1 a_2 - \partial_2 a_1$ belongs to $W^{4,1}(\mathbb{R}^2_+) \cap W^{4,2}(\mathbb{R}^2_+)$. Assume also that*

$$(1.2)\qquad d_0 = \mathrm{dist}\ (\partial\mathbb{R}^2_+, \mathrm{supp}\ b) > 0.$$

Then there are positive constants C and T such that the following estimate holds for $0 < \nu \ll 1$.

$$(1.3)\qquad \sup_{0<t<T} \|u_{NS}^{(\nu)}(t) - u_E(t) - u_P^{(\nu)}(t)\|_{L^\infty(\mathbb{R}^2_+)} \le C\nu^{\frac12}.$$

Here $u_{NS}^{(\nu)}$ is the solution of (NS_ν), u_E is the solution of the Euler equations with the initial velocity a, and $u_P^{(\nu)}$ describes the boundary layer of the form

$$(1.4)\qquad u_P^{(\nu)}(t,x) = \big(v_{P,1}(t,x_1,\frac{x_2}{\nu^{\frac12}}),\ \nu^{\frac12} v_{P,2}(t,x_1,\frac{x_2}{\nu^{\frac12}})\big),$$

where $v_P = (v_{P,1}, v_{P,2})$ *is the solution of the (modified) Prandtl equations. Moreover,* T *is estimated from below as* $T \geq c \min\{d_0, 1\}$*, where* c *is a positive constant depending only on* $\|b\|_{W^{4,1}(\mathbb{R}^2_+) \cap W^{4,2}(\mathbb{R}^2_+)}$.

The space $\dot{W}^{1,p}_{0,\sigma}(\mathbb{R}^2_+)$ is the completion with respect to the norm $\|\nabla f\|_{L^p(\mathbb{R}^2_+)}$ of the space of all smooth, divergence-free vector fields with compact support in $\mathbb{R}^2_+$, and $W^{k,p}(\mathbb{R}^2_+)$ is a usual Sobolev space.

The velocity field $u_E = (u_{E,1}, u_{E,2})$ of the ideal incompressible flows is subject to the Euler equations

$$\text{(E)} \qquad \begin{cases} \partial_t u_E + u_E \cdot \nabla u_E + \nabla p_E = 0 & t > 0,\ x \in \mathbb{R}^2_+, \\ \operatorname{div} u_E = 0 & t \geq 0,\ x \in \mathbb{R}^2_+, \\ u_{E,2} = 0 & t \geq 0,\ x \in \partial\mathbb{R}^2_+, \\ u_E|_{t=0} = a & x \in \mathbb{R}^2_+. \end{cases}$$

Since the initial velocity a in Theorem 1.1 possesses an enough regularity the existence and the uniqueness of the classical solution of (E) are verified by the known approach [39, 41, 11, 4].

The Prandtl equations for the boundary layer profile $\tilde{v}_P = (\tilde{v}_{P,1}, \tilde{v}_{P,2})$ are written as follows.

(P)

$$\begin{cases} (\partial_t - \partial^2_{X_2})\tilde{v}_{P,1} + \tilde{v}_{P,1}\partial_1\tilde{v}_{P,1} + \tilde{v}_{P,2}\partial_{X_2}\tilde{v}_{P,1} + \partial_1\tilde{\pi}_P = 0 & t > 0,\ (x_1, X_2) \in \mathbb{R}^2_+ \\ \partial_1\tilde{v}_{P,1} + \partial_{X_2}\tilde{v}_{P,2} = 0,\ \partial_{X_2}\tilde{\pi}_P = 0 & t \geq 0,\ (x_1, X_2) \in \mathbb{R}^2_+ \\ \tilde{v}_P(t, x_1, 0) = 0 & t \geq 0,\ x_1 \in \mathbb{R}, \\ \lim_{X_2 \to \infty} \tilde{v}_{P,1}(t, x_1, X_2) = u_{E,1}(t, x_1, 0) & t \geq 0,\ x_1 \in \mathbb{R}, \\ \lim_{X_2 \to \infty} \tilde{\pi}_P(t, x_1, X_2) = p_E(t, x_1, 0) & t \geq 0,\ x_1 \in \mathbb{R}, \\ \tilde{v}_P|_{t=0} = 0 & (x_1, X_2) \in \mathbb{R}^2_+. \end{cases}$$

The velocity field $v_P = (v_{P,1}, v_{P,2})$ for the modified Prandtl equations is defined by $v_{P,1}(t, x_1, X_2) = \tilde{v}_{P,1}(t, x_1, X_2) - u_{E,1}(t, x_1, 0)$, $v_{P,2}(t, x_1, X_2) = \int_{X_2}^{\infty} \partial_1 v_{P,1}(t, x_1, Y_2)\, \mathrm{d}Y_2$; cf. [34]. Under the assumptions on the monotonicity of the data the solvability of the Prandtl equations is proved by [30, 25, 40] using the Crocco transformation, and recently also by [1, 23] whose proofs are based on a direct energy method. Without the monotonicity conditions so far we need the analyticity of the initial data to get the local-in-time solvability of the Prandtl equations [3, 33], and this analyticity is in fact required only in the tangential direction [17, 16]. The solvability of the Prandtl equations for general initial data in a Sobolev class is still an open issue, although the ill-posedness is strongly suggested. Indeed, for the linearized Prandtl equations the ill-posedness in the Sobolev framework is shown in [7].

The lower bound of T in Theorem 1.1 is of the order $\mathcal{O}(d_0)$ when d_0 is small. This order seems to be natural and optimal to ensure (1.3) in our setting, for the vorticity of the Euler flows keeps the distance $\mathcal{O}(d_0)$ from the boundary among the time period $0 \leq t \leq \mathcal{O}(d_0)$. After the time period ensured by Theorem 1.1 the separation of the boundary layer is expected to occur in general and the vorticity will exhibit rather complicated behaviors; [15, 29]. The mathematical description of these phenomena is a challenging problem.

The idea to establish the asymptotic expansion (1.3) is explained as follows. The proof is based on two key observations. Firstly we observe that the solution should be analytic at least near the boundary because so is at the initial time. Thus in our setting the solvability of the Prandtl equations is already ensured by the previous works. But we note here that the solvability of the Prandtl equations itself does not necessarily imply the desired asymptotic expansion, as in the counter example by [9]. Moreover, our solution should lose the analyticity as it leaves the boundary, and it is important to estimate how to lose it precisely. We overcome this difficulty by introducing a suitable weighted function space which represents this loss of analyticity. Secondly we use the fact that the vorticity field of the Euler flows satisfies the transport equations and hence its support is away from the boundary even in positive time. Then the vorticity of the Navier-Stokes flows is expected to be small exponentially in ν^{-1} in the region between the boundary layer and the support of the vorticity of the Euler flows. This implies that the strong and uncontrollable interaction does not occur between the vorticity produced in the boundary layer and the outer vorticity originated from the initial one, resulting the classical thickness $\mathcal{O}(\nu^{1/2})$ of the boundary layer at least for a short time. These two mechanisms, the analyticity near the boundary and the weak interaction between the boundary vorticity and the outer vorticity, exclude the possibility of the instability of the boundary layer observed by [9]. The approach based on the vorticity formulation is a key to reveal these mechanisms.

In the present article we recall the vorticity formulation in the next section and state three key lemmas used in [21] to prove Theorem 1.1; compatibility of weighted function spaces (Lemma 3.1), pointwise estimate of fundamental solutions to the heat-transport equations (Lemma 3.2), ACK theorem (Lemma 3.3). The ACK theorem, which itself is an interesting object of research, used in [21] is a slightly extended version of [28, 10]; see also [27, 32]. For convenience to the reader we give a proof of this ACK theorem in Section 3.3.

2. Results from vorticity formulation

2.1. **Vorticity equations.** Let $\omega = \mathrm{Rot}\ u = \partial_1 u_2 - \partial_2 u_1$ be the vorticity field. Then the Biot-Savart law in $\mathbb{R}^2_+$ is expressed as

$$u = J(\omega) = \big(J_1(\omega),\ J_2(\omega)\big) := \nabla^\perp(-\Delta_D)^{-1}\omega, \tag{2.1}$$

where $\nabla^\perp = (\partial_2, -\partial_1)$ and $h = (-\Delta_D)^{-1} f$ denotes the solution of the Poisson equation $-\Delta h = f$ in $\mathbb{R}^2_+$ subject to the Dirichlet boundary condition $h = 0$ on $\partial\mathbb{R}^2_+$. We introduce the bilinear forms

$$B(f,h) = J(f)\cdot\nabla h, \ \ N(f,h) = J_1\big(B(f,h)\big)\big|_{x_2=0}. \tag{2.2}$$

Then the vorticity equations for the Navier-Stokes flows are described as follows.

$$(\mathrm{V}_\nu) \qquad \begin{cases} \partial_t\omega - \nu\Delta\omega + B(\omega,\omega) = 0 & t>0,\ x\in\mathbb{R}^2_+, \\ \nu(\partial_2\omega + (-\partial_1^2)^{\frac12}\omega) = -N(\omega,\omega) & t>0,\ x\in\partial\mathbb{R}^2_+, \\ \omega|_{t=0} = b := \mathrm{Rot}\ a. & x\in\mathbb{R}^2_+. \end{cases}$$

The first equation of (V_ν) is obtained by taking the Rot in the first equation of (NS_ν). The boundary condition in (V_ν) is imposed so as to keep the no-slip boundary condition on $u = J(\omega)$ under the time-evolution of the vorticity field; cf. [2, 20].

The vorticity field of the Euler flows, denoted by ω_E, satisfies the equations

$$(\mathrm{V_E}) \qquad \begin{cases} \partial_t\omega_E + B(\omega_E,\omega_E) = 0 & t>0,\ x\in\mathbb{R}^2_+, \\ \omega_E|_{t=0} = b & x\in\mathbb{R}^2_+. \end{cases}$$

When $b \in W^{4,1}(\mathbb{R}^2_+)\cap W^{4,2}(\mathbb{R}^2_+)$ the global solvability of $(\mathrm{V_E})$ is classical and in particular we have $\omega_E \in C^1([0,T]\times\overline{\mathbb{R}^2_+})\cap L^\infty(0,T;W^{4,1}(\mathbb{R}^2_+)\cap W^{4,2}(\mathbb{R}^2_+))$ for any $T>0$. Moreover, the support condition (1.2) implies that

$$\cup_{0\le t\le T_0}\ \mathrm{supp}\ \omega_E(t) \subset \{x\in\mathbb{R}^2_+ \mid x_2 \ge 2^5 d_E\},\ d_E = \min\{2^{-6}d_0, 2^{-1}\} \tag{2.3}$$

for some $T_0 \ge C d_E$ with $C>0$ depending only on $\|b\|_{W^{4,1}\cap W^{4,2}}$.

By taking into account the asymptotic expansion at $\nu\to 0$ it is natural to define the vorticity field w_P of the Prandtl flows $\tilde{v}_P$ by the relation

$w_P = -\partial_2 \tilde v_{P,1}$. Thus the Biot-Savart law in this case is

(2.4)

$$\begin{aligned}\tilde v_{P,1}(t,x_1,X_2) &= v_{E,1}(t,x_1,X_2) + v_{P,1}(t,x_1,X_2)\\ &:= u_{E,1}(t,x_1,0) + \int_{X_2}^\infty w_P(t,x_1,Y_2)dY_2,\end{aligned}$$

(2.5)

$$\begin{aligned}\tilde v_{P,2}(t,x_1,X_2) &= v_{E,2}(t,x_1,X_2) + v_{P,2}(t,x_1,X_2)\\ &:= X_2\partial_2 u_{E,2}(t,x_1,0)\\ &\quad - \partial_1\Big(\int_0^{X_2} Y_2 w_P(t,x_1,Y_2)dY_2 + X_2\int_{X_2}^\infty w_P(t,x_1,Y_2)dY_2\Big).\end{aligned}$$

Set $\nabla_X = (\partial_1, \partial_{X_2})$. Then the equation for $w_P = w_P(t,x_1,X_2)$ is given by

$(\mathrm{V_P})$

$$\begin{cases}\partial_t w_P - \partial_{X_2}^2 w_P = -\tilde v_P\cdot\nabla_X w_P & t>0,\ (x_1,X_2)\in\mathbb{R}^2_+,\\ \partial_{X_2} w_P = -\int_0^\infty \tilde v_P\cdot\nabla_X w_P dY_2 - N(\omega_E,\omega_E) & t>0,\ (x_1,X_2)\in\partial\mathbb{R}^2_+,\\ w_P|_{t=0} = 0 \quad (x_1,X_2)\in\mathbb{R}^2_+.\end{cases}$$

The boundary condition of w_P in $(\mathrm{V_P})$ is observed in [2], or one can directly derive it from (V_ν) by performing the formal expansion $\omega(t,x) = \omega_E(t,x) + \nu^{-1/2} w_P(t,x_1,x_2/\nu^{1/2})$ + remainder. This boundary condition is actually replaced by $\partial_{X_2} w_P = -\partial_1 p_E$ in view of (P).

The key structure of the outer part w_{II} in (1.1) is that it satisfies the heat-transport equations with the homogeneous Neumann boundary condition

$$(\mathrm{V}_{\mathrm{II}_\nu})\qquad \begin{cases}\partial_t w_{II} - \nu\Delta w_{II} + B(\omega, w_{II}) = -B(\omega-\omega_E,\omega_E) + \nu\Delta\omega_E,\\ \partial_2 w_{II}|_{x_2=0} = 0,\\ w_{II}|_{t=0} = 0.\end{cases}$$

It should be emphasized that each term in the right-hand side of $(\mathrm{V}_{\mathrm{II}_\nu})$ is supported away from the boundary.

2.2. **Representation formula for solutions of the linearized problem.** In this section we recall the solution formula to the linear problem

$$(\mathrm{LV})\qquad \begin{cases}\partial_t\omega - \nu\Delta\omega = f & t>0,\ x\in\mathbb{R}^2_+,\\ \omega|_{t=0} = b & x\in\mathbb{R}^2_+,\end{cases}$$

subject to the boundary condition

$$(\mathrm{LBC})\qquad \nu\big(\partial_2 + (-\partial_1^2)^{\frac12}\big)\omega = g \qquad t>0,\ x\in\partial\mathbb{R}^2_+.$$

Here f, g, b are assumed to be smooth and decay fast enough at spatial infinity. We denote by G and E the two-dimensional Gaussian and Newton potential, respectively, i.e., $G(t,x) = (4\pi t)^{-1}\exp\big(-|x|^2/(4t)\big)$ and $E(x) =$

$-(2\pi)^{-1}\log|x|$. Let $*$ be the standard convolution in $\mathbb{R}^2$. Following [20], we set

$$\Gamma(t,x) = \big(\Xi E * G(t)\big)(x),\ \ \Xi = 2\big(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}}\partial_2\big). \tag{2.6}$$

We also use the notation $(h_1 \star h_2)(x) = \displaystyle\int_{\mathbb{R}^2_+} h_1(x-y^*)h_2(y)\,\mathrm{d}y$, where $y^* = (y_1, -y_2)$.

Lemma 2.1 ([20]). *The integral equation for (LV)-(LBC) is given by*

$$\begin{aligned}\omega(t) &= e^{\nu t\Delta_N}b + \Gamma(\nu t)\star b - \Gamma(0)\star b + \int_0^t e^{\nu(t-s)\Delta_N}\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s\\ &\quad + \int_0^t \Gamma(\nu(t-s))\star\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s\\ &\quad - \int_0^t \Gamma(0)\star\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s.\end{aligned} \tag{2.7}$$

Here $e^{t\Delta_N}$ is the semigroup for the heat equation (with the unit viscosity) in $\mathbb{R}^2_+$ subject to the homogeneous Neumann boundary condition, $\Gamma(0)\star := \lim_{t\downarrow 0}\Gamma(t)\star$, and $g\mathcal{H}^1_{\{x_2=0\}}$ is a one-dimensional Hausdorff measure with density g defined by $\langle h, g\mathcal{H}^1_{\{x_2=0\}}\rangle = \displaystyle\int_{\mathbb{R}} h(x_1,0)g(x_1)\,\mathrm{d}x_1$ for $h \in C_0(\overline{\mathbb{R}^2_+})$.

The formula (2.7) is a basic tool to define the solution mapping for the nonlinear problem (V_ν) and to establish various estimates of it. The reader is referred to [35, 37] for the solution formula of the (Navier-)Stokes equations. We note that $\Gamma(0)\star h = \Xi E \star h$ in $\mathbb{R}^2_+$.

2.3. **Function spaces.** One of the key ingredient in [21] is to set up a suitable family of Banach spaces. Recalling the definition of $d_E \in (0,1/2)$ in (2.3), we set

$$\varphi_P^{(\mu,\rho)}(\xi_1, X_2) = \exp\left(\frac{\mu|\xi_1|}{4} + \rho X_2^2\right), \tag{2.8}$$

$$\varphi_{IP,\nu}^{(\mu,\rho)}(\xi_1, X_2) = \exp\left(\frac{(\mu - \nu^{\frac{1}{2}}X_2)_+|\xi_1|}{4} + \rho X_2^2\right), \tag{2.9}$$

$$\varphi_{E,\nu}^{(\mu,\theta)}(\xi_1, x_2) = \exp\left(\frac{(\mu - x_2)_+|\xi_1|}{4} + \frac{\theta}{\nu}(6d_E - x_2)_+^2\right), \tag{2.10}$$

where $\mu,\rho,\theta \ge 0$ and $(\alpha)_+ = \max\{\alpha, 0\}$ for $\alpha \in \mathbb{R}$. Let

$$\langle\xi_1\rangle = (1+\xi_1^2)^{\frac{1}{2}},\ \ \hat{f}(\xi_1,x_2) = \mathcal{F}(f)(\xi_1,x_2) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} f(x_1,x_2)e^{-ix_1\xi_1}\,\mathrm{d}x_1. \tag{2.11}$$

We denote by $\|\hat{f}\|_{L^p_{\xi_1}L^q_{x_2}}$ the norm $\Big(\int_{\mathbb{R}}\big(\int_0^\infty |\hat{f}(\xi_1,x_2)|^q\,\mathrm{d}x_2\big)^{p/q}\,\mathrm{d}\xi_1\Big)^{1/p}$. We set

(2.12)

$$\|f\|_{X_P^{(\mu,\rho)}} = \sum_{k=0,1}\Big(\|\varphi_P^{(\mu,\rho)}X_2^{\frac{k}{2}}\langle\xi_1\rangle^2\hat{f}(\xi_1,X_2)\|_{L^2_{\xi_1}L^{1+k}_{X_2}} + \|\varphi_P^{(\mu,\rho)}X_2^{1+\frac{k}{2}}\langle\xi_1\rangle\partial_{X_2}\hat{f}(\xi_1,X_2)\|_{L^2_{\xi_1}L^{1+k}_{X_2}}\Big),$$

(2.13)

$$\|f\|_{X_{IP_\nu}^{(\mu,\rho)}} = \sum_{k=0,1}\Big(\|\varphi_{IP,\nu}^{(\mu,\rho)}X_2^{\frac{k}{2}}\langle\xi_1\rangle\hat{f}(\xi_1,X_2)\|_{L^2_{\xi_1}L^{1+k}_{X_2}} + \|\varphi_{IP,\nu}^{(\mu,\rho)}X_2^{1+\frac{k}{2}}\partial_{X_2}\hat{f}(\xi_1,X_2)\|_{L^2_{\xi_1}L^{1+k}_{X_2}}\Big),$$

(2.14)

$$\|f\|_{X_{E,\nu}^{(\mu,\theta)}} = \|\varphi_{E,\nu}^{(\mu,\theta)}\langle\xi_1\rangle\hat{f}(\xi_1,x_2)\|_{L^2_{\xi_1}L^2_{x_2}} + \|\varphi_{E,\nu}^{(\mu,\theta)}\partial_2\hat{f}(\xi_1,x_2)\|_{L^2_{\xi_1}L^2_{x_2}} + \|\varphi_{E,\nu}^{(0,\theta)}f\|_{L^1_x}.$$

The spaces $X_P^{(\mu,\rho)}$, $X_{IP,\nu}^{(\mu,\rho)}$, $X_{E,\nu}^{(\mu,\theta)}$, are then naturally defined as the subspaces of $L^2(\mathbb{R}^2_+)$ equipped with the norms $\|\cdot\|_{X_P^{(\mu,\rho)}}$, $\|\cdot\|_{X_{IP,\nu}^{(\mu,\rho)}}$, $\|\cdot\|_{X_{E,\nu}^{(\mu,\theta)}}$, respectively. The space $X_P^{(\mu,\rho)}$ is applied for w_P, and $X_{IP,\nu}^{(\mu,\rho)}$ and $X_{E,\nu}^{(\mu,\theta)}$ used for w_{IP} and w_{II}.

By the definition of the weights (2.9) - (2.10) the functions in $X_{IP,\nu}^{(\mu,\rho)}$ or $X_{E,\nu}^{(\mu,\theta)}$ with $\mu>0$ are analytic in the tangential direction near the boundary. The form $(\mu-x_2)_+|\xi_1|$ represents how the analyticity is lost as the function leaves the boundary, and $\nu^{-1}(6d_E-x_2)_+^2$ expresses the smallness exponentially in ν^{-1} near the boundary. The weight $X_2^{k/2}$ for the space $L^{1+k}_{X_2}$ in (2.12) - (2.13) reflects the relation with the scaling

$$(R_s f)(x) = s^{\frac{1}{2}}f(x_1,s^{\frac{1}{2}}x_2)\qquad s>0, \tag{2.15}$$

which seems to be important to make the estimates sharp and to derive the lower bound of T in Theorem 1.1. These weights are compatible with the heat equations and essential in our arguments; see Lemma 3.1. The counterpart of Theorem 1.1 in terms of the vorticity formulation is described as follows.

Theorem 2.1 ([21]). *There are C, T, μ, ρ, $\theta>0$ such that the solution $\omega_{NS}^{(\nu)}$ to* (V_ν) *is constructed in the form* (1.1), *where*

$$\sup_{0<t<T}\|w_P(t)\|_{X_P^{(\mu,\rho/t)}}\le 1$$

$$\sup_{0<t<T}\|w_{IP}^{(\nu)}(t)\|_{X_{IP,\nu}^{(\mu,\rho/t)}} + \sup_{0<t<T}\|w_{II}^{(\nu)}(t)\|_{X_{E,\nu}^{(\mu,\theta/t)}}\le C\nu^{\frac{1}{2}}.$$

3. Key lemmas

3.1. **Invariant property of function spaces under the action of the heat semigroup.** In view of the solution formula (2.7) it is essential to establish the estimates for the heat semigroup $\{e^{\nu t\Delta_N}\}_{t\geq 0}$ in our functional setting.

Lemma 3.1 ([21, Proposition 3.1]). *Let $t > s \geq 0$, $\mu \geq 0$, $0 \leq \rho \leq 2^{-4}$, and $0 \leq \theta \leq 2^{-4}$. Then it follows that*

$$\|\varphi_P^{(\mu,\frac{\rho}{t})}\mathcal{F}\big(R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f\big)\|_{L^2_{\xi_1}L^1_{X_2}} \leq C\,\|\varphi_P^{(\mu,\frac{\rho}{s})}\mathcal{F}(f)\|_{L^2_{\xi_1}L^1_{X_2}}, \tag{3.1}$$

$$\|\varphi_{IP,\nu}^{(\mu,\frac{\rho}{t})}\mathcal{F}\big(R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f\big)\|_{L^2_{\xi_1}L^1_{X_2}} \leq C\,\|\varphi_{IP,\nu}^{(\mu,\frac{\rho}{s})}\mathcal{F}(f)\|_{L^2_{\xi_1}L^1_{X_2}}, \tag{3.2}$$

$$\|\varphi_{E,\nu}^{(\mu,\frac{\theta}{t})}\mathcal{F}\big(e^{\nu(t-s)\Delta_N}f\big)\|_{L^2_{\xi_1}L^2_{x_2}} \leq C\,\|\varphi_{E,\nu}^{(\mu,\frac{\theta}{s})}\mathcal{F}(f)\|_{L^2_{\xi_1}L^2_{x_2}}. \tag{3.3}$$

Remark 3.1. The proof of Lemma 3.1 implies that

$$\begin{aligned}
\sup_{0<t<T}\|R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f\|_{X_P^{(\mu,\rho/t)}} &\leq C\sup_{0<t<T}\|f\|_{X_P^{(\mu,\rho/t)}},\\
\sup_{0<t<T}\|R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f\|_{X_{IP,\nu}^{(\mu,\rho/t)}} &\leq C\sup_{0<t<T}\|f\|_{X_{IP,\nu}^{(\mu,\rho/t)}},\\
\sup_{0<t<T}\|e^{\nu(t-s)\Delta_N}f\|_{X_{E,\nu}^{(\mu,\theta/t)}} &\leq C\sup_{0<t<T}\|f\|_{X_{E,\nu}^{(\mu,\theta/t)}}.
\end{aligned}$$

That is, the function spaces described in Theorem 2.1 are invariant under the action of the heat semigroup.

Sketch of the proof of Lemma 3.1. Here we give a sketch of the proof only for (3.2). The other estimates are obtained in the similar manner. Set $g(t,X_2) = (4\pi t)^{-1/2}\exp\big(-X_2^2/(4t)\big)$. Then

$$|\mathcal{F}\big(R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f\big)(\xi_1,X_2)| \lesssim e^{-\nu(t-s)\xi_1^2}\int_0^\infty g(t-s,X_2-Y_2)|\hat{f}(\xi_1,Y_2)|\,\mathrm{d}Y_2.$$

From the inequalities

$$\begin{aligned}
(\mu-\nu^{\frac{1}{2}}X_2)_+|\xi_1| &\leq (\mu-\nu^{\frac{1}{2}}Y_2)_+|\xi_1| + \nu^{\frac{1}{2}}|X_2-Y_2||\xi_1|,\\
\nu^{\frac{1}{2}}|X_2-Y_2|\,|\xi_1| &\leq \nu(t-s)\xi_1^2 + \frac{|X_2-Y_2|^2}{4(t-s)},
\end{aligned}$$

we have

$$\begin{aligned}
|\mathcal{F}\big(R_\nu e^{\nu(t-s)\Delta}R_{\frac{1}{\nu}}f\big)(\xi_1,X_2)| &\lesssim e^{-\frac{3}{4}\nu(t-s)\xi_1^2-\frac{1}{4}(\mu-\nu^{\frac{1}{2}}X_2)_+|\xi_1|}\\
&\quad\cdot\int_0^\infty g(2(t-s),X_2-Y_2)e^{-\frac{\rho}{s}Y_2^2}|(\varphi_{IP,\nu}^{(\mu,\frac{\rho}{s})}\hat{f})(\xi_1,Y_2)|\,\mathrm{d}Y_2.
\end{aligned}$$

Thus the desired estimate follows by applying the inequality ([21, Lemma 7.1])

$$\|e^{\frac{\beta}{t}X_2^2} g(t-s) * h(X_2)\|_{L^1_{X_2}} \lesssim \|e^{\frac{\beta}{s}X_2^2} h(X_2)\|_{L^1_{X_2}}, \quad 0 < \beta < \frac{1}{4},$$

and then by taking the L^2 norm with respect to ξ_1. The proof is complete.

3.2. **Fundamental solution to the heat-transport equations.** To establish Theorem 2.1 the estimate of the influence on the boundary vorticity by the outer vorticity is the most important issue and requires the mathematical technicality. In particular, it is important to obtain a sharp pointwise estimate for solutions to $(\mathrm{V}_{\mathrm{II}_\nu})$ near the boundary. For this purpose the following lemma on the fundamental solution to the heat-transport equations is used in [21]. Set

$$H^{(\nu)}(t) = -B(\omega - \omega_E, \omega_E) + \nu\Delta\omega_E.$$

Lemma 3.2 ([21, Lemma 7.2]). *We denote by $P_u^{(\nu)}(t,s)$ the evolution operator for $\partial_t - \nu\Delta + u\cdot\nabla$ in $\mathbb{R}^2_+$ with the homogeneous Neumann boundary condition. Then the solution $w_{II}^{(\nu)}$ to $(\mathrm{V}_{\mathrm{II}_\nu})$ is represented as*

$$w_{II}^{(\nu)}(t) \;=\; \int_0^t P_{u_{NS}}^{(\nu)}(t,s) H^{(\nu)}(s)\,\mathrm{d}s,$$

and the kernel of $P_{u_{NS}}^{(\nu)}(t,s)$ satisfies

$$0 < P_{u_{NS}}^{(\nu)}(t,x;s,y) \le \frac{1}{2\pi\nu(t-s)} \exp\left(-\frac{\left(|x-y| - \int_s^t \|u_{NS}(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right)_+^2}{4\nu(t-s)}\right).$$

Remark 3.2. We note that the support of $H^{(\nu)}(t)$ is away from the boundary when the initial vorticity is located away from the boundary. The above pointwise estimate then yields the exponential smallness of $w_{II}^{(\nu)}$ in ν^{-1} near the boundary.

3.3. **Abstract Cauchy-Kowalewski theorem.** Let $\mu_0 \in (0,1)$. We assume that there are two-parameter families of Banach spaces $\{X_\mu^t\}_{0<\mu,t\le\mu_0}$ and $\{Y_\mu^t\}_{0<\mu,t\le\mu_0}$ such that

$$\begin{aligned} &X_\mu^t \hookrightarrow Y_\mu^t \qquad \text{for all} \quad 0 < \mu \le \mu_0, \\ &X_{\mu_2}^t \hookrightarrow X_{\mu_1}^t \quad \text{and} \quad Y_{\mu_2}^t \hookrightarrow Y_{\mu_1}^t \qquad \text{if} \quad \mu_1 \le \mu_2. \end{aligned}$$

Here $\hookrightarrow$ represents the continuous embedding. We consider the integral equation of the form

$$w(t) = \int_0^t \Lambda(t,s,w)\,\mathrm{d}s + F(t), \tag{3.4}$$

where F is a given function satisfying

(3.5)
$$\sup_{0<t\le\mu_0}\Big(\frac{\mu_0}{t}\Big)^m \sup_{0<s<t}\|F(s)\|_{X^s_{\mu_0}} \le R < \infty \qquad \text{for some } R>0 \text{ and } m\in(0,1).$$

For a time dependent function $f = f(t)$ we set
$$\|f\|_{X_\mu(t)} = \sup_{0<s<t}\|f(s)\|_{X^s_\mu}, \qquad \|f\|_{Y_\mu(t)} = \sup_{0<s<t}\|f(s)\|_{Y^s_\mu}.$$

In order to construct the remainder terms $w^{(\nu)}_{IP}$, $w^{(\nu)}_{II}$ in (1.1) the abstract Cauchy-Kowalewski theorem of the following type is used. The new ingredient is that the topology for the convergence of the iteration sequence has to be weaker than the one for the uniform bound, in order to handle the hyperbolic nature of the equations at the inviscid limit and the lack of the analyticity away from the boundary.

Lemma 3.3. *Let $R>0$ and $m\in(0,1)$ be the numbers in (3.5). Assume that there are positive constants C_1, C_2, σ_1, and σ_2 such that $m<\sigma_i\le 1$, $i=1,2$, and that the following statement holds: if*
$$\sup_{\frac{\mu_0}{2}\le\mu<\mu_0}\ \sup_{0<s<c(1-\frac{\mu}{\mu_0})}\|v\|_{X_\mu(s)}\Big(\frac{c(1-\frac{\mu}{\mu_0})}{s}-1\Big)^m + \sup_{0<s<\frac{c}{2}}\|v\|_{X_{\frac{\mu_0}{4}}(s)} \le 8R,$$
$$\sup_{\frac{\mu_0}{2}\le\mu<\mu_0}\ \sup_{0<s<c(1-\frac{\mu}{\mu_0})}\|w\|_{X_\mu(s)}\Big(\frac{c(1-\frac{\mu}{\mu_0})}{s}-1\Big)^m + \sup_{0<s<\frac{c}{2}}\|w\|_{X_{\frac{\mu_0}{4}}(s)} \le 8R$$
hold for a fixed $c\in(0,\mu_0)$ then

(3.6)
$$\|\Lambda(t,s,w)\|_{X^t_{\mu'}} \le C_1\Big(\frac{1}{\mu-\mu'}+\frac{1}{(\mu-\mu')^{\sigma_1}(t-s)^{1-\sigma_1}}\Big)\|w\|_{X_\mu(s)} + h(t,s),$$

(3.7)
$$\begin{aligned}&\|\Lambda(t,s,v)-\Lambda(t,s,w)\|_{Y^t_{\mu'}}\\ &\le C_2\Big(\frac{1}{\mu-\mu'}+\frac{1}{(\mu-\mu')^{\sigma_2}(t-s)^{1-\sigma_2}}\Big)\|v-w\|_{Y_\mu(s)},\end{aligned}$$

for $\mu_0/4\le\mu'<\mu<\mu_0$ and $0<t<c(1-\mu/\mu_0)$. Here $h(t,s)$ is assumed to be a nonnegative function satisfying

$$\int_0^t h(t,s)\,ds \le \Big(\frac{t}{\mu_0}\Big)^m R. \tag{3.8}$$

Under the above assumptions there is $T_0\in(0,\mu_0)$ such that there exists a unique solution w to (3.4) satisfying
$$\sup_{1/2\le\kappa<1}\ \sup_{0<t<T_0(1-\kappa)}\|w\|_{X_{\kappa\mu_0}(t)}\Big(\frac{T_0(1-\kappa)}{t}-1\Big)^m + \sup_{0<t<T_0/2}\|w\|_{X_{\frac{\mu_0}{4}}(t)} \le 8R.$$

Proof. As usual, we consider the iteration sequence $\{w^{(k)}\}$ defined by

$$w^{(0)}(t) = F(t), \qquad w^{(k+1)}(t) = \int_0^t \Lambda(t, s, w^{(k)})\, ds + w^{(0)}(t).$$

Then for a fixed $\gamma_0 \in (0, \mu_0)$ we set $\gamma_{k+1} = \gamma_k\big(1 - (k+2)^{-2}\big)$ and $\gamma = \lim_{k\to\infty} \gamma_k = \gamma_0 \Pi_{k=0}^{\infty}\big(1 - (k+2)^{-2}\big) > 0$. We also set

$$\lambda_k = \sup_{\frac{1}{2}\le\kappa<1} \sup_{0<t<\gamma_k(1-\kappa)} \|w^{(k)}\|_{X_{\kappa\mu_0}(t)}\big(\frac{\gamma_k(1-\kappa)}{t} - 1\big)^m,$$

$$\eta_k = \sup_{0<t<\gamma_k/2} \|w^{(k)}\|_{X_{\frac{\mu_0}{4}}(t)},$$

$$\zeta_k = \sup_{\frac{1}{2}\le\kappa<1} \sup_{0<t<\gamma_k(1-\kappa)} \|w^{(k+1)} - w^{(k)}\|_{Y_{\kappa\mu_0}(t)}\big(\frac{\gamma_k(1-\kappa)}{t} - 1\big)^m.$$

We will show that if γ_0 is sufficiently small then $\lambda_k \le 4R$, $\eta_k \le 4R$, and $\zeta_k \le \delta_0^k \zeta_0$ for all k and for some $\delta_0 \in (0,1)$. First we consider λ_k and η_k. The case $k = 0$ is clear from the assumption on F. Assume that the estimates hold for k. Then we see from $\gamma_{k+1} < \gamma_k$ that

$$\sup_{\frac{\mu_0}{2}\le\mu<\mu_0} \sup_{0<s<\gamma_{k+1}(1-\frac{\mu}{\mu_0})} \|w^{(k)}\|_{X_\mu(s)}\big(\frac{\gamma_{k+1}(1-\frac{\mu}{\mu_0})}{s} - 1\big)^m + \sup_{0<s<\frac{\gamma_{k+1}}{2}} \|w^{(k)}\|_{X_{\frac{\mu_0}{4}}(s)}$$
$$\le 8R.$$

Hence we have for $1/4 \le \kappa < 1$ and $0 < t < \gamma_{k+1}(1-\kappa)$,

(3.9)

$$\|w^{(k+1)}(t)\|_{X^t_{\kappa\mu_0}}$$
$$\le \frac{C_1}{\mu_0}\int_0^t \big(\frac{1}{\kappa(s)-\kappa} + \frac{\mu_0^{1-\sigma_1}}{(\kappa(s)-\kappa)^{\sigma_1}(t-s)^{1-\sigma_1}}\big)\|w^{(k)}\|_{X_{\kappa(s)\mu_0}(s)}\, ds + (\frac{t}{\mu_0})^m R.$$

Here $\kappa(s)$ has to be chosen so that $\kappa < \kappa(s) < 1$ and $s < \gamma_k(1-\kappa(s))$. First let us take $1/2 \le \kappa < 1$ and $\kappa(s) = 2^{-1}\big(1 - s/\gamma_{k+1} + \kappa\big)$. Then we have from $\|w^{(k)}\|_{X_{\kappa(s)\mu_0}(s)} \le \big(\gamma_k(1-\kappa(s))/s - 1\big)^{-m}\lambda_k$ and $\gamma_{k+1} < \gamma_k$,

$$\int_0^t \big(\frac{1}{\kappa(s)-\kappa} + \frac{\mu_0^{1-\sigma_1}}{(\kappa(s)-\kappa)^{\sigma_1}(t-s)^{1-\sigma_1}}\big)\|w^{(k)}\|_{X_{\kappa(s)\mu_0}(s)}\, ds$$
$$\le C\lambda_k\big(\frac{t}{\gamma_{k+1}(1-\kappa)-t}\big)^m(\gamma_{k+1} + \mu_0^{1-\sigma_1}\gamma_{k+1}^{\sigma_1}).$$

Thus by taking $\gamma_0 = \epsilon_0\mu_0$ with sufficiently small $\epsilon_0 \in (0,1)$, we get $\lambda_{k+1} \le 4R$. Next we take $\kappa = 1/4$ and $\kappa(s) = 2^{-1}(3/2 - s/\gamma_{k+1})$ in (3.9). Then

$\kappa(s)-\kappa\geq 1/4$ for $s<\gamma_{k+1}/2$, and thus, when $0<t<\gamma_{k+1}/2$ we have

$$\begin{aligned}
&\int_0^t \Big(\frac{1}{\kappa(s)-\kappa}+\frac{\mu_0^{1-\sigma_1}}{(\kappa(s)-\kappa)^{\sigma_1}(t-s)^{1-\sigma_1}}\Big)\|w^{(k)}\|_{X_{\kappa(s)\mu_0}(s)}\,\mathrm{d}s\\
&\leq C\lambda_k\int_0^t \Big(1+\frac{\mu_0^{1-\sigma_1}}{(t-s)^{1-\sigma_1}}\Big)\Big(\frac{s}{\gamma_k(1-\kappa(s))-s}\Big)^m\,\mathrm{d}s\\
&\leq C\lambda_k t^m\int_0^t \Big(1+\frac{\mu_0^{1-\sigma_1}}{(t-s)^{1-\sigma_1}}\Big)(t-s)^{-m}\,\mathrm{d}s\leq C\lambda_k t^m(t+\mu_0^{1-\sigma_1}t^{\sigma_1-m}),
\end{aligned}$$

for $0<m<\sigma_1$. Thus $\eta_{k+1}\leq 4R$ holds by taking $\gamma_0=\epsilon_0\mu_0$ with sufficiently small $\epsilon_0\in(0,1)$. By the induction on k we have now achieved the desired estimates of λ_k and η_k. Next we estimate ζ_k. Let $1/2\leq\kappa<1$, $0<t<\gamma_{k+1}(1-\kappa)$. By the assumption we have

$$\begin{aligned}
&\|w^{(k+2)}(t)-w^{(k+1)}(t)\|_{Y^t_{\kappa\mu_0}}\\
&\leq\frac{C_2}{\mu_0}\int_0^t \Big(\frac{1}{\kappa(s)-\kappa}+\frac{\mu_0^{1-\sigma_2}}{(\kappa(s)-\kappa)^{\sigma_2}(t-s)^{1-\sigma_2}}\Big)\|w^{(k+1)}-w^{(k)}\|_{Y_{\kappa(s)\mu_0}(s)}\,\mathrm{d}s.
\end{aligned}$$

Let us take $\kappa(s)=2^{-1}(1-s/\gamma_{k+1}+\kappa)$, which is larger than κ and less than 1. Then the similar calculation as in the case of λ_k implies that $\zeta_{k+1}\leq\delta\zeta_k$ for some $\delta\in(0,1)$ if $\gamma=\epsilon_0\mu_0$ with small ϵ_0. Collecting these, we see that $\{w^{(k)}\}$ is a Cauchy sequence in the space endowed with the norm $\|f\|=\sup_{1/2\leq\kappa<1}\sup_{0<t<\gamma(1-\kappa)}\|f\|_{Y_{\kappa\mu_0}(t)}\big(\gamma(1-\kappa)/t-1\big)^m$. Thus there is a limit w of $\{w^{(k)}\}$ with $\|w\|<\infty$. Moreover, from the uniform bound of λ_k and η_k we also have

$$\sup_{1/2\leq\kappa<1}\ \sup_{0<t<\gamma(1-\kappa)}\|w\|_{X_{\kappa\mu_0}(t)}\Big(\frac{\gamma(1-\kappa)}{t}-1\Big)^m+\sup_{0<t<\gamma/2}\|w\|_{X_{\frac{\mu_0}{4}}(t)}\leq 8R.$$

It is not difficult to see that w satisfies (3.4) for each $0<t<\gamma/2$. The uniqueness of solutions satisfying the above estimate is proved by using the topology of Y^t_μ and the details are omitted here. The proof is complete.

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