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Global Classical and Weak Solutions to the Three-Dimensional Full Compressible Navier-Stokes System with Vacuum and Large Oscillations *

Xiangdi HUANG[a], Jing LI[b]

[a] NCMIS, Academy of Mathematics and System Sciences,
Chinese Academy of Sciences, Beijing 100190,P. R. China

[b] Institute of Applied Mathematics, AMSS,
& Hua Loo-Keng Key Laboratory of Mathematics,
Chinese Academy of Sciences, Beijing 100190, P. R. China

The motion of compressible viscous, heat-conductive polytropic fluid reads

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(u_t + u\cdot\nabla u) = \mu\Delta u + (\mu+\lambda)\nabla(\operatorname{div} u) - \nabla P, \\ \frac{R}{\gamma-1}\rho(\theta_t + u\cdot\nabla\theta) = \kappa\Delta\theta - P\operatorname{div}u + \lambda(\operatorname{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2, \end{cases} \tag{0.1}$$

where $\mathfrak{D}(u)$ is the deformation tensor:

$$\mathfrak{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{tr}}).$$

Here $\rho, u = \left(u^1, u^2, u^3\right)^{\mathrm{tr}}, e, P(\rho, e)$, and θ represent respectively the fluid density, velocity, specific internal energy, pressure, and absolute temperature. The constant viscosity coefficients μ and λ satisfy the physical restrictions:

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. \tag{0.2}$$

We study the ideal polytropic fluids so that P and e are given by the state equations:

$$P(\rho, e) = (\gamma-1)\rho e = R\rho\theta, \quad e = \frac{R\theta}{\gamma-1}, \tag{0.3}$$

where $\gamma > 1$ is the adiabatic constant, and R, κ are both positive constants.

Let $\tilde\rho, \tilde\theta$ both be fixed positive constants. We look for the solutions $(\rho(x,t), u(x,t), \theta(x,t))$, with the far field behavior:

$$(\rho, u, \theta)(x,t) \to (\tilde\rho, 0, \tilde\theta), \quad \text{as } |x|\to\infty,\ t>0, \tag{0.4}$$

$$(\rho, \rho u, \rho\theta)(x, t=0) = (\rho_0, \rho_0 u_0, \rho_0\theta_0)(x), \quad x\in\mathbb{R}^3, \tag{0.5}$$

Moreover, for classical solutions, we replace the initial condition with

$$(\rho, u, \theta)(x, t=0) = (\rho_0, u_0, \theta_0), \quad x\in\mathbb{R}^3, \quad \text{with } \rho_0\geq 0, \theta_0 \geq 0. \tag{0.6}$$

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We assume that $\tilde{\rho} = \tilde{\theta} = 1.$ and define the initial energy C_0 as follows:

$$
\begin{aligned}
C_0 \triangleq & \frac{1}{2} \int \rho_0 |u_0|^2 dx + R \int (\rho_0 \log \rho_0 - \rho_0 + 1) \, dx \\
& + \frac{R}{\gamma - 1} \int \rho_0 \, (\theta_0 - \log \theta_0 - 1) \, dx + \frac{R}{2(\gamma - 1)} \int \rho_0 (\theta_0 - 1)^2 dx.
\end{aligned} \tag{0.7}
$$

Then the first main result in this paper can be stated as follows:

Theorem 0.1 *For given numbers $M > 0$ (not necessarily small), $q \in (3, 6)$, and $\bar{\rho} > 2$, suppose that the initial data (ρ_0, u_0, θ_0) satisfies*

$$
\rho_0 - 1 \in H^2 \cap W^{2,q}, \quad u_0 \in H^2, \quad \theta_0 - 1 \in H^2, \tag{0.8}
$$

$$
0 \le \inf \rho_0 \le \sup \rho_0 < \bar{\rho}, \quad \inf \theta_0 \ge 0, \quad \|\nabla u_0\|_{L^2} \le M, \tag{0.9}
$$

and the compatibility conditions:

$$
-\mu \Delta u_0 - (\mu + \lambda) \nabla \mathrm{div} u_0 + R \nabla (\rho_0 \theta_0) = \sqrt{\rho_0} g_1, \tag{0.10}
$$

$$
\kappa \Delta \theta_0 + \frac{\mu}{2} |\nabla u_0 + (\nabla u_0)^{\mathrm{tr}}|^2 + \lambda (\mathrm{div} u_0)^2 = \sqrt{\rho_0} g_2, \tag{0.11}
$$

with $g_1, g_2 \in L^2$. Then there exists a positive constant ε depending only on μ, λ, κ, R, γ, $\bar{\rho}$, and M such that if

$$
C_0 \le \varepsilon, \tag{0.12}
$$

the Cauchy problem (0.1) (0.4) (0.6) has a unique global classical solution (ρ, u, θ) in $\mathbb{R}^3 \times (0, \infty)$ satisfying

$$
0 \le \rho(x, t) \le 2\bar{\rho}, \quad \theta(x, t) \ge 0, \quad x \in \mathbb{R}^3, \, t \ge 0, \tag{0.13}
$$

$$
\begin{cases}
\rho - 1 \in C([0, T]; H^2 \cap W^{2,q}), \quad (u, \theta - 1) \in C([0, T]; H^2), \\
u \in L^\infty(\tau, T; H^3 \cap W^{3,q}), \quad \theta - 1 \in L^\infty(\tau, T; H^4), \\
(u_t, \theta_t) \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1),
\end{cases} \tag{0.14}
$$

and the following large-time behavior:

$$
\lim_{t \to \infty} (\|\rho(\cdot, t) - 1\|_{L^p} + \|\nabla u(\cdot, t)\|_{L^r} + \|\nabla \theta(\cdot, t)\|_{L^r}) = 0, \tag{0.15}
$$

with any

$$
0 < \tau < T < \infty, \quad p \in (2, \infty), \quad r \in [2, 6). \tag{0.16}
$$

The next result of this paper will treat the weak solutions. To begin with, we give the definition of weak solutions.

Definition 0.1 *We say that $(\rho, u, E = \frac{1}{2}|u|^2 + \frac{R}{\gamma - 1}\theta)$ is a weak solution to Cauchy problem (0.1) (0.4) (0.5) provided that*

$$
\rho - 1 \in L^\infty_{\mathrm{loc}}([0, \infty); L^2 \cap L^\infty(\mathbb{R}^3)), \quad u, \theta - 1 \in L^2(0, \infty; H^1(\mathbb{R}^3)),
$$

and that for all test functions $\psi \in \mathcal{D}(\mathbb{R}^3 \times (-\infty, \infty))$,

$$
\int_{\mathbb{R}^3} \rho_0 \psi(\cdot, 0) dx + \int_0^\infty \int_{\mathbb{R}^3} (\rho \psi_t + \rho u \cdot \nabla \psi) \, dx dt = 0, \tag{0.17}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \rho_0 u_0^j \psi(\cdot,0)dx + \int_0^\infty \int_{\mathbb{R}^3} \left(\rho u^j \psi_t + \rho u^j u \cdot \nabla\psi + P(\rho,\theta)\psi_{x_j}\right) dxdt \\
&- \int_0^\infty \int_{\mathbb{R}^3} \left(\mu \nabla u^j \cdot \nabla\psi + (\mu+\lambda)(\mathrm{div} u)\psi_{x_j}\right) dxdt = 0, \quad j = 1,2,3,
\end{aligned} \tag{0.18}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left(\frac{1}{2}\rho_0|u_0|^2 + \frac{R}{\gamma-1}\rho_0\theta_0\right) \psi(\cdot,0)dx \\
&= \int_0^\infty \int_{\mathbb{R}^3} (\rho E\psi_t + (\rho E + P)u \cdot \nabla\psi)\, dxdt \\
&\quad - \int_0^\infty \int_{\mathbb{R}^3} \left(\kappa\nabla\theta + \frac{1}{2}\mu\nabla(|u|^2) + \mu u \cdot \nabla u + \lambda u \mathrm{div} u\right) \cdot \nabla\psi dxdt.
\end{aligned} \tag{0.19}
$$

We also define

$$
\dot{f} \triangleq f_t + u \cdot \nabla f, \quad G \triangleq (2\mu+\lambda)\mathrm{div} u - R(\rho\theta - 1), \quad \omega \triangleq \nabla \times u, \tag{0.20}
$$

which are the material derivative of f, the effective viscous flux, and the vorticity respectively. We now state our second main result as follows:

Theorem 0.2 *For given numbers $M > 0$ (not necessarily small), and $\bar{\rho} > 2$, there exists a positive constant ε depending only on μ, λ, κ, R, γ, $\bar{\rho}$, and M such that if the initial data (ρ_0, u_0, θ_0) satisfies (0.9) and*

$$
C_0 \le \varepsilon, \tag{0.21}
$$

with C_0 as in (0.7), there is a global weak solution (ρ, u, θ) to the Cauchy problem (0.1) (0.4) (0.5) satisfying

$$
\rho - 1 \in C([0,\infty); L^2 \cap L^p), \quad (\rho u,\ \rho|u|^2,\ \rho(\theta-1)) \in C([0,\infty); H^{-1}), \tag{0.22}
$$

$$
u \in C((0,\infty); L^2), \quad \theta - 1 \in C((0,\infty); W^{1,r}), \tag{0.23}
$$

$$
u(\cdot,t),\ \omega(\cdot,t),\ G(\cdot,t),\ \nabla\theta(\cdot,t) \in H^1, \quad t > 0, \tag{0.24}
$$

$$
\rho \in [0, 2\bar{\rho}] \quad \text{a.e.}, \quad \theta \ge 0 \quad \text{a.e.}, \tag{0.25}
$$

and the following large-time behavior:

$$
\lim_{t\to\infty} \left(\|\rho(\cdot,t) - 1\|_{L^p} + \|u(\cdot,t)\|_{L^p\cap L^\infty} + \|\nabla\theta(\cdot,t)\|_{L^r}\right) = 0, \tag{0.26}
$$

with any p, r as in (0.16). *In addition, there exists some positive constant C depending only on μ, λ, κ, R, γ, $\bar{\rho}$, and M, such that, for $\sigma(t) \triangleq \min\{1,t\}$, the following estimates hold*

$$
\sup_{t\in(0,\infty)} \|u\|_{H^1} + \int_0^\infty \int |(\rho u)_t + \mathrm{div}(\rho u \otimes u)|^2\, dxdt \le C, \tag{0.27}
$$

$$
\begin{aligned}
&\sup_{t\in(0,\infty)} \int \left((\rho-1)^2 + \rho|u|^2 + \rho(\theta-1)^2\right) dx \\
&\quad + \int_0^\infty \left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) dt \le C C_0^{1/4},
\end{aligned} \tag{0.28}
$$

$$
\begin{aligned}
&\sup_{t\in(0,\infty)} \left(\sigma^2\|\nabla u\|_{L^6}^2 + \sigma^4\|\theta - 1\|_{H^2}^2\right) \\
&+ \int_0^\infty \left(\sigma^2\|u_t\|_{L^2}^2 + \sigma^2\|\nabla\dot{u}\|_{L^2}^2 + \sigma^4\|\theta_t\|_{H^1}^2\right) dt \le C C_0^{1/8}.
\end{aligned} \tag{0.29}
$$

Corollary 0.3 ([12]) *In addition to the conditions of Theorem 0.1, assume further that there exists some point $x_0 \in \mathbb{R}^3$ such that $\rho_0(x_0) = 0$. Then the unique global classical solution (ρ, u, θ) to the Cauchy problem (0.1) (0.4) (0.6) obtained in Theorem 0.1 has to blow up as $t \to \infty$, in the sense that for any $r > 3$,*

$$\lim_{t\to\infty} \|\nabla\rho(\cdot,t)\|_{L^r} = \infty.$$

A few remarks are in order:

Remark 0.1 *Theorem 0.1 is the first result concerning the global existence of classical solutions with vacuum to the full compressible Navier-Stokes system. Moreover, the conclusions in Theorem 0.1 generalize the classical theory of Matsumura-Nishida ([17]) to the case of large oscillations since in this case, the requirement of small energy, (0.12), is equivalent to smallness of the mean-square norm of $(\rho_0 - 1, u_0, \theta_0 - 1)$. In addition, the initial density is allowed to vanish and the initial temperature may be zero. However, although the large-time asymptotic behavior (0.15) is similar to that in [17], yet our solution may contain vacuum states, whose appearance leads to the large time blowup behavior stated in Corollary 0.3, this is in sharp contrast to that in [17] where the gradients of the density are suitably small uniformly for all time.*

Remark 0.2 *It should be noted here that Theorem 0.2 is the first result concerning the global existence of weak solutions to (0.1) in the presence of vacuum and extends the global weak solutions of Hoff ([10]) to the case of large oscillations and non-negative initial density. Moreover, the initial temperature is allowed to be zero.*

Remark 0.3 *Simple calculations yield that if*

$$\sup_{x\in\mathbb{R}^3} \theta_0(x) \le \bar\theta, \tag{0.30}$$

we have

$$\int \rho_0(\theta_0-1)^2 dx \le 2(\bar\theta+1)\int \rho_0\left(\theta_0 - \log\theta_0 - 1\right)dx,$$

which implies $\tilde C_0 \le C_0 \le (\bar\theta+2)\tilde C_0$, where

$$\begin{aligned}\tilde C_0 \triangleq & \frac12\int \rho_0|u_0|^2dx + R\int\left(\rho_0\log\rho_0 - \rho_0 + 1\right)dx \\ & + \frac{R}{\gamma-1}\int \rho_0\left(\theta_0 - \log\theta_0 - 1\right)dx\end{aligned}$$

is the usual initial energy. In other words, if we replace C_0 with the usual initial energy $\tilde C_0$, the ε in Theorems 0.1 and 0.2 will also depend on the upper bound of the initial temperature.

We now comment on the analysis of this paper. Note that though the local existence and uniqueness of strong solutions to (0.1) in the presence of vacuum was obtained by Cho-Kim ([6]), the local existence of classical solutions with vacuum to (0.1) still remains unknown. Some of the main new difficulties to obtain the classical solutions to (0.1) (0.4) (0.6) for initial data in the class satisfying (0.8)–(0.11) are due to the appearance of vacuum. Thus, we take the strategy that we first extend the standard local classical solutions with strictly positive initial density (see Lemma 1.1) globally in time just under the condition that the initial energy is suitably small (see Proposition 4.1), then let the lower bound of the initial density go to zero. To do so, one needs to establish global a priori estimates, which are independent of the lower bound of the density, on smooth solutions to (0.1) (0.4) (0.6) in suitable higher norms. It turns out that the key issue in this paper is to derive both the time-independent upper bound for the density and the time-dependent higher norm estimates of the smooth solution (ρ, u, θ). Compared to the isentropic case ([12]), the first main difficulty lies in the fact that the basic energy estimate cannot yield directly the bounds on the L^2-norm (in both time and space) of the spatial derivatives of both the velocity and the temperature since the super norm of the temperature is just assumed to satisfy the a priori bound $(\min\{1,t\})^{-3/2}$ (see (2.6)), which in fact could be arbitrarily large for small time. To overcome this difficulty, based on careful analysis on the basic energy estimate, we succeed in deriving a new estimate of the temperature which shows that the spatial L^2-norm of the deviation of the temperature from its far field value can be bounded by the combination of the initial energy with the spatial L^2-norm of the spatial derivatives of the temperature (see (2.10)). This estimate, which will play a crucial role in the analysis of this paper, together with elaborate analysis on the bounds of the energy, then yields the key energy-like estimate, provided that the initial energy is suitably small (see Lemma 2.3). We remark that one of the key issues to obtain such an energy-like estimate lies in the positivity of the far field density, which excludes the case of compactly supported initial density.

Next, the second main difficulty is to obtain the time-independent upper bound of the density. Based on careful initial layer analysis and making a full use of the structure of (0.1), we succeed in deriving the weighted spatial mean estimates of the material derivatives of both the velocity and the temperature, which are independent of the lower bound of density, provided that the initial energy is suitably small (see Lemmas 2.4 and 2.5). This approach is motivated by the basic estimates of the material derivatives of both the velocity and the temperature, which are developed by Hoff ([10]) in the theory of weak solutions with strictly positive initial density. Having all these estimates at hand, we are able to obtain the desired estimates of $L^1(0,\min\{1,T\};\, L^\infty(\mathbb{R}^3))$-norm and the time-independent ones of $L^2(\min\{1,T\},T;\, L^\infty(\mathbb{R}^3))$-norm of both the effective viscous flux (see (0.20) for the definition) and the deviation of the temperature from its far field value. It follows from these key estimates and a Gronwall-type inequality (see Lemma 1.5) that we are able to obtain a time-uniform upper bound of the density which is crucial for global estimates of classical solutions. This approach to estimate a uniform upper bound for the density is new compared to our previous analysis on the isentropic compressible Navier-Stokes equations in [12].

Then, the third main step is to bound the gradients of the density, the velocity, and the temperature. Motivated by our recent studies ([11]) on the blow-up criteria of strong (or classical) solutions to the barotropic compressible Navier-Stokes equations, such bounds can be obtained by solving a logarithm Gronwall inequality based on a Beale-Kato-Majda-type inequality (see Lemma 1.6) and the a priori estimates we have just derived. Moreover, such a derivation simultaneously yields the bound for $L^{3/2}(0,T;L^\infty(\mathbb{R}^3))$-norm of the gradient of the velocity(see Lemma 3.1 and its proof). It should be noted here that we do not require smallness of the gradient of the initial density which prevents the appearance of vacuum ([17]).

Finally, with these a priori estimates of the gradients of the solutions at hand, one can obtain the desired higher order estimates by careful initial layer analysis on the time derivatives and then the spatial ones of the density, the velocity and the temperature. It should be emphasized here that all these a priori estimates are independent of the lower bound of the density. Therefore, we can build proper approximate solutions with strictly positive initial density then take appropriate limits by letting the lower bound of the initial density go to zero. The limiting functions having exactly the desired properties are shown to be the global classical solutions to the Cauchy problem (0.1) (0.4) (0.6). In addition, the initial density is allowed to vanish. We can also establish the global weak solutions almost the same way as we established the classical one with a new modified approximating initial data.

The rest of the paper is organized as follows: In Section 1, we collect some elementary facts and inequalities which will be needed in later analysis. Section 2 is devoted to deriving the lower-order a priori estimates on classical solutions which are needed to extend the local solution to all time. Based on the previous results, higher-order estimates are established in Section 3. Then finally, the main results, Theorems 0.1 and 0.2, are proved in Section 4.

1 Preliminaries

The following well-known local existence theory, where the initial density is strictly away from vacuum, can be shown by the standard contraction mapping argument (see for example [17,18], in particular, [17, Theorem 5.2]).

Lemma 1.1 *Assume that* (ρ_0, u_0, θ_0) *satisfies*

$$(\rho_0 - 1, u_0, \theta_0 - 1) \in H^3, \quad \inf_{x\in\mathbb{R}^3} \rho_0(x) > 0. \tag{1.1}$$

Then there exist a small time $T_0 > 0$ *and a unique classical solution* (ρ, u, θ) *to the Cauchy problem (0.1) (0.4) (0.6) on* $\mathbb{R}^3 \times (0, T_0]$ *such that*

$$\inf_{(x,t)\in\mathbb{R}^3\times(0,T_0]} \rho(x,t) \geq \frac{1}{2} \inf_{x\in\mathbb{R}^3} \rho_0(x), \tag{1.2}$$

$$\begin{cases} (\rho - 1, u, \theta - 1) \in C([0, T_0]; H^3), \quad \rho_t \in C([0, T_0]; H^2), \\ (u_t, \theta_t) \in C([0, T_0]; H^1), \quad (u, \theta - 1) \in L^2(0, T_0; H^4), \end{cases} \tag{1.3}$$

and

$$\begin{cases} (\sigma u_t, \sigma\theta_t) \in L^2(0, T_0; H^3), \quad (\sigma u_{tt}, \sigma\theta_{tt}) \in L^2(0, T_0; H^1), \\ (\sigma^2 u_{tt}, \sigma^2\theta_{tt}) \in L^2(0, T_0; H^2), \quad (\sigma^2 u_{ttt}, \sigma^2\theta_{ttt}) \in L^2(0, T_0; L^2), \end{cases} \tag{1.4}$$

where $\sigma(t) \triangleq \min\{1, t\}$. *Moreover, for any* $(x,t) \in \mathbb{R}^3 \times [0, T_0]$, *the following estimate holds*

$$\theta(x,t) \geq \inf_{x\in\mathbb{R}^3} \theta_0(x) \exp\left\{-(\gamma - 1)\int_0^{T_0} \|\mathrm{div} u\|_{L^\infty} dt\right\}, \tag{1.5}$$

provided $\inf_{x\in\mathbb{R}^3} \theta_0(x) \geq 0$.

Next, the following well-known Gagliardo-Nirenberg-Sobolev-type inequality will be used later frequently (see [19]).

Lemma 1.2 *For $p \in (1,\infty)$ and $q \in (3,\infty)$, there exists some generic constant $C > 0$ which may depend on p and q such that for $f \in D^1(\mathbb{R}^3)$, $g \in L^p(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$, and $\varphi, \psi \in H^2(\mathbb{R}^3)$, we have*

$$\|f\|_{L^6} \le C\|\nabla f\|_{L^2}, \tag{1.6}$$

$$\|g\|_{C(\overline{\mathbb{R}^3})} \le C\|g\|_{L^p}^{p(q-3)/(3q+p(q-3))}\|\nabla g\|_{L^q}^{3q/(3q+p(q-3))}, \tag{1.7}$$

$$\|\varphi\psi\|_{H^2} \le C\|\varphi\|_{H^2}\|\psi\|_{H^2}. \tag{1.8}$$

Then, the following inequality is an easy consequence of (1.6) and will be used frequently later.

Lemma 1.3 *Let the function $g(x)$ defined in $\mathbb{R}^3$ be non-negative and satisfy $g(\cdot) - 1 \in L^2(\mathbb{R}^3)$. Then there exists a universal positive constant C such that for $r \in [1,2]$ and any open set $\Sigma \subset \mathbb{R}^3$, the following estimate holds*

$$\int_\Sigma |f|^r dx \le C\int_\Sigma g|f|^r dx + C\|g-1\|_{L^2(\mathbb{R}^3)}^{(6-r)/3}\|\nabla f\|_{L^2(\mathbb{R}^3)}^r, \tag{1.9}$$

for all $f \in \left\{f \in D^1(\mathbb{R}^3) \mid g|f|^r \in L^1(\Sigma)\right\}$.

Next, it follows from $(0.1)_2$ that G, ω defined in (0.20) satisfy

$$\triangle G = \operatorname{div}(\rho\dot{u}), \quad \mu\triangle\omega = \nabla\times(\rho\dot{u}). \tag{1.10}$$

Applying the standard L^p-estimate to the elliptic systems (1.10) together with (1.6) yields the following elementary estimates (see [12, Lemma 2.3]).

Lemma 1.4 *Let (ρ, u, θ) be a smooth solution of (0.1) (0.4). Then there exists a generic positive constant C depending only on μ, λ, and R such that, for any $p \in [2,6]$,*

$$\|\nabla u\|_{L^p} \le C\left(\|G\|_{L^p} + \|\omega\|_{L^p}\right) + C\|\rho\theta - 1\|_{L^p}, \tag{1.11}$$

$$\|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{1.12}$$

$$\|G\|_{L^p} + \|\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)}\left(\|\nabla u\|_{L^2} + \|\rho\theta - 1\|_{L^2}\right)^{(6-p)/(2p)}, \tag{1.13}$$

$$\|\nabla u\|_{L^p} \le C\|\nabla u\|_{L^2}^{(6-p)/(2p)}\left(\|\rho\dot{u}\|_{L^2} + \|\rho\theta - 1\|_{L^6}\right)^{(3p-6)/(2p)}. \tag{1.14}$$

Next, the following Gronwall-type inequality will be used to get the uniform (in time) upper bound of the density ρ.

Lemma 1.5 *Let the function $y \in W^{1,1}(0,T)$ satisfy*

$$y'(t) + \alpha y(t) \le g(t) \text{ on } [0,T], \quad y(0) = y^0, \tag{1.15}$$

where α is a positive constant and $g \in L^p(0,T_1) \cap L^q(T_1,T)$ for some $p \ge 1, q \ge 1$, and $T_1 \in [0,T]$. Then

$$\sup_{0\le t\le T} y(t) \le |y^0| + (1+\alpha^{-1})\left(\|g\|_{L^p(0,T_1)} + \|g\|_{L^q(T_1,T)}\right). \tag{1.16}$$

Finally, we state the following Beale-Kato-Majda-type inequality whose proof can be found in [2,11] and will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\rho\|_{L^2\cap L^6}$.

Lemma 1.6 ([2,11]) *For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$:*

$$\begin{aligned}\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le\ & C\left(\|\operatorname{div} u\|_{L^\infty(\mathbb{R}^3)} + \|\nabla\times u\|_{L^\infty(\mathbb{R}^3)}\right)\log(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^3)}) \\ & + C\|\nabla u\|_{L^2(\mathbb{R}^3)} + C.\end{aligned} \tag{1.17}$$

2 A priori estimates (I): Lower-order estimates

In this section, we will establish a priori bounds for the smooth, local-in-time solution to (0.1) (0.4) (0.6) obtained in Lemma 1.1. We thus fix a smooth solution (ρ, u, θ) of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0, T]$ for some time $T > 0$, with initial data (ρ_0, u_0, θ_0) satisfying (1.1).

For $\sigma(t) \triangleq \min\{1, t\}$, we define $A_i(T)(i = 1, \cdots, 4)$ as follows:

$$A_1(T) = \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho|\dot{u}|^2 dxdt, \tag{2.1}$$

$$A_2(T) = \frac{R}{2(\gamma-1)} \sup_{t\in[0,T]} \int \rho(\theta-1)^2 dx + \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) dt, \tag{2.2}$$

$$\begin{aligned} A_3(T) = &\sup_{t\in(0,T]} \left(\sigma\|\nabla u\|_{L^2}^2 + \sigma^2 \int \rho|\dot{u}|^2 dx + \sigma^2\|\nabla\theta\|_{L^2}^2\right) \\ &+ \int_0^T \int \left(\sigma\rho|\dot{u}|^2 + \sigma^2|\nabla\dot{u}|^2 + \sigma^2\rho(\dot{\theta})^2\right) dxdt, \end{aligned} \tag{2.3}$$

$$A_4(T) = \sup_{t\in(0,T]} \sigma^4 \int \rho|\dot{\theta}|^2 dx + \int_0^T \int \sigma^4|\nabla\dot{\theta}|^2 dxdt. \tag{2.4}$$

We have the following key a priori estimates on (ρ, u, θ).

Proposition 2.1 *For given numbers $M > 0$ (not necessarily small), and $\bar{\rho} > 2$, assume that (ρ_0, u_0, θ_0) satisfies*

$$0 < \inf_{x\in\mathbb{R}^3} \rho_0(x) \le \sup_{x\in\mathbb{R}^3} \rho_0(x) < \bar{\rho}, \quad \inf_{x\in\mathbb{R}^3} \theta_0(x) > 0, \quad \|\nabla u_0\|_{L^2} \le M. \tag{2.5}$$

Then there exist positive constants K and ε_0 both depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0, T]$ satisfying

$$0 < \rho \le 2\bar{\rho}, \;\; A_1(\sigma(T)) \le 3K, \;\; A_i(T) \le 2C_0^{1/(2i)} \;\; (i = 2, 3, 4), \tag{2.6}$$

the following estimates hold

$$0 < \rho \le 3\bar{\rho}/2, \;\; A_1(\sigma(T)) \le 2K, \;\; A_i(T) \le C_0^{1/(2i)} \;\; (i = 2, 3, 4), \tag{2.7}$$

provided

$$C_0 \le \varepsilon_0. \tag{2.8}$$

Proof. Proposition 2.1 is an easy consequence of the following Lemmas 2.2, 2.3, 2.6–2.8.

In this section, we let C denote some generic positive constant depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M.

Lemma 2.1 *Under the conditions of Proposition 2.1, there exists a positive constant $C = C(\bar{\rho})$ depending only on $\mu, \lambda, \kappa, R, \gamma$, and $\bar{\rho}$ such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0, T]$ satisfying $0 < \rho \le 2\bar{\rho}$, the following estimates hold*

$$\sup_{0\le t\le T} \int \left(\rho|u|^2 + (\rho-1)^2\right) dx \le C(\bar{\rho})C_0, \tag{2.9}$$

and

$$\|(\theta-1)(\cdot, t)\|_{L^2} \le C(\bar{\rho})C_0^{1/2} + C(\bar{\rho})C_0^{1/3}\|\nabla\theta(\cdot, t)\|_{L^2}, \tag{2.10}$$

for all $t \in (0, T]$.

Next, the following lemma will give an estimate on the term $A_1(\sigma(T))$.

Lemma 2.2 *Under the conditions of Proposition 2.1, there exist positive constants $K \geq M+1$ and $\varepsilon_1 \leq 1$ both depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying*

$$0 < \rho \leq 2\bar{\rho}, \quad A_2(\sigma(T)) \leq 2C_0^{1/4}, \tag{2.11}$$

the following estimate holds

$$A_1(\sigma(T)) \leq 2K, \tag{2.12}$$

provided $A_1(\sigma(T)) \leq 3K$ and $C_0 \leq \varepsilon_1$.

The following elementary L^2 bounds are crucial for deriving the desired estimate on $A_2(T)$ (see Lemma 2.3 below).

Lemma 2.3 *Under the conditions of Proposition 2.1, there exists a positive constant ε_2 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying (2.6) with K as in Lemma 2.2, the following estimate holds*

$$A_2(T) \leq C_0^{1/4}, \tag{2.13}$$

provided $C_0 \leq \varepsilon_2$.

Next, to estimate $A_3(T)$, we first establish the following Lemmas 2.4 and 2.5 concerning some elementary estimates on $\dot{u}$ and $\dot{\theta}$.

Lemma 2.4 *In addition to the conditions of Proposition 2.1, assume that $C_0 \leq 1$. Let (ρ, u, θ) be a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying (2.6) with K as in Lemma 2.2. Then there exist positive constants C and C_1 both depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that, for any $\beta \in (0,1]$, the following estimates hold*

$$\begin{aligned}&(\sigma B_1)'(t) + \frac{3}{2}\int \sigma\rho|\dot{u}|^2dx \\ &\leq CC_0^{1/4}\sigma' + 2\beta\sigma^2\|\rho^{1/2}\dot{\theta}\|_{L^2}^2 + C\beta^{-1}\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + C\sigma^2\|\nabla u\|_{L^4}^4,\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\left(\sigma^2\int \rho|\dot{u}|^2dx\right)_t + \frac{3\mu}{2}\int \sigma^2|\nabla\dot{u}|^2dx \\ &\leq 2\sigma\int \rho|\dot{u}|^2dx + C_1\sigma^2\|\rho^{1/2}\dot{\theta}\|_{L^2}^2 + C\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + C\sigma^2\|\nabla u\|_{L^4}^4,\end{aligned} \tag{2.15}$$

where

$$B_1(t) \triangleq \mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2 + 2R\int \mathrm{div}u(\rho\theta - 1)dx. \tag{2.16}$$

Lemma 2.5 *In addition to the conditions of Proposition 2.1, assume that $C_0 \leq 1$. Let (ρ, u, θ) be a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying (2.6) with K as in Lemma 2.2. Then there exists a positive constant C depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that the following estimate holds*

$$\begin{aligned}&\left(\sigma^2\varphi\right)'(t) + \sigma^2\int\left(\mu|\nabla\dot{u}|^2 + \rho(\dot{\theta})^2\right)dx \\ &\leq C\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + 2\sigma\int \rho|\dot{u}|^2dx + C\sigma^2\|\nabla u\|_{L^4}^4,\end{aligned} \tag{2.17}$$

where $\varphi(t)$ is defined by

$$\varphi(t) \triangleq \int \rho|\dot{u}|^2(x,t)dx + (C_1+1)B_2(t), \tag{2.18}$$

with C_1 as in Lemma 2.4 and

$$B_2(t) \triangleq \frac{\gamma-1}{R}\left(\kappa\|\nabla\theta\|_{L^2}^2 - 2\lambda\int(\mathrm{div}u)^2\theta dx - 4\mu\int|\mathfrak{D}(u)|^2\theta dx\right). \tag{2.19}$$

Next, we will use Lemmas 2.4 and 2.5 to obtain the following estimate on $A_3(T)$.

Lemma 2.6 *Under the conditions of Proposition 2.1, there exists a positive constant ε_3 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying (2.6) with K as in Lemma 2.2, the following estimate holds*

$$A_3(T) \le C_0^{1/6}, \tag{2.20}$$

provided $C_0 \le \varepsilon_3$.

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtain all the higher order estimates and thus to extend the classical solution globally.

Lemma 2.7 *Under the conditions of Proposition 2.1, there exists a positive constant ε_4 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying (2.6) with K as in Lemma 2.2, the following estimate holds*

$$\sup_{0\le t\le T}\|\rho(\cdot,t)\|_{L^\infty} \le \frac{3\bar{\rho}}{2}, \tag{2.21}$$

provided $C_0 \le \varepsilon_4$.

Next, the following Lemma 2.8 will give an estimate on $A_4(T)$, which together with Lemmas 2.2, 2.3, 2.6 and 2.7 finishes the proof of Proposition 2.1.

Lemma 2.8 *Under the conditions of Proposition 2.1, there exists a positive constant ε_0 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ satisfying (2.6) with K as in Lemma 2.2, the following estimate holds*

$$A_4(T) \le C_0^{1/8}, \tag{2.22}$$

provided $C_0 \le \varepsilon_0$.

3 A priori estimates (II): Higher-order estimates

In this section, we will derive the higher order estimates of a smooth solution (ρ, u, θ) of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ with smooth (ρ_0, u_0, θ_0) satisfying (0.8) and (2.5). Moreover, we shall always assume that (ρ, u, θ) and (ρ_0, u_0, θ_0) satisfy respectively (2.6) and (2.8). To proceed, we define $\tilde{g}_1$ and $\tilde{g}_2$ as

$$\tilde{g}_1 \triangleq \rho_0^{-1/2}\left(-\mu\Delta u_0 - (\mu+\lambda)\nabla\mathrm{div}u_0 + R\nabla(\rho_0\theta_0)\right) \tag{3.1}$$

and
$$\tilde{g_2} \triangleq \rho_0^{-1/2}\left(\kappa\Delta\theta_0 + \frac{\mu}{2}|\nabla u_0 + (\nabla u_0)^{\mathrm{tr}}|^2 + \lambda(\mathrm{div}u_0)^2\right), \tag{3.2}$$
respectively. It thus follows from (0.8) and (2.5) that
$$\tilde{g_1} \in L^2, \quad \tilde{g_2} \in L^2. \tag{3.3}$$
From now on, the generic constant C will depend only on
$$T,\ \|\tilde{g_1}\|_{L^2},\ \|\tilde{g_2}\|_{L^2},\ \|u_0\|_{H^2},\ \|\rho_0 - 1\|_{H^2\cap W^{2,q}},\ \|\theta_0 - 1\|_{H^2},$$
besides $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M.

We begin with the important estimates on the smooth solution (ρ, u, θ).

Lemma 3.1 *The following estimates hold*
$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho^{1/2}\dot{u}\|_{L^2} + \|\theta - 1\|_{H^1}\right) + \int_0^T\int \rho(\dot{\theta})^2 dxdt \\ &+ \int_0^T \left(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 + \|\mathrm{div}u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2\right) dt \le C,\end{aligned} \tag{3.4}$$
$$\sup_{0\le t\le T}\left(\|\rho - 1\|_{H^1\cap W^{1,6}} + \|u\|_{H^2}\right) + \int_0^T \|\nabla u\|_{L^\infty}^{3/2} dt \le C. \tag{3.5}$$

Lemma 3.2 *The following estimates hold*
$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho_t\|_{H^1} + \|\theta - 1\|_{H^2} + \|\rho - 1\|_{H^2} + \|u\|_{H^2}\right) \\ &+ \int_0^T \left(\|u_t\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\rho u_t\|_{H^1}^2 + \|\rho\theta_t\|_{H^1}^2\right) dt \le C,\end{aligned} \tag{3.6}$$
$$\int_0^T \left(\|(\rho u_t)_t\|_{H^{-1}}^2 + \|(\rho\theta_t)_t\|_{H^{-1}}^2\right) dt \le C. \tag{3.7}$$

Lemma 3.3 *The following estimate holds:*
$$\sup_{0\le t\le T}\sigma\left(\|\nabla u_t\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2\right) + \int_0^T \sigma\int \rho|u_{tt}|^2 dxdt \le C. \tag{3.8}$$

Lemma 3.4 *For $q \in (3, 6)$ as in Theorem 0.1, it holds that*
$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho - 1\|_{W^{2,q}} + \sigma\|u\|_{H^3}^2\right) \\ &+ \int_0^T \left(\|u\|_{H^3}^2 + \|\nabla^2 u\|_{W^{1,q}}^{p_0} + \sigma\|\nabla u_t\|_{H^1}^2\right) dt \le C,\end{aligned} \tag{3.9}$$
where
$$p_0 \triangleq \frac{1}{2}\min\left\{\frac{5q-6}{3(q-2)}, \frac{9q-6}{5q-6}\right\} \in (1, 7/6). \tag{3.10}$$

Lemma 3.5 *For $q \in (3, 6)$ as in Theorem 0.1, the following estimate holds*
$$\sup_{0\le t\le T}\sigma\left(\|\theta_t\|_{H^1} + \|\nabla^3\theta\|_{L^2} + \|u_t\|_{H^2} + \|u\|_{W^{3,q}}\right) + \int_0^T \sigma^2\|\nabla u_{tt}\|_{L^2}^2 dt \le C. \tag{3.11}$$

Lemma 3.6 *The following estimate holds*
$$\sup_{0\le t\le T}\sigma^2\left(\|\nabla^2\theta\|_{H^2} + \|\theta_t\|_{H^2}\right) + \int_0^T \sigma^4\|\nabla\theta_{tt}\|_{L^2}^2 dt \le C. \tag{3.12}$$

4 Proofs of Theorems 0.1 and 0.2

With all the a priori estimates in Sections 2 and 3 at hand, we are ready to prove the main results of this paper in this section.

Proposition 4.1 *For given numbers $M > 0$ (not necessarily small), $\bar{\rho} > 2$, assume that (ρ_0, u_0, θ_0) satisfies (1.1), (2.5), and (2.8). Then there exists a unique classical solution (ρ, u, θ) of (0.1) (0.4) (0.6) in $\mathbb{R}^3 \times (0, \infty)$ satisfying (1.3)–(1.5) with T_0 replaced by any $T \in (0, \infty)$. Moreover, (2.9), (2.6) hold for any $T \in (0, \infty)$.*

Proof of Theorem 0.1. Let (ρ_0, u_0, θ_0) satisfying (0.8)–(0.11) be initial data as described in Theorem 0.1. Assume that C_0 satisfies (0.12), where

$$\varepsilon \triangleq \varepsilon_0/2, \tag{4.1}$$

with ε_0 as in Proposition 2.1. For constants

$$\delta, \eta \in (0, \min\{1, \bar{\rho} - \sup_{x \in \mathbb{R}^3} \rho_0(x)\}), \tag{4.2}$$

we define

$$\rho_0^{\delta,\eta} \triangleq \frac{j_\delta * \rho_0 + \eta}{1+\eta}, \quad u_0^{\delta,\eta} \triangleq j_\delta * u_0, \quad \theta_0^{\delta,\eta} \triangleq \frac{j_\delta * \theta_0 + \eta}{1+\eta}, \tag{4.3}$$

where j_δ is the standard mollifying kernel of width δ. Then, $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$ satisfies

$$\begin{cases} (\rho_0^{\delta,\eta} - 1, u_0^{\delta,\eta}, \theta_0^{\delta,\eta} - 1) \in H^\infty, \\ \dfrac{\eta}{1+\eta} \le \rho_0^{\delta,\eta} \le \dfrac{\bar{\rho}+\eta}{1+\eta} < \bar{\rho}, \quad \theta_0^{\delta,\eta} \ge \dfrac{\eta}{\bar{\rho}+\eta}, \quad \|\nabla u_0^{\delta,\eta}\|_{L^2} \le M, \end{cases} \tag{4.4}$$

and

$$\begin{cases} \lim\limits_{\delta+\eta \to 0} \left(\|\rho_0^{\delta,\eta} - \rho_0\|_{H^2 \cap W^{2,q}} + \|u_0^{\delta,\eta} - u_0\|_{H^2} + \|\theta_0^{\delta,\eta} - \theta_0\|_{H^2} \right) = 0, \\ \|\nabla(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})\|_{H^1} \le \|\nabla(\rho_0, u_0, \theta_0)\|_{H^1}, \quad \|\nabla \rho_0^{\delta,\eta}\|_{W^{1,q}} \le \|\nabla \rho_0\|_{W^{1,q}}, \end{cases} \tag{4.5}$$

due to (0.8) and (0.9). Moreover, the initial norm $C_0^{\delta,\eta}$ for $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$, i.e., the right hand side of (0.7) with (ρ_0, u_0, θ_0) replaced by $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$, satisfies

$$\lim_{\eta \to 0} \lim_{\delta \to 0} C_0^{\delta,\eta} = C_0.$$

Therefore, there exists an $\eta_0 \in (0, \min\{1, \bar{\rho} - \sup_{x \in \mathbb{R}^3} \rho_0(x)\})$ such that, for any $\eta \in (0, \eta_0)$, we can find some $\delta_0(\eta) > 0$ such that

$$C_0^{\delta,\eta} \le C_0 + \varepsilon_0/2 \le \varepsilon_0, \tag{4.6}$$

provided that

$$0 < \eta \le \eta_0, \quad 0 < \delta \le \delta_0(\eta). \tag{4.7}$$

We assume that δ, η satisfy (4.7). Proposition 4.1 together with (4.6) and (4.4) thus yields that there exists a smooth solution $(\rho^{\delta,\eta}, u^{\delta,\eta}, \theta^{\delta,\eta})$ of (0.1) (0.4) (0.6) with initial data $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$ on $\mathbb{R}^3 \times [0, T]$ for all $T > 0$. Moreover, (2.9) and (2.6) both hold with (ρ, u, θ) being replaced by $(\rho^{\delta,\eta}, u^{\delta,\eta}, \theta^{\delta,\eta})$.

Next, for the initial data $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$, the $\tilde{g}_1$ in (3.1) in fact is

$$\begin{aligned}\tilde{g}_1 &\triangleq (\rho_0^{\delta,\eta})^{-1/2}\left(-\mu\Delta u_0^{\delta,\eta} - (\mu+\lambda)\nabla \mathrm{div} u_0^{\delta,\eta} + R\nabla(\rho_0^{\delta,\eta}\theta_0^{\delta,\eta})\right)\\ &= (\rho_0^{\delta,\eta})^{-1/2}(j_\delta * \rho_0)^{1/2} g_1 + (\rho_0^{\delta,\eta})^{-1/2}\left(j_\delta * (\sqrt{\rho_0} g_1) - \sqrt{j_\delta * \rho_0}\, g_1\right)\\ &\quad + R(\rho_0^{\delta,\eta})^{-1/2}\nabla\left(j_\delta * (\rho_0\theta_0) - (1+\eta)^{-2}(j_\delta * \rho_0)(j_\delta * \theta_0)\right)\\ &\quad + R\eta(1+\eta)^{-2}(\rho_0^{\delta,\eta})^{-1/2}\nabla(\rho_0^{\delta,\eta} + \theta_0^{\delta,\eta}),\end{aligned} \tag{4.8}$$

where in the second equality we have used (0.10). Similarly, the $\tilde{g}_2$ in (3.2) is

$$\begin{aligned}\tilde{g}_2 &\triangleq (\rho_0^{\delta,\eta})^{-1/2}\left(\kappa\Delta\theta_0^{\delta,\eta} + \frac{\mu}{2}|\nabla u_0^{\delta,\eta} + (\nabla u_0^{\delta,\eta})^{\mathrm{tr}}|^2 + \lambda(\mathrm{div} u_0^{\delta,\eta})^2\right)\\ &= (\rho_0^{\delta,\eta})^{-1/2}(j_\delta * \rho_0)^{1/2} g_2 + (\rho_0^{\delta,\eta})^{-1/2}\left(j_\delta * (\sqrt{\rho_0} g_2) - \sqrt{j_\delta * \rho_0}\, g_2\right)\\ &\quad - \frac{\mu}{2}(\rho_0^{\delta,\eta})^{-1/2}\left(j_\delta * \left(|\nabla u_0 + (\nabla u_0)^{\mathrm{tr}}|^2\right) - |\nabla(j_\delta * u_0) + (\nabla(j_\delta * u_0))^{\mathrm{tr}}|^2\right)\\ &\quad - \lambda(\rho_0^{\delta,\eta})^{-1/2}\left(j_\delta * \left((\mathrm{div} u_0)^2\right) - (\mathrm{div}(j_\delta * u_0))^2\right),\end{aligned} \tag{4.9}$$

due to (0.11). Since $g_1, g_2 \in L^2$, one deduces from (4.8), (4.9), (4.4), (4.5), and (0.8) that there exists some positive constant C independent of δ and η such that

$$\begin{cases} \|\tilde{g}_1\|_{L^2} \le (1+\eta)^{1/2}\|g_1\|_{L^2} + C\eta^{-1/2}m_1(\delta) + C\sqrt{\eta},\\ \|\tilde{g}_2\|_{L^2} \le (1+\eta)^{1/2}\|g_2\|_{L^2} + C\eta^{-1/2}m_2(\delta),\end{cases} \tag{4.10}$$

with $0 \le m_i(\delta) \to 0$ $(i = 1,2)$ as $\delta \to 0$. Hence, for any $0 < \eta < \eta_0$, there exists some $0 < \delta_1(\eta) \le \delta_0(\eta)$ such that

$$m_1(\delta) + m_2(\delta) < \eta, \tag{4.11}$$

for any $0 < \delta < \delta_1(\eta)$. We thus obtain from (4.10) and (4.11) that there exists some positive constant C independent of δ and η such that

$$\|\tilde{g}_1\|_{L^2} + \|\tilde{g}_2\|_{L^2} \le 2\|g_1\|_{L^2} + 2\|g_2\|_{L^2} + C, \tag{4.12}$$

provided that

$$0 < \eta < \eta_0, \quad 0 < \delta < \delta_1(\eta). \tag{4.13}$$

Now, we assume that η, δ satisfy (4.13). It thus follows from (4.6), Proposition 2.1, (4.5), (4.12), and Lemmas 3.1–3.6 that for any $T > 0$, there exists some positive constant C independent of δ and η such that (2.9), (2.6), (3.6), (3.7), (3.9), (3.11), and (3.12) hold for $(\rho^{\delta,\eta}, u^{\delta,\eta}, \theta^{\delta,\eta})$. Then passing to the limit first $\delta \to 0$, then $\eta \to 0$, together with standard arguments yields that there exists a solution (ρ, u, θ) of (0.1) (0.4) (0.6) on $\mathbb{R}^3 \times (0,T]$ for all $T > 0$, such that (ρ, u, θ) satisfies (2.9), (2.6), (3.6), (3.7), (3.9), (3.11) and (3.12). Hence, (ρ, u, θ) satisfies (0.13), $(0.14)_2$, $(0.14)_3$, and

$$\rho - 1 \in L^\infty(0,T; H^2 \cap W^{2,q}), \quad (u, \theta - 1) \in L^\infty(0,T; H^2). \tag{4.14}$$

Moreover, (0.1) holds in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$.

Next, to finish the existence part of Theorem 0.1, it remains to prove

$$\rho - 1 \in C([0,T]; H^2 \cap W^{2,q}), \quad u,\ \theta - 1 \in C([0,T]; H^2). \tag{4.15}$$

It follows from (3.6) and (4.14) that

$$\rho - 1 \in C([0,T]; H^1 \cap W^{1,\infty}) \cap C([0,T]; H^2 \cap W^{2,q}\text{ -weak}), \tag{4.16}$$

and for all $r \in [2,6)$,

$$u,\ \theta - 1 \in C([0,T]; H^1 \cap W^{1,r}). \tag{4.17}$$

Since (0.1) holds in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$ for all $T \in (0,\infty)$, one derives from [15, Lemma 2.3] that, for $j_\nu(x)$ being the standard mollifying kernel of width ν, $\rho^\nu \triangleq \rho * j_\nu$ satisfies

$$(\Delta\rho^\nu)_t + \operatorname{div}(u\Delta\rho^\nu) = -\operatorname{div}(\rho\Delta u) * j_\nu - 2\operatorname{div}(\partial_i\rho \cdot \partial_i u) * j_\nu + R_\nu, \tag{4.18}$$

where R_ν satisfies

$$\int_0^T \|R_\nu\|_{L^2\cap L^q}^{3/2} dt \le C \int_0^T \|u\|_{W^{1,\infty}}^{3/2} \|\Delta\rho\|_{L^2\cap L^q}^{3/2} dt \le C, \tag{4.19}$$

due to (3.5), (3.6), and (3.9). Multiplying (4.18) by $q|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu$, we obtain after integration by parts that

$$\begin{aligned}
&(\|\Delta\rho^\nu\|_{L^q}^q)'(t) \\
&= (1-q)\int |\Delta\rho^\nu|^q \operatorname{div} u dx - q\int (\operatorname{div}(\rho\Delta u) * j_\nu)|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu dx \\
&\quad - 2q\int (\operatorname{div}(\partial_i\rho \cdot \partial_i u) * j_\nu)|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu dx + q\int R_\nu |\Delta\rho^\nu|^{q-2}\Delta\rho^\nu dx,
\end{aligned}$$

which together with (3.6), (3.9), and (4.19) yields that, for p_0 as in (3.10),

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\Delta\rho^\nu\|_{L^q} + \int_0^T |(\|\Delta\rho^\nu\|_{L^q}^q)'(t)|^{p_0} dt \\
&\le C + C\int_0^T \left(\|\nabla u\|_{W^{2,q}}^{p_0} + \|R_\nu\|_{L^2\cap L^q}^{p_0}\right) dt \\
&\le C.
\end{aligned}$$

This fact combining with the Ascoli-Arzela theorem thus leads to

$$\|\Delta\rho^\nu(\cdot,t)\|_{L^q} \to \|\Delta\rho(\cdot,t)\|_{L^q} \text{ in } C([0,T]), \text{ as } \nu \to 0^+.$$

In particular, we have

$$\|\nabla^2\rho(\cdot,t)\|_{L^q} \in C([0,T]). \tag{4.20}$$

Similarly, one can obtain that

$$\|\nabla^2\rho(\cdot,t)\|_{L^2} \in C([0,T]). \tag{4.21}$$

Therefore, the continuity of $\nabla^2\rho$ in L^p $(p = 2, q)$, i.e.,

$$\nabla^2\rho \in C([0,T]; L^2 \cap L^q), \tag{4.22}$$

follows directly from (4.16), (4.20), and (4.21).

It follows from (3.6) and (3.7) that

$$\rho u_t, \rho\theta_t \in C([0,T]; L^2), \tag{4.23}$$

which together with (4.16), (4.17), and (4.22) gives

$$u \in C([0,T]; H^2). \tag{4.24}$$

This fact combining with (4.23), (4.22), (4.17), and (3.6) leads to

$$\theta - 1 \in C([0,T]; H^2),$$

which as well as (4.16), (4.22), and (4.24) leads to (4.15).

Finally, since the proof of the uniqueness of (ρ, u, θ) is similar to that of [4, Theorem 1], to finish the proof of Theorem 0.1, it remains to prove (0.15). We will only show

$$\lim_{t\to\infty} \|\nabla u\|_{L^2} = 0, \tag{4.25}$$

since the other terms in (0.15) follow directly from (0.26). It follows from (2.6) that

$$\begin{aligned}
&\int_1^\infty |(\|\nabla u\|_{L^2}^2)'(t)|dt \\
&= 2\int_1^\infty \left|\int \partial_j u^i \partial_j u_t^i dx\right| dt \\
&= 2\int_1^\infty \left|\int \partial_j u^i \partial_j (\dot{u}^i - u^k \partial_k u^i) dx\right| dt \\
&= 2\int_1^\infty \left|\int (\partial_j u^i \partial_j \dot{u}^i - \partial_j u^i \partial_j u^k \partial_k u^i - \partial_j u^i u^k \partial_{kj} u^i) dx\right| dt \\
&= \int_1^\infty \left|\int (2\partial_j u^i \partial_j \dot{u}^i - 2\partial_j u^i \partial_j u^k \partial_k u^i + |\nabla u|^2 \mathrm{div} u) dx\right| dt \\
&\le C\int_1^\infty \left(\|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^3}^3\right) dt \\
&\le C\int_1^\infty \left(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4\right) dt \\
&\le C,
\end{aligned}$$

which together with (2.6) implies (4.25). We finish the proof of Theorem 0.1.

Proof of Theorem 0.2. We will prove Theorem 0.2 in three steps.

Step 1. Construction of approximate solutions. Let (ρ_0, u_0, θ_0) satisfying (0.9) be initial data as described in Theorem 0.2. Assume that C_0 satisfies (0.21) with ε as in (4.1). Let δ and η be as in (4.2) and j_δ be the standard mollifier. We define

$$\hat{\rho}_0^{\delta,\eta} \triangleq \frac{j_\delta * \rho_0 + \eta}{1+\eta}, \quad \hat{u}_0^{\delta,\eta} \triangleq j_\delta * u_0, \quad \hat{\theta}_0^{\delta,\eta} \triangleq \frac{j_\delta * (\rho_0\theta_0) + \eta}{j_\delta * \rho_0 + \eta}. \tag{4.26}$$

Then, $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$ satisfies

$$\begin{cases} (\hat{\rho}_0^{\delta,\eta} - 1, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta} - 1) \in H^\infty, \\ 0 < \dfrac{\eta}{1+\eta} \le \hat{\rho}_0^{\delta,\eta} \le \dfrac{\bar{\rho}+\eta}{1+\eta} < \bar{\rho}, \quad \hat{\theta}_0^{\delta,\eta} \ge \dfrac{\eta}{\bar{\rho}+\eta} > 0, \quad \|\nabla \hat{u}_0^{\delta,\eta}\|_{L^2} \le M, \end{cases} \tag{4.27}$$

due to (0.9). Moreover, it follows from (0.9) and (0.21) that

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\left(\|\hat{\rho}_0^{\delta,\eta} - \rho_0\|_{L^2} + \|\hat{u}_0^{\delta,\eta} - u_0\|_{H^1} + \|\hat{\rho}_0^{\delta,\eta}\hat{\theta}_0^{\delta,\eta} - \rho_0\theta_0\|_{L^2}\right) = 0. \tag{4.28}$$

We claim that the initial norm $\hat{C}_0^{\delta,\eta}$ for $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$, i.e., the right hand side of (0.7) with (ρ_0, u_0, θ_0) replaced by $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$, satisfies

$$\lim_{\eta\to 0}\lim_{\delta\to 0} \hat{C}_0^{\delta,\eta} \le C_0, \tag{4.29}$$

which yields that there exists an $\hat{\eta} > 0$ such that, for any $\eta \in (0, \hat{\eta})$, there exists some $\hat{\delta}(\eta) > 0$ such that

$$\hat{C}_0^{\delta,\eta} \leq C_0 + \varepsilon_0/2 \leq \varepsilon_0, \tag{4.30}$$

provided

$$0 < \eta \leq \hat{\eta}, \quad 0 < \delta \leq \hat{\delta}(\eta). \tag{4.31}$$

We assume that δ, η always satisfy (4.31). Proposition 4.1 as well as (4.27) and (4.30) thus yields that there exists a smooth solution $(\hat{\rho}^{\delta,\eta}, \hat{u}^{\delta,\eta}, \hat{\theta}^{\delta,\eta})$ of (0.1) (0.4) (0.6) with initial data $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$ on $\mathbb{R}^3 \times [0,T]$ for all $T > 0$. Moreover, for any $T > 0$, $(\hat{\rho}^{\delta,\eta}, \hat{u}^{\delta,\eta}, \hat{\theta}^{\delta,\eta})$ satisfies (2.9), (2.6) with (ρ, u, θ) replaced by $(\hat{\rho}^{\delta,\eta}, \hat{u}^{\delta,\eta}, \hat{\theta}^{\delta,\eta})$.

It remains to prove (4.29). In fact, we only have to show

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\int \hat{\rho}_0^{\delta,\eta}\left(\hat{\theta}_0^{\delta,\eta} - \log\hat{\theta}_0^{\delta,\eta} - 1\right)dx \leq \int \rho_0\left(\theta_0 - \log\theta_0 - 1\right)dx, \tag{4.32}$$

since the other terms in (4.29) can be proved in a similar and even simpler way. Noticing that

$$\begin{aligned}
&\hat{\rho}_0^{\delta,\eta}\left(\hat{\theta}_0^{\delta,\eta} - \log\hat{\theta}_0^{\delta,\eta} - 1\right)\\
&= \hat{\rho}_0^{\delta,\eta}(\hat{\theta}_0^{\delta,\eta} - 1)^2\int_0^1 \frac{\alpha}{\alpha(\hat{\theta}_0^{\delta,\eta} - 1) + 1}d\alpha\\
&= \frac{(j_\delta * (\rho_0\theta_0 - \rho_0))^2}{1+\eta}\int_0^1 \frac{\alpha}{\alpha(j_\delta * (\rho_0\theta_0) - j_\delta * \rho_0) + j_\delta * \rho_0 + \eta}d\alpha\\
&\in \left[0,\ \eta^{-1}(j_\delta * (\rho_0\theta_0 - \rho_0))^2\right],
\end{aligned}$$

we deduce from (4.28) and Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&\lim_{\delta\to 0}\int \hat{\rho}_0^{\delta,\eta}\left(\hat{\theta}_0^{\delta,\eta} - \log\hat{\theta}_0^{\delta,\eta} - 1\right)dx\\
&= \int \frac{\rho_0 + \eta}{1+\eta}\left(\frac{\rho_0\theta_0 + \eta}{\rho_0 + \eta} - \log\frac{\rho_0\theta_0 + \eta}{\rho_0 + \eta} - 1\right)dx\\
&= (1+\eta)^{-1}\int_{(\rho_0\theta_0 < 1/2)\cup(\rho_0\theta_0 > 2)}\left(\rho_0\theta_0 - \rho_0 - (\rho_0 + \eta)\log\frac{\rho_0\theta_0 + \eta}{\rho_0 + \eta}\right)dx\\
&\quad + (1+\eta)^{-1}\int_{(1/2 \leq \rho_0\theta_0 \leq 2)}(\rho_0 + \eta)\left(\frac{\rho_0\theta_0 + \eta}{\rho_0 + \eta} - \log\frac{\rho_0\theta_0 + \eta}{\rho_0 + \eta} - 1\right)dx\\
&\triangleq (1+\eta)^{-1}I_1 + (1+\eta)^{-1}I_2,
\end{aligned} \tag{4.33}$$

where we have used the following simple fact that, for $f \in L^p (1 \leq p < \infty)$,

$$\lim_{\delta\to 0}\|j_\delta * f - f\|_{L^p} = 0, \quad \lim_{\delta\to 0} j_\delta * f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^3.$$

It follows from (0.21) that

$$\begin{aligned}
&|(\rho_0\theta_0 < 1/2) \cup (\rho_0\theta_0 > 2)|\\
&\leq 4\int(\rho_0\theta_0 - 1)^2 dx\\
&\leq 8\int(\rho_0\theta_0 - \rho_0)^2 dx + 8\int(\rho_0 - 1)^2 dx\\
&\leq C,
\end{aligned}$$

which combining with Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
I_1 &= \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} (\rho_0\theta_0 - \rho_0\log(\rho_0\theta_0+\eta) - \eta\log(\rho_0\theta_0+\eta))\,dx \\
&\quad + \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} ((\rho_0+\eta)\log(\rho_0+\eta) - \rho_0)\,dx \\
&\le \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} (\rho_0\theta_0 - \rho_0\log(\rho_0\theta_0) - \eta\log\eta)\,dx \\
&\quad + \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} (\rho_0\log(\rho_0+\eta) + \eta\log(\rho_0+\eta) - \rho_0)\,dx \\
&\to \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} \rho_0\,(\theta_0 - \log\theta_0 - 1)\,dx, \quad \text{as } \eta \to 0.
\end{aligned} \tag{4.34}$$

Noticing that

$$\begin{aligned}
&(\rho_0+\eta)\left(\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - \log\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - 1\right) \\
&= (\rho_0\theta_0-\rho_0)^2\int_0^1 \frac{\alpha}{\alpha(\rho_0\theta_0-\rho_0)+\rho_0+\eta}\,d\alpha \\
&\in \left[0, 2\,(\rho_0\theta_0-\rho_0)^2\right],
\end{aligned}$$

provided $\rho_0\theta_0 \ge 1/2$, we deduce from Lebesgue's dominated convergence theorem that

$$\lim_{\eta\to 0} I_2 = \int_{(1/2\le\rho_0\theta_0\le 2)} \rho_0\,(\theta_0 - \log\theta_0 - 1)\,dx,$$

which together with (4.33) and (4.34) gives (4.32).

Step 2. Compactness results. For the approximate solutions $(\hat\rho^{\delta,\eta}, \hat u^{\delta,\eta}, \hat\theta^{\delta,\eta})$ obtained in the previous step, we will pass to the limit first $\delta \to 0$, then $\eta \to 0$ and apply (2.6) to obtain the global existence of weak solutions. Since the two steps are similar, we will only sketch the arguments for $\delta \to 0$. Thus, we fix $\eta \in (0,\hat\eta)$ and simply denote $(\hat\rho^{\delta,\eta}, \hat u^{\delta,\eta}, \hat\theta^{\delta,\eta})$ by $(\rho^\delta, u^\delta, \theta^\delta)$. For $R \in (0,\infty)$, let $B_R(x_0) \triangleq \{x \in \mathbb{R}^3 | |x - x_0| < R\}$ denote a ball centered at $x_0 \in \mathbb{R}^3$ with radius R. We claim that there exists some appropriate subsequence $\delta_j \to 0$ of $\delta \to 0$ such that, for any $0 < \tau < T < \infty$ and $0 < R < \infty$, we have

$$\begin{cases} \theta^{\delta_j} - 1 \rightharpoonup \theta - 1 \ \text{ weakly in } \ L^2(0,T;H^1(\mathbb{R}^3)), \\ u^{\delta_j} \rightharpoonup u \ \text{ weakly star in } \ L^\infty(0,T;H^1(\mathbb{R}^3)), \end{cases} \tag{4.35}$$

$$\begin{cases} \rho^{\delta_j} - 1 \to \rho - 1 \quad \text{in } \ C([0,T];L^2(\mathbb{R}^3)\text{-weak}), \\ \rho^{\delta_j} - 1 \to \rho - 1 \quad \text{in } \ C([0,T];H^{-1}(B_R(0))), \end{cases} \tag{4.36}$$

$$\begin{cases} \rho^{\delta_j}u^{\delta_j} \to \rho u, \ \rho^{\delta_j}(\theta^{\delta_j} - 1) \to \rho(\theta - 1) \quad \text{in } \ C([0,T];L^2(\mathbb{R}^3)\text{-weak}), \\ \rho^{\delta_j}u^{\delta_j} \to \rho u \ \text{ in } \ C([0,T];H^{-1}(B_R(0))), \end{cases} \tag{4.37}$$

$$\rho^{\delta_j}|u^{\delta_j}|^2 \to \rho|u|^2 \ \text{ in } \ C([0,T];L^3\text{-weak}), \tag{4.38}$$

and

$$\begin{cases} u^{\delta_j} \to u, \ G^{\delta_j} \to G, \ \omega^{\delta_j} \to \omega, \ \nabla\theta^{\delta_j} \to \nabla\theta \ \text{ in } \ C([\tau,T];H^1(\mathbb{R}^3)\text{-weak}), \\ u^{\delta_j} \to u, \ G^{\delta_j} \to G, \ \omega^{\delta_j} \to \omega, \ \nabla\theta^{\delta_j} \to \nabla\theta \ \text{ in } \ C([\tau,T];L^2(B_R(0))). \end{cases} \tag{4.39}$$

We thus write (0.1) in the weak forms for the approximate solutions $(\rho^\delta, u^\delta, \theta^\delta)$, then let $\delta = \delta_j$ and take appropriate limits. Standard arguments as well as (4.35)–(4.39) thus yield that the limit (ρ, u, θ) is a weak solution of (0.1) (0.4) (0.5) in the sense of Definition 0.1 and satisfies (0.22)–(0.25) except $\rho - 1 \in C([0,\infty), L^2)$ which in fact can be obtained by similar arguments leading to (4.22). In addition, the estimates (0.27)–(0.29) follows direct from (2.9), (2.6), and (4.35)–(4.39).

It remains to prove (4.36)–(4.39) since (4.35) is a direct consequence of (2.6). It follows from (2.9), (2.6), and $(0.1)_1$ that

$$\sup_{t\in[0,\infty)} \|\rho^\delta_t\|_{H^{-1}(\mathbb{R}^3)} \le C,$$

which as well as (2.6), [15, Lemma C.1], and the Aubin-Lions lemma yields that there exists a subsequence of $\delta_j \to 0$, still denoted by δ_j, such that (4.36) holds. Moreover, one deduces that (extract a subsequence)

$$\rho^{\delta_j} - 1 \rightharpoonup \rho - 1, \ \nabla u^{\delta_j} \rightharpoonup \nabla u \ \text{ weakly in } \ L^4(\mathbb{R}^3 \times (1,\infty)),$$

with $\rho - 1$ and ∇u satisfying

$$\int_1^\infty \left(\|\rho - 1\|_{L^4}^4 + \|\nabla u\|_{L^4}^4\right) dt \le C. \tag{4.40}$$

Then, simple calculations together with (2.6) yield that, for any $0 < T < \infty$, there exists some $C(T)$ independent of δ and η such that

$$\|(\rho^\delta u^\delta)_t\|_{L^2(0,T;H^{-1}(\mathbb{R}^3))} + \|(\rho^\delta \theta^\delta)_t\|_{L^2(0,T;H^{-1}(\mathbb{R}^3))} \le C(T), \tag{4.41}$$

which together with (2.6), (4.36), and (4.35) gives (4.37).

Next, to prove (4.38), one deduces from (2.6) and $(0.1)_1$ that, for any $\zeta \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}
&\left|\int (\rho^\delta |u^\delta|^2)_t \zeta dx\right| \\
&= \left| - \int \operatorname{div}(\rho^\delta u^\delta)|u^\delta|^2 \zeta dx + 2\int \rho^\delta u^\delta \cdot u^\delta_t \zeta dx\right| \\
&= \left|\int \rho^\delta u^\delta \cdot \nabla(|u^\delta|^2\zeta) dx + 2\int \rho^\delta u^\delta \cdot (\dot{u}^\delta - u^\delta \cdot \nabla u^\delta)\zeta dx\right| \\
&\le C\int \rho^\delta |u^\delta|^3 |\nabla \zeta| dx + C\int \rho^\delta |u^\delta|^2 |\nabla u^\delta||\zeta| dx + C\int \rho^\delta |u^\delta||\dot{u}^\delta||\zeta| dx \\
&\le C\|u^\delta\|_{L^6}^3 \|\nabla\zeta\|_{L^2} + C\|u^\delta\|_{L^6}^2 \|\nabla u^\delta\|_{L^2}\|\zeta\|_{L^6} + C\|u^\delta\|_{L^6}\|(\rho^\delta)^{1/2}\dot{u}^\delta\|_{L^2}\|\zeta\|_{L^3} \\
&\le C\left(\|\nabla u^\delta\|_{L^2} + \|(\rho^\delta)^{1/2}\dot{u}^\delta\|_{L^2}\right)\|\zeta\|_{H^1},
\end{aligned}$$

which together with (2.6) gives

$$\int_0^\infty \|(\rho^\delta |u^\delta|^2)_t\|_{H^{-1}}^2 dt \le C. \tag{4.42}$$

It follows from (2.6) that

$$\sup_{t\in[0,\infty)} \|\rho^\delta |u^\delta|^2\|_{L^1\cap L^3} \le C,$$

which combining with (4.42), (4.35), and (4.37) yields (4.38).

Finally, we prove (4.39) which implies the strong limits of u^δ and θ^δ. We deduce from (2.6), (1.12), (4.41) that

$$\sup_{t\in[0,\infty)} (\|u^\delta\|_{H^1} + \sigma^2\|G^\delta\|_{H^1} + \sigma^2\|\omega^\delta\|_{H^1} + \sigma^2\|\nabla\theta^\delta\|_{H^1}) \le C, \tag{4.43}$$

and

$$\int_\tau^T \left(\|u_t^\delta\|^2_{L^2(\mathbb{R}^3)} + \|G_t^\delta\|^2_{H^{-1}(\mathbb{R}^3)} + \|\omega_t^\delta\|^2_{H^{-1}(\mathbb{R}^3)} + \|\theta_t^\delta\|^2_{H^1(\mathbb{R}^3)}\right) dt \le C(\tau,T), \tag{4.44}$$

for all $0<\tau<T<\infty$. The Aubin-Lions lemma together with (4.43) and (4.44) thus gives (4.39).

Step 3. Proofs of (0.26).

Finally, to finish the proof of Theorem 0.2, it remains to prove (0.26). Since (ρ,u) satisfies (0.17), for the standard mollifier $j_\nu(x)(\nu>0)$, $\rho^\nu \triangleq \rho * j_\nu$ satisfies

$$\begin{cases} \rho^\nu_t + \operatorname{div}(u\rho^\nu) = r_\nu, \\ \rho^\nu(x,t=0) = \rho_0 * j_\nu, \end{cases} \tag{4.45}$$

where r_ν satisfies, for any $T>0$,

$$\lim_{\nu\to 0^+} \int_0^T \|r_\nu\|^2_{L^2} dt = 0, \tag{4.46}$$

due to (2.9), (2.6), and [15, Lemma 2.3]. Multiplying (4.45) by $4(\rho^\nu-1)^3$, we obtain after integration by parts that, for $t\ge 1$,

$$\begin{aligned} &(\|\rho^\nu - 1\|^4_{L^4})' \\ &= -4\int(\rho^\nu-1)^3\operatorname{div}u dx - 3\int(\rho^\nu-1)^4\operatorname{div}u dx + 4\int r_\nu(\rho^\nu-1)^3 dx \\ &\le C\|\rho^\nu-1\|^4_{L^4} + C\|\nabla u\|^4_{L^4} + C\|r_\nu\|_{L^2}, \end{aligned} \tag{4.47}$$

which implies that, for all $1\le N\le s\le N+1\le t\le N+2$,

$$\begin{aligned} \|\rho^\nu(\cdot,t)-1\|^4_{L^4} \le& \|\rho^\nu(\cdot,s)-1\|^4_{L^4} + C\int_N^{N+2} \left(\|\rho^\nu-1\|^4_{L^4} + \|\nabla u\|^4_{L^4}\right) dt \\ &+ C\int_N^{N+2} \|r_\nu\|_{L^2} dt. \end{aligned} \tag{4.48}$$

Letting $\nu\to 0^+$ in (4.48) together with (4.46) and (0.22) yields that

$$\|\rho(\cdot,t)-1\|^4_{L^4} \le \|\rho(\cdot,s)-1\|^4_{L^4} + C\int_N^{N+2} \left(\|\rho-1\|^4_{L^4} + \|\nabla u\|^4_{L^4}\right) dt. \tag{4.49}$$

Integrating (4.49) with respect to s over $[N,N+1]$ leads to

$$\begin{aligned} \sup_{t\in[N+1,N+2]} \|\rho(\cdot,t)-1\|^4_{L^4} &\le C\int_N^{N+2} \left(\|\rho-1\|^4_{L^4} + \|\nabla u\|^4_{L^4}\right) dt \\ &\to 0, \text{ as } N\to\infty, \end{aligned} \tag{4.50}$$

due to (4.40). This together with (0.25) and (0.28) implies that, for all $p \in (2,\infty)$,

$$\lim_{t\to\infty}\int |\rho-1|^p dx = 0. \tag{4.51}$$

Finally, we will prove

$$\lim_{t\to\infty}\left(\|u\|_{L^4}+\|\nabla\theta\|_{L^2}\right)=0, \tag{4.52}$$

which combining with (4.51), (0.25), (0.27)–(0.29), and the Gagliardo-Nirenberg inequality thus gives (0.26). In fact, one deduces from (0.27)–(0.29) that

$$\begin{aligned}&\int_1^\infty \left(\|u\|_{L^4}^4+\|\nabla\theta\|_{L^2}^2\right)dt\\ &\le C\int_1^\infty \|u\|_{L^2}\|\nabla u\|_{L^2}^3 dt+\int_1^\infty \|\nabla\theta\|_{L^2}^2 dt\\ &\le C,\end{aligned} \tag{4.53}$$

$$\begin{aligned}\int_1^\infty \left|\frac{d}{dt}\left(\|u(\cdot,t)\|_{L^4}^4\right)\right|dt &= 4\int_1^\infty \left|\int |u|^2 u\cdot u_t dx\right|dt\\ &\le C\int_1^\infty \|u\|_{L^\infty}\|u\|_{L^4}^2\|u_t\|_{L^2}dt\\ &\le C,\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}\int_1^\infty \left|\frac{d}{dt}\left(\|\nabla\theta(\cdot,t)\|_{L^2}^2\right)\right|dt &= 2\int_1^\infty \left|\int \nabla\theta\cdot\nabla\theta_t dx\right|dt\\ &\le C\int_1^\infty \|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}dt\\ &\le C.\end{aligned} \tag{4.55}$$

Thus, we derive (4.52) easily from (4.53)–(4.55). The proof of Theorem 0.2 is finished.

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