数理解析研究所講究録
第 1831 巻 2013 年 130-157

Exhibition of Mathematical Methods
English Translation of Sanpō-Hakki 算法発揮

Matumoto, Takao[1] (松本 堯生) and Chijiwa, Tomohiro[2] (千々和 智大)
Professor Emeritus, Hiroshima Univ. Doctor of Science, Hiroshima Univ.

Translators' Preface

Sanpō-hakki [算法発揮] is the first published book in the world on determinants. It was published in 1690 at Osaka. Three volumes are devoted to the theory of elimination using determinants, applied to the problems with answers and formation of equations. We can see the original texts at Wasan DB[3] in Degital Collections of Tohoku University. This is an abridged translation of Sanpō-hakki with call number 7.20306.1 of Kano(狩野) collection. There we can see one more published Sanpō-hakki with call number 63 of Okamoto-kan (岡本刊) collection which has an advertisement of some medicine at the end of volume 2. Japan Academy [日本学士院] has an original Sanpō-hakki owned by Endo with claim number 1705 and one more but without volume 3. Mathematics Department of Kyoto University also has a beautiful Sanpō-hakki[4]. In 1710 it was published again, deleting the name of the author at the first page of each volume. Wasan Institute [和算研究所] and Koju Bunko[5] [高樹文庫] have this version. So, we know now at least six complete original texts with two versions. There are also some written copies and in 1935 Sūgaku-koten-shoin [数学古典書院] published copies produced on a mimeograph.

The name of the author of Sanpō-hakki is explicitly written at each volume as IZEKI Tomotoki [井関知辰] but by the custom of that time his teacher SHIMADA Naomasa [嶋田尚政] can be considered to be the actual author.

The volume 2 and the main part of volume 3 are written only by Chinese characters, but the other parts including the volume 1 which may be considered as explanations are written by Chinese characters and Japanese alphabets katakana mixed.

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[1]Additional affiliation: Vice-Director of Seki-Kowa Mathematical Institute at Yokkaichi University and Long-term Researcher at RIMS Kyoto University, e-mail: matumoto@math.sci.hiroshima-u.ac.jp
[2]Additional affiliation: Yamaguchi-Konan junior high school, e-mail: chijiwa.tomohiro@yamaguchi-ygc.ed.jp

[3]http://dbr.library.tohoku.ac.jp/infolib/meta_pub/G0000002wasan (Choose English if necessary.)
[4]http://edb.math.kyoto-u.ac.jp/wasan/109
[5]Due to Ishigurao [石黒] and c/o Imizu-city Shinminato Museum [射水市新湊博物館].

In Sanpō-hakki the new unknown is intorduced by the name certain unknown [何某 bo] in the volume 1, **celestial element** [天元 Tengen] in the volume 2 and Tengen with some adjective in the volume 3. The **side-notation** [傍書法 bōsho-ho] used in Sanpō-hakki is different from that of Seki school. One vertical segment indicates the term of each degree of the unknown in vertical order and the side-notation contains not only literal coefficients but also numerical coefficients expressed by Chinese characters and $\pm$ signs. But in this translation we use a horizontal segment instead of vertical one and side-notation becomes up-notation. We add sometimes the corresponding modern notation or phrases in [ ] like [$a + bx + cx^2 = 0$] or [eliminated expression]. Note that the same notation was used to represent the formula $a + bx + cx^2$ and the equation $a+bx+cx^2 = 0$. Note also that an equal sign = and a parenthesis () were not invented at that time in Japan. The footnotes are the translators' explanations or comments, while the explanatory notes at the top of the volume 3 is the author's.

For us the volume 1 is rather easy to understand but the volumes 2 and 3 seem unfamiliar. So, we add some comment as an appendix at the end.

0 Sanpō-Hakki, Foreword

What occurs from heaven odd and earth even is the number. The world is not mixed up due to it. Today and the past are also calm. The operation of the nation is managed well and the difference of people goes without a deadlock. Though the purification keeps changing day and night, the transition of time does not differ at all. It runs well according to the objects. There is a person named IZEKI Tomotoki. He learned Mathematics from SHIMADA Naomasa. He has been intelligent since he was young and it is hard to explain that in words. He understands theory in one step and develops his wisdom in a half. At last, he discovered what the ancients could not find and mentioned what the predecessors could not say. That is as if Yellow Emperor meets Reishu's reputation. How can Mathematics in Wei-Tang dynasty reach his readiness? Now, he edited Sanpō-Hakki in 3 volumes for people. The "theory and application" are simply outlined and economically detailed. The people around the world, who try to open their eyes to Mathematics, can escape from the pitch-dark cave. It can be said that he is a person who knows the unearth reason and the obvious phenomena. He, however, became thirty years old. Why does he stop here?

Written[6] by hokusui lordless samurai Ichijiken Ichu[7] in May 1690

with two seals (一時軒 岡西惟中).

[6]This line and the next line are deleted in the 2nd version.

[7]OKANISHI Ichu (1639-1711). This foreword written by an old haiku poet [俳人 haijin] is difficult to understand and translate.

1 Sanpō-Hakki, Volume 1 edited by IZEKI Jubeejo Tomotoki

a student of SHIMADA Naomasa

When we want to solve the problem, sometimes it is difficult to obtain an answer equation [in an asking unknown quantity x] directly. In such a case considering every quantity in the possible answer equation as known, we introduce a new unknown quantity [y] by the celestial element method[8] and form two equations [in y] specifying their coefficients by the names of quantities, numerical factor and plus or minus at the side of each degree of unknown as usual[9]. Distinguish them as the former equation and the latter equation. Using them we would get an answer equation[10] [in x]. If it is hard to do it in one step, we repeat it several times and get the answer process [to find x].

1.1 Case of two quadratic equations

[Let two quadratic equations in y be given:][11]

The former equation	$\underline{+a}$	$\underline{+b}$	$\underline{+c}$	$[a+by+cy^2=0]$	in the new unknown [y],
The latter equation	$\underline{+d}$	$\underline{+e}$	$\underline{+f}$	$[d+ey+fy^2=0]$	in the new unknown [y].

Multiplying the latter equation by a in the former equation, we obtain $\underline{+ad}$ $\underline{+ae}$ $\underline{+af}$ from which we subtract the equation $\underline{+ad}$ $\underline{+bd}$ $\underline{+cd}$, the former equation multiplied by d in the latter equation. Let the remainder[12] be *the first equation.*

Multiplying the former equation by f in the latter equation, we obtain $\underline{+af}$ $\underline{+bf}$ $\underline{+cf}$ from which we subtract the equation $\underline{+cd}$ $\underline{+ce}$ $\underline{+cf}$, the latter equation multiplied by c in the former equation. Let the remainder be *the second equation.*

[8]The celestial element method is a method to form an algebraic equation in one variable x with numerical coefficients. This method enables us to manipulate polynomials in x; a polynomial $a+bx+cx^2+dx^3=0$ is represented by the vector of its coefficients: $\underline{+a}$ $\underline{+b}$ $\underline{+c}$ $\underline{+d}$. In the same way the unknown quantity y is represented by the vector $\bigcirc$ $\underline{+1}$ for example in the volume 3.

[9]This part seems to come from Hatusbi-sanpō-endangenkai [発微算法演段諺解]. Tanaka's Sōshiki-ikkan-no-jutsu [双式一貫之術] in Sangaku-funkai [算学紛解] 第 1 巻 has almost the same sentence.

[10]How to get this equation in x is the theme of the volume 1.

[11]The former equation $a+by+cy^2=0$ is represented by a vector $\underline{+a}$ $\underline{+b}$ $\underline{+c}$, where a, b, and c are polynomials in x with numerical coefficients; the latter equation is represented similarly. In the original text, the author uses *kana* (Japanese alphabets) *i*(イ), *ro*(ロ), *ha*(ハ), $\cdots$, which are rendered by alphabets a, b, c, $\cdots$.

[12]Actually it is the remainder divided by the unknown [y].

[Quadratic converted[13] expression[14] :]

Quadratic
Yō-ritsu

$-bd$ $+ae$	$-cd$ $+af$
g	h
$-cd$ $+af$	$-ce$ $+bf$
i	j

The first equation in the new unknown [y].
[$(ae-bd)+(af-cd)y=0$
expressed as $g+hy=0$.]

The second equation in the new unknown [y].
[$(af-cd)+(bf-ce)y=0$
expressed as $i+jy=0$.]

[Quadratic eliminated[15] expression:]

Quadratic *In-ritsu* $\oplus\, gj$ $\ominus\, hi$ [The unknown y is now elimited:
$gj-hi=0$ is an equation in the unknown x.]

The solution is the following: g in the first equation is the unknown[16] [y] multiplied by h. So, g multiplied by j is the unknown [y] multiplied by h and j. Move it to the left-hand side[17] as *Left* [$gj=-hjy \to$ *Left*] . Hence it is positive [$+gj$].

i in the second equation is the unknown [y] multiplied by j. So, i multiplied by h is the unknown [y] multiplied by h and j. It cancels out the formula *Left* at the left-hand side [$gj-hi=$ *Left* $+hjy=-hjy+hjy=0$] . Hence it is negative[18] [$-hi$].

1.2 Case of two cubic equations

[Let two cubic equations in y be given:]

The former equation $\underline{+a}$ $\underline{+b}$ $\underline{+c}$ $\underline{+d}$ in the new unknown [y].
The latter equation $\underline{+e}$ $\underline{+f}$ $\underline{+g}$ $\underline{+h}$ in the new unknown [y].

Multiplying the latter equation by a in the former equation, we obtain the equation $\underline{+ae}$ $\underline{+af}$ $\underline{+ag}$ $\underline{+ah}$ from which we subtract the equation $\underline{+ae}$ $\underline{+be}$ $\underline{+ce}$ $\underline{+de}$, the former equation multiplied by e in the latter equation. Let the remainder be *the first equation*.

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13 Sanpō-hakki uses the term *Yō-ritsu* [陽率] but the term Kanshiki [換式], which means the converted equations, used in Kaifukudai-no-hō [解伏題之法] seems better for us to understand.

14 The encircled one in the table, for example g indicates that g is defined by the above, i.e., $g := ae - bd$.

15 Sanpō-hakki uses the term *In-ritsu* [陰率]. We find that the *In-ritsu* equation is obtained by eliminating the extra unknown y from two equations in y. In fact, this is the determinant of the above *Yōritsu* [陽率] matrix and the resultant of the former and latter equations.

16 The signs are ignored. In the modern notation $g=-hy$.

17 The operation is originally done on the counting board. When the formula on the board is saved in a memory called the left-hand side, the board is reset to start a new operation.

18 If a formula A is saved in the left-hand side and a formula B is on the board, a new equation $A-B=0$ can be formed by cancellation.

Multiplying the latter equation by b in the former equation, we obtain the equation $\underline{+be}\ \underline{+bf}\ \underline{+bg}\ \underline{+bh}$ from which we subtract the equation $\underline{+af}\ \underline{+bf}\ \underline{+cf}\ \underline{+df}$, the former equation multiplied by f in the latter equation. Then, add the first equation to the remainder $\underline{\begin{smallmatrix}-af\\+be\end{smallmatrix}}\ \underline{}\ \underline{\begin{smallmatrix}-cf\\+bg\end{smallmatrix}}\ \underline{\begin{smallmatrix}-df\\+bh\end{smallmatrix}}$. Let the sum be *the second equation.*

Multiplying the former equation by h in the latter equation, we obtain the equation $\underline{+ah}\ \underline{+bh}\ \underline{+ch}\ \underline{+dh}$ from which we subtract the equation $\underline{+de}\ \underline{+df}\ \underline{+dg}\ \underline{+dh}$, the latter equation multiplied by d in the former equation. Let the remainder be *the third equation.*

[Cubic converted expression :]

	$\underline{\begin{smallmatrix}-be\\+af\end{smallmatrix}}$ i	$\underline{\begin{smallmatrix}-ce\\+ag\end{smallmatrix}}$ j	$\underline{\begin{smallmatrix}-de\\+ah\end{smallmatrix}}$ k	The first equation in the new unknown [y].
Cubic *Yō-ritsu*	$\underline{\begin{smallmatrix}-ce\\+ag\end{smallmatrix}}$ l	$\underline{\begin{smallmatrix}-de\\+ah\\-cf\\+bg\end{smallmatrix}}$ m	$\underline{\begin{smallmatrix}-df\\+bh\end{smallmatrix}}$ n	The second equation in the new unknown [y].
	$\underline{\begin{smallmatrix}-de\\+ah\end{smallmatrix}}$ o	$\underline{\begin{smallmatrix}-df\\+bh\end{smallmatrix}}$ p	$\underline{\begin{smallmatrix}-dg\\+ch\end{smallmatrix}}$ q	The third equation in the new unknown [y].

[Cubic eliminated expression:]

Cubic *In-ritsu*	⊕ 3 terms:	imq	jno	klp	[$imq + jno + klp$
	⊖ 3 terms:	inp	jlq	kmo	$-inp - jlq - kmo = 0$]

The solution is the following: To the coefficients[19] of the first equation we attach their signs as $\underline{+i}\ \underline{-j}\ \underline{+k}$. Then, we can get the cubic *In-ritsu* by multiplying them by quadratic *In-ritsu.*

> In the following three figures the letters in ○ are those in the cubic *Yō-ritsu.* The letters out of ○ are those in the quadratic *Yō-ritsu.* The meaning of the figures should be understood by comparing them.

[19]The words "the coefficients of" are necessary in the context, although not contained literally.

+ ⓘ		
	ⓜ g	ⓝ h
	ⓟ i	ⓠ j

Reciting the quadratic *In-ritsu* according to the letters out of ○, we write the right figure by using the letters in ○.

$+\,mq$
$-\,np$

Multiply by +ⓘ.

$+\,imq$
$-\,inp$

	− ⓙ	
ⓛ g		ⓝ h
ⓞ i		ⓠ j

The same as above.

$+\,lq$
$-\,no$

Multiply by −ⓙ.

$-\,jlq$
$+\,jno$

		+ ⓚ
ⓛ g	ⓜ h	
ⓞ i	ⓟ j	

The same as above.

$+\,lp$
$-\,mo$

Multiply by +ⓚ.

$+\,klp$
$-\,kmo$

The sum of three terms with + sign and the sum of three terms with − sign are equal. Thus, this is the cubic *In-ritsu*.

Although it is clear by the above figures, those who doubt it should understand it as follows:

Multiplying the second equation by ⓘ in the first equation, we obtain the equation $\underline{+\,il}$ $\underline{+\,im}$ $\underline{+\,in}$ from which we subtract the equation $\underline{+\,il}$ $\underline{+\,jl}$ $\underline{+\,kl}$, the first equation multiplied by ⓛ in the second equation. Let the remainder be *the top equation.*

Multiplying the third equation by ⓘ in the first equation, we obtain the equation $\underline{+\,io}$ $\underline{+\,ip}$ $\underline{+\,iq}$ from which we subtract the equation $\underline{+\,io}$ $\underline{+\,jo}$ $\underline{+\,ko}$, the first equation multiplied by ⓞ in the third equation. Let the remainder be *the bottom equation.*

The top equation	$-\,jl$ $\underline{+\,im}$ ⓖ	$-\,kl$ $\underline{+\,in}$ ⓗ	in the new unknown $[y]$.
The bottom equation	$-\,jo$ $\underline{+\,ip}$ ⓘ	$-\,ko$ $\underline{+\,iq}$ ⓙ	in the new unknown $[y]$.

We obtain the following equation from quadratic *In-ritsu*. The multiplication of g and j together is

$$\begin{array}{ll} + & jklo \\ - & ijlq \\ - & ikmo \\ + & iimq\,. \end{array}$$

Move the sum to the left-hand side as *Left*. The multiplication of h and i together is $\begin{array}{cl} + & jklo \\ - & iklp \\ - & ijno \\ + & iinp\ . \end{array}$ The sum cancels out the formula *Left* at the left-hand side. Hence we subtract it from *Left* and get $\begin{array}{cl} - & ⓘjlq \\ - & ⓘkmo \\ + & ⓘimq \\ + & ⓘklp \\ + & ⓘjno \\ - & ⓘinp\ . \end{array}$ We obtain the cubic *In-ritsu* by dividing all of them by ⓘ.

Although we know how to obtain the *In-ritsu* in this way, it becomes complicated in the case of higher degree. We, therefore, calculate the *In-ritsu* by the previous method. Since we can obtain the *In-ritsu* of arbitrary degree in the same way, we omit this kind of explanation in the cases of 4th, 5th and 6th degree. We follow only the previous method.

1.3 Case of two quartic[20] equations

[Let two quartic equations in y be given:]

The former equation	$\underline{+a}$	$\underline{+b}$	$\underline{+c}$	$\underline{+d}$	$\underline{+e}$	in the new unknown [y].
The latter equation	$\underline{+f}$	$\underline{+g}$	$\underline{+h}$	$\underline{+i}$	$\underline{+j}$	in the new unknown [y].

From the equation obtained by multiplying the latter equation by a in the former equation we subtract the equation obtained by multiplying the former equation by f in the latter equation. Let the remainder be *the first equation.*

From the equation obtained by multiplying the latter equation by b in the former equation we subtract the equation obtained by multiplying the former equation by g in the latter equation. Then, add the first equation to the remainder. Let the sum be *the second equation.*

From the equation obtained by multiplying the latter equation by c in the former equation we subtract the equation obtained by multiplying the former equation by h in the latter equation. Then, add the second equation to the remainder. Let the sum be *the third equation.*

From the equation obtained by multiplying the former equation by j in the latter equation we subtract the equation obtained by multiplying the latter equation by e in

[20]The equation of degree $n+1$ is written as n-jō-hōshiki [n 乗方式] when $n \geq 3$.

the former equation. Let the remainder be *the fourth equation.*

[Quartic converted expression:]

Quartic *Yō-ritsu*

$-bf$ $+ag$ k	$-cf$ $+ah$ l	$-df$ $+ai$ m	$-ef$ $+aj$ n	The first equation in the new unknown [y].
$-cf$ $+ah$ o	$-df$ $+ai$ $-cg$ $+bh$ p	$-ef$ $+aj$ $-dg$ $+bi$ q	$-eg$ $+bj$ r	The second equation in the new unknown [y].
$-df$ $+ai$ s	$-ef$ $+aj$ $-dg$ $+bi$ t	$-eg$ $+bj$ $-dh$ $+ci$ u	$-eh$ $+cj$ v	The third equation in the new unknown [y].
$-ef$ $+aj$ w	$-eg$ $+bj$ x	$-eh$ $+cj$ y	$-ei$ $+dj$ z	The fourth equation in the new unknown [y].

[Quartic eliminated expression:]

Quartic *In-ritsu*

⊕ 12 terms:	*kpuz*	*kqvx*	*krty*
	lovy	*lqsz*	*lruw*
	motz	*mpvw*	*mrsx*
	noux	*npsy*	*nqtw*
⊖ 12 terms:	*kpvy*	*kqtz*	*krux*
	louz	*lqvw*	*lrsy*
	movx	*mpsz*	*mrtw*
	noty	*npuw*	*nqsx*

The solution is the following: To the coefficients of the first equation we attach their signs as $\underline{+k}\ \underline{-l}\ \underline{+m}\ \underline{-n}$. Then, we can get the quartic *In-ritsu* by multiplying them by cubic *In-ritsu.*

In the following four figures the letters in ◯ are those in the quartic *Yō-ritsu.* The letters out of ◯ are those in the cubic *Yō-ritsu.* The meaning of the figures should be understood by comparing them.

$+$ ⓚ			
	ⓟ i	ⓠ j	ⓡ k
	ⓣ l	ⓤ m	ⓥ n
	ⓧ o	ⓨ p	ⓩ q

Reciting the cubic *In-ritsu* according to the letters out of ○, we write the right figure by using the letters in ○.

	Multiply by + ⓚ.	
$+puz$		$+kpuz$
$+qvx$		$+kqvx$
$+rty$		$+krty$
$-pvy$		$-kpvy$
$-qtz$		$-kqtz$
$-rux$		$-krux$

	$-$ ⓛ		
ⓞ i		ⓠ j	ⓡ k
ⓢ l		ⓤ m	ⓥ n
ⓦ o		ⓨ p	ⓩ q

The same as above.

	Multiply by − ⓛ.	
$+ouz$		$-louz$
$+qvw$		$-lqvw$
$+rsy$		$-lrsy$
$-ovy$		$+lovy$
$-qsz$		$+lqsz$
$-ruw$		$+lruw$

		$+$ ⓜ	
ⓞ i	ⓟ j		ⓡ k
ⓢ l	ⓣ m		ⓥ n
ⓦ o	ⓧ p		ⓩ q

The same as above.

	Multiply by + ⓜ.	
$+otz$		$+motz$
$+pvw$		$+mpvw$
$+rsx$		$+mrsx$
$-ovx$		$-movx$
$-psz$		$-mpsz$
$-rtw$		$-mrtw$

			$-$ ⓝ
ⓞ i	ⓟ j	ⓠ k	
ⓢ l	ⓣ m	ⓤ n	
ⓦ o	ⓧ p	ⓨ q	

The same as above.

	Multiply by − ⓝ.	
$+oty$		$-noty$
$+puw$		$-npuw$
$+qsx$		$-nqsx$
$-oux$		$+noux$
$-psy$		$+npsy$
$-qtw$		$+nqtw$

The sum of 12 terms with + sign and the sum of 12 terms with − sign are equal. Thus, this is the quartic *In-ritsu*.

1.4 Case of two quintic equations

[Let two quintic equations in y be given:]

The former equation	$\underline{+a}\ \underline{+b}\ \underline{+c}\ \underline{+d}\ \underline{+e}\ \underline{+f}$	in the new unknown [y].
The latter equation	$\underline{+g}\ \underline{+h}\ \underline{+i}\ \underline{+j}\ \underline{+k}\ \underline{+l}$	in the new unknown [y].

From the equation obtained by multiplying the latter equation by a in the former equation, we subtract the equation obtained by multiplying the former equation by g in the latter equation. Let the remainder be *the first equation.*

From the equation obtained by multiplying the latter equation by b in the former equation, we subtract the equation obtained by multiplying the former equation by h in the latter equation. Then, add the first equation to the remainder. Let the sum be *the second equation.*

From the equation obtained by multiplying the latter equation by c in the former equation we subtract the equation obtained by multiplying the former equation by i in the latter equation. Then, add the second equation to the remainder. Let the sum be *the third equation.*

From the equation obtained by multiplying the latter equation by d in the former equation we subtract the equation obtained by multiplying the former equation by j in the latter equation . Then, add the third equation to the remainder. Let the sum be *the fourth equation.*

From the equation obtained by multiplying the former equation by l in the latter equation we subtract the equation obtained by multiplying the latter equation by f in the former equation . Let the remainder be *the fifth equation.*

[Quintic converted expression:]

Quintic
Yō-ritsu

$-bg$ $+ah$ m	$-cg$ $+ai$ n	$-dg$ $+aj$ o	$-eg$ $+ak$ p	$-fg$ $+al$ q	The first equation in the new unknown $[y]$.
$-cg$ $+ai$ r	$-dg$ $+aj$ $-ch$ $+bi$ s	$-eg$ $+ak$ $-dh$ $+bj$ t	$-fg$ $+al$ $-eh$ $+bk$ u	$-fh$ $+bl$ v	The second equation in the new unknown $[y]$.
$-dg$ $+aj$ w	$-eg$ $+ak$ $-dh$ $+bj$ x	$-fg$ $+al$ $-eh$ $+bk$ $-di$ $+cj$ y	$-fh$ $+bl$ $-ei$ $+ck$ z	$-fi$ $+cl$ A	The third equation in the new unknown $[y]$.
$-eg$ $+ak$ B	$-fg$ $+al$ $-eh$ $+bk$ C	$-fh$ $+bl$ $-ei$ $+ck$ D	$-fi$ $+cl$ $-ej$ $+dk$ E	$-fj$ $+dl$ F	The fourth equation in the new unknown $[y]$.
$-fg$ $+al$ G	$-fh$ $+bl$ H	$-fi$ $+cl$ I	$-fj$ $+dl$ J	$-fk$ $+el$ K	The fifth equation in the new unknown $[y]$.

[Quintic eliminated expression:]

Quintic *In-ritsu* ⊕ 60 terms:

$msyEK$	$mszFI$	$msADJ$	$mtxFJ$
$mtzCK$	$mtAEH$	$muxDK$	$muyFH$
$muACI$	$mvxEI$	$mvyCJ$	$mvzDH$
$nryFJ$	$nrzDK$	$nrAEI$	$ntwEK$
$ntzFG$	$ntABJ$	$nuwFI$	$nuyBK$
$nuADG$	$nvwDJ$	$nvyEG$	$nvzBI$
$orxEK$	$orzFH$	$orACJ$	$oswFJ$
$oszBK$	$osAEG$	$ouwCK$	$ouxFG$
$ouABH$	$ovwEH$	$ovxBJ$	$ovzCG$
$prxFI$	$pryCK$	$prADH$	$pswDK$
$psyFG$	$psABI$	$ptwFH$	$ptxBK$
$ptACG$	$pvwCI$	$pvxDG$	$pvyBH$
$qrxDJ$	$qryBH$	$qrzCI$	$qswEI$
$qsyBJ$	$qszDG$	$qtwCJ$	$qtxEG$
$qtzBH$	$quwDH$	$quxBI$	$quyCG$

Quintic *In-ritsu* ⊖ 60 terms:

$msyFJ$	$mszDK$	$msAEI$	$mtxEK$
$mtzFH$	$mtACJ$	$muxFI$	$muyCK$
$muADH$	$mvxDJ$	$mvyEH$	$mvzCI$
$nryEK$	$nrzFI$	$nrADJ$	$ntwFJ$
$ntzBK$	$ntAEG$	$nuwDK$	$nuyFG$
$nuABI$	$nvwEI$	$nvyBJ$	$nvzDG$
$orxFJ$	$orzCK$	$orAEH$	$oswEK$
$oszFG$	$osABJ$	$ouwFH$	$ouxBK$
$ouACG$	$ovwCJ$	$ovxEG$	$ovzBH$
$prxDK$	$pryFH$	$prACI$	$pswFI$
$psyBK$	$psADG$	$ptwCK$	$ptxFG$
$ptABH$	$pvwDH$	$pvxBI$	$pvyCG$
$qrxEI$	$qryCJ$	$qrzDH$	$qswDJ$
$qsyEG$	$qszBI$	$qtwEH$	$qtxBJ$
$qtzCG$	$quwCI$	$quxDG$	$quyBH$

(*We omit to translate the solution with 5 figures similar to the case of quartic equations except the following last sentence.*)

The sum of 60 terms with + sign and the sum of 60 terms with − sign are equal. Thus, this is the quintic *In-ritsu*.

1.5 Case of two sextic equations

[Let two sextic equations in y be given:]

The former equation $\underline{+a}\ \underline{+b}\ \underline{+c}\ \underline{+d}\ \underline{+e}\ \underline{+f}\ \underline{+g}$ in the new unknown [y].
The latter equation $\underline{+h}\ \underline{+i}\ \underline{+j}\ \underline{+k}\ \underline{+l}\ \underline{+m}\ \underline{+n}$ in the new unknown [y].

Since how to obtain the *Yo-ritsu* is the same as before, we omit it.

Sextic *Yō-ritsu*

$-bh$ $+ai$ ○	$-ch$ $+aj$ ○	$-dh$ $+ak$ ○	$-eh$ $+al$ ○	$-fh$ $+am$ ○	$-gh$ $+an$ ○	The first equation.
$-ch$ $+aj$ ○	$-dh$ $+ak$ $-ci$ $+bj$ ○	$-eh$ $+al$ $-di$ $+bk$ ○	$-fh$ $+am$ $-ei$ $+bl$ ○	$-gh$ $+an$ $-fi$ $+bm$ ○	$-gi$ $+bn$ ○	The second equation.
$-dh$ $+ak$ ○	$-eh$ $+al$ $-di$ $+bk$ ○	$-fh$ $+am$ $-ei$ $+bl$ $-dj$ $+ck$ ○	$-gh$ $+an$ $-fi$ $+bm$ $-ej$ $+cl$ ○	$-gi$ $+bn$ $-fj$ $+cm$ ○	$-gj$ $+cn$ ○	The third equation.
$-eh$ $+al$ ○	$-fh$ $+am$ $-ei$ $+bl$ ○	$-gh$ $+an$ $-fi$ $+bm$ $-ej$ $+cl$ ○	$-gi$ $+bn$ $-fj$ $+cm$ $-ek$ $+dl$ ○	$-gj$ $+cn$ $-fk$ $+dm$ ○	$-gk$ $+dn$ ○	The fourth equation.
$-fh$ $+am$ ○	$-gh$ $+an$ $-fi$ $+bm$ ○	$-gi$ $+bn$ $-fj$ $+cm$ ○	$-gj$ $+cn$ $-fk$ $+dm$ ○	$-gk$ $+dn$ $-fl$ $+em$ ○	$-gl$ $+en$ ○	The fifth equation.
$-gh$ $+an$ ○	$-gi$ $+bn$ ○	$-gj$ $+cn$ ○	$-gk$ $+dn$ ○	$-gl$ $+en$ ○	$-gm$ $+fn$ ○	The sixth equation.

Sextic *In-ritsu*	⊕ 360 terms. ⊖ 360 terms.	By the diagram of sextic *Yo-ritsu* above we make six figures and then obtain the sextic *In-ritsu* by multiplying by the quintic *In-ritsu*. The method is the same as in the case of quadratic, cubic, quartic or quintic *In-ritsu*. Thus, we omit the details.

- We omit the case when the former and latter equations are of degree greater than 6. You can guess the method through the above examples. Although there are other methods to obtain the *In-ritsu* directly besides two methods given in this book, we do not mention about them because it is not easy for the beginners to understand. After understanding this book, you can find out the methods by yourselves.

- As stated at first, after finding the former and latter equations you will get the answer equation that means the equation in the true unknown by using *Yō-ritsu* and *In-ritsu* [the converted and eliminated expressions] for the appropriate degree. Now for the better understanding of the beginners we add the problems and their answers in the next volume and how to form the equations in the final volume. The able people should not follow the details and try to solve by themselves. Even with three volumes, which teach rules, answer processes and how to form the equations, it might be difficult for the usual people to understand without oral teaching. But if you never give up by the hardness and read them every day and night, you will understand them someday.

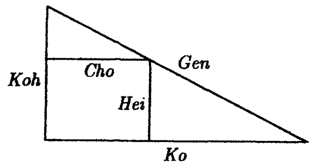
The end of Sanpo-Hakki Volume 1.

2 Sanpō-Hakki, Volume 2 edited by IZEKI Jubeejo Tomotoki

a student of SHIMADA Naomasa

2.1 Problem 1

[Problem:] There is a rectangle inside a right-angled triangle as shown in the figure. Let A be the sum of the hypotenuse [*Gen*, 弦] and the area [*Gai*, 外] outside the rectangle.[21] Let B be the sum of the height [*Koh*, 勾] and the long side [*Cho*, 長]. Let C be the sum of the base [*Ko*, 股] and the short side [*Hei*, 平]. Find the values of *Koh*, *Ko*, *Gen*, *Cho* and *Hei*.



Answer: By the process stated below, we can find *Koh*.

Answer process: By the method of celestial element, let *Koh* be the unknown. Subtracting it from B, we get *Cho* as the remainder $[Cho = B - Koh]$. Subtract it from C and multiply the remainder by *Koh*. Call the product α $[\alpha = Koh \times (C - Cho)]$.[22]

Take *Koh* and add C to it. Call the sum β $[\beta = Koh + C]$.[23]

Take 2 times A, from which we subtract C multiplied by *Koh* and call the remainder γ $[\gamma = 2A - Koh \times C]$.

Take *Koh* and add 2 times *Cho* to it. Call the sum δ $[\delta = Koh + 2Cho]$.

Take the sum of 4 times *Koh* squared[24] and 4 times C squared, from which we subtract γ squared and call the remainder ε $[\varepsilon = 4Koh^2 + 4C^2 - \gamma^2]$.

Take the sum of 8 times C and 2 times δ multiplied γ.[25] Call it ζ $[\zeta = 8C + 2\delta \times \gamma]$.

Take δ squared, from which we subtract 4 rods[26] and call the remainder η $[\eta = \delta^2 - 4]$.

---

[21] In the original text, given quantities are called *tada-iu-sū*(只云数), *mata-iu-sū*(又云数) and *betsu-iu-sū*(別云数), which are rendered by A, B and C.

[22] In the original text, the author uses the name of 28 constellations *kaku*(角), *kō*(亢), *tei*(氐), *bō*(房), *shin*(心), *bi*(尾), *ki*(箕), *kei*(計) (The author uses this instead of *tou*(斗).), *gyū*(牛), *jo*(女), $\cdots$, which are rendered by greek alphabets α, β, γ, δ, ε, ζ, η, θ, ι, κ, $\cdots$.

[23] The first operation is originally to arrange a polynomial on the counting board. But there is no comment on the arrangement in the case of taking the 4 sum. So, to take is used for every first operation in this translation.

[24] We use the term A squared rather than the square of A, because it is closer to the original expression A 巾.

[25] Later we will use the expression '2 times the product of γ and δ' as well.

[26] The number is counted by rods [算 (木)]. So, 4 rods is used for number 4 here.

Take the product of β and ε, from which we subtract the product of α and ζ. Call the remainder θ [$\theta = \beta \times \varepsilon - \alpha \times \zeta$].

Take the product of α and η and add ε to it. Call the sum ι [$\iota = \alpha \times \eta + \varepsilon$].

Take the product of β and η and add ζ to it. Call the sum κ [$\kappa = \beta \times \eta + \zeta$].

Take the product of κ and θ and move it to the left-hand side as *Left* [$Left = \kappa \times \theta$].

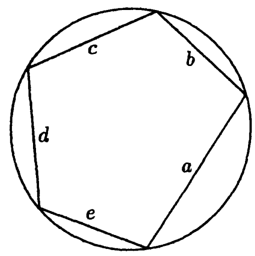
Take ι squared. Since it cancels out [the formula] *Left* at the left-hand side, we obtain an equation [$Left - \iota^2 = 0$]. Solving the equation of degree 8, we obtain *Koh*. It answers the problem.

2.2 Problems 2 to 6

(*We omit to translate the subsections 2.2 - 2.6.*)

2.7 Problem 7

[Problem:] There is a pentagon inside a circle as shown in the figure. When the length of five sides a, b, c, d and e are given, find the diameter of the circle.



Answer: By the process stated below, we can find the diameter of the circle.

Answer process: By the celestial element method let the diameter of the circle be the unknown. Its square is called α.

Take the 4th power of b and add the 4th power of c to it. From the sum subtract 2 times the product of b squared and c squared. Multiply the remainder by α, and call the product β.

Take the product of α and b squared and add the product of α and c squared to it. From the sum we subtract 2 times the product of b squared and c squared, and call the remainder γ.

Take the 4th power of d and add the 4th power of e to it. From the sum we subtract 2 times the product of d squared and e squared, and call the remainder δ.

Take the product of α and d squared and add the product of α and e squared to it. From the sum we subtract 2 times the product of d squared and e squared, and call the remainder ε.

Take α from which we subtract 2 times a squared, and call the remainder ζ.

Take 2 times the product of α squared and a squared and add 2 times the product of ε and ζ to it. From the sum we subtract the product of α and γ, and call the remainder η.

Take the 4 sum of the 6 times the product of α squared and the 4th power of a, 8 times the product of ε, ζ and a squared, 4 times the product of δ and ζ squared and 4 times ε squared. From the sum we subtract the sum of 2 times the product of α squared and δ, 4 times the product of α, ε and a squared and the product of α and β. Then, we call the remainder θ.

Take the 4 sum of the product of α squared and the 6th power of a, the product of δ, ε and ζ, the product of ε, ζ and the 4th power of a and 2 times the product of ε squared and a squared. From the sum we subtract the sum of the product of α squared, δ and a squared, 2 times the product of α, δ, ζ and a squared and 2 times the product of α, ε and the 4th power of a. Then, we call the remainder ι.

Take the 4 sum of the product of α squared and δ squared, the product of α squared and the 8th power of a, 4 times the product of ε squared and the 4th power of a and 2 times the product of α squared, δ and the 4th power of a. From the sum we subtract the sum of 4 times the product of α, δ, ε and a squared and 4 times the product of α, ε and the 6th power of a. Then, we call the remainder κ.

Take 4 times the product of β and ι from which we subtract 2 times the product of γ and κ, and call the remainder λ.

Take the product of β and θ from which we subtract the product of α and κ, and call the remainder μ.

Take the product of β and η and add the product of γ and θ to it. From the sum we subtract 2 times the product of α and ι, and we call the remainder ν.

Take the sum of the product of α and μ squared, 4 times the product of β squared, η and ν and 8 times the product of γ squared, η and λ and move it to the left-hand side as *Left*.

Take the sum of 2 times the product of α, ν and λ and 8 times the product of β, γ, η and μ. Since it cancels out [the formula] *Left* at the left-hand side, we obtain an equation. Solving the equation[27] of degree 14, we obtain the diameter. It answers the problem.

The end of Sanpo-Hakki Volume 2.

[27] Actually the equation of degree 7 in α.

3 Sanpō-Hakki, Volume 3

3.0 Explanatory notes

- The changes of plus and minus, addition and subtraction are as usual.

- Each encircled name of twenty-eight constellations[28] below[29] the horizontal segment ____ represents the place where the formula described above the horizontal segment ____ are moved to. Even if there are 2, 3, 5 or 10 names above ____, we abbreviate them into one name to simplify the sentences. In many books we find "寄何位"[30] but we omit "位" to simplify the sentences in the volume 2 of this book.

- There is $\dfrac{\begin{array}{l}+1 Koh\,Cho\\ -1 KohC\end{array}}{}$ in the former equation of Problem 1. It is an abbreviation of *KohCho* Multi. 1+ [勾長相乗一段正] and *KohC* Multi. 1− [勾別云数相乗一段負]; we omit the letters Multi. [相乗], the unit [段] and the value [云数] in the value C [別云数]. The letters *Eho* [法] for Ensekiho [円積法] in Problem 3 and *Gai* [外] for the area of complement in Problem 5 are the same kind of abbreviations.

- If there is a common name in all the coefficients of the former or latter equation, we can omit the name. If there is a common numerical divisor in all the coefficients of the former or latter equation, we can reduce the common divisor. We can apply the same to the [converted] *Yō*- or [eliminated] *In*-diagrams. This method is called reduction method of all common names and common divisors [遍省遍約之法 Henshō-henyaku-no-hō]. You will see this operation in the volume 3.

- In the answer process to Problem 3, we wrote: "Take 8 times the product of *Koh* and *Eho* from which we subtract α, and call the remainder β. " On the other hand, we have $\dfrac{\begin{array}{l}-2\\ +8\ KohEho\\ -1\ Koh\end{array}}{+\ \beta}$ in the former equation. Since the sum of one *Koh* and number 2 equals α, we can call β as in the volume 2. This is another method to simplify the sentences. There is a further example for the term θ of Problem 3, and so on.

[28]They are rendered by Greek letters.
[29]At the left-hand side of the vertical segment in the original text.
[30]It means "move it to so-and-so place". The letter "位" here means a place.

- In[31] Hatsubi-genkai, Meigen, Ikkyoku-sanpō and others, it is written the answer of second to seventh power formula but not written how to get it. So, the beginners cannot get the higher power formula. Here, we show how to get it.

  △ The second power formula:
  By the celestial element method we take a new unknown ○ ____ $[y]$ and move its square ○ ○ ____ $[y^2]$ to the left-hand side.
  The square of the new unknown[32] is supposed to be known and cancels out [the formula] *Left* at the left-hand side. Hence [we get]

  $$\textbf{the former equation}\quad \frac{+1\,何^2}{ⓐ}\quad ○\quad \frac{-1}{ⓒ}\qquad [何^2 - y^2 = 0].$$

  By the given problem we get

  $$\textbf{the latter equation}\quad \frac{\pm\ 実}{ⓓ}\quad \frac{\pm\ 法}{ⓔ}\quad ○\qquad [\pm 実 \pm 法 y = 0]$$

  in the same unknown [y]. The part of the degree 2 and greater is a multiple of the square of the unknown [y]. So, substituting the square we add it by sliding 2 places to left[33] using plus or minus as usual.
  Applying Case of two quadratic equations in the volume 1, we get the second power formula.

  △ The third power formula:
  By the celestial element method we take a new unknown ○ ____ $[y]$ and move its cube ○ ○ ○ ____ $[y^3]$ to the left-hand side.
  The cube of the unknown [y] is supposed to be known and cancels out [the formula] *Left* at the left-hand side. Hence [we get]

  $$\textbf{the former equation}\quad \frac{+1\,何^3}{ⓐ}\quad ○\quad ○\quad \frac{-1}{ⓓ}.$$

  By the given problem we get

  $$\textbf{the latter equation}\quad \frac{\pm\ 実}{ⓔ}\quad \frac{\pm\ 法}{ⓕ}\quad \frac{\pm\ 廉}{ⓖ}\quad ○$$

  in the same unknown [y]. The part of the degree 3 and greater is a multiple of the cube of the unknown [y]. So, substituting the cube we add it by sliding 3 places to left, using plus or minus as usual.
  Applying Case of two cubic equations in the volume 1, we get the third power formula.

---

[31] These names certainly stand for Hatsubi-sanpō-endangenkai [発微算法演段諺解] (1685), Meigen-sanpō [明元算法] (1689) and [一極算法] (1689).
[32] The square $何^2$ of the unknown y is usually given by a formula in the true unknown x.
[33] Of course "up" in the original text.

In the case of degree greater than 3, the method to get power formula is the same as the above examples and we omit further details.

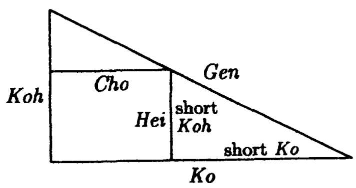
The end of Sanpo-Hakki Volume[34] 3: Explanatory notes.

3 Sanpō-Hakki, Volume 3 edited by IZEKI Jubeejo Tomotoki

a student of SHIMADA Naomasa

3.1 Problem 1, Formation of equations

- The sum of *Gai* and *Gen* is given as A $[A = Gai + Gen]$.
- The sum of *Ko* and *Hei* is given as C $[C = Ko + Hei]$.
- *Koh* is [supposed to be] given. It is the unknown in the answer process $[x = Koh]$.
- The *Cho* is given as the remainder of the subtraction of *Koh* from B $[Cho = B - Koh]$.



Temporarily let *Hei* be a new unknown $\bigcirc\ \underline{\quad\quad}$ $[y]$ by the celestial element method. Subtract it from C. Then, the remainder equals *Ko*; $\underline{+1\,C}\ \ \underline{-1\quad}$ $[C - y]$. Add *Koh* to it. Then, the sum equals the sum of *Koh* and *Ko*; $\begin{array}{ll} +1\,Koh & \\ \underline{+1\,C\quad} & \underline{-1\quad} \end{array}$ $[Koh + C - y]$.

Multiply it by *Hei* and move the product to the left-hand side; $\bigcirc\ \begin{array}{ll} +1\,Koh & \\ \underline{+1\,C\quad} & \underline{-1\quad} \end{array}$ $[Left = (Koh + C)y - y^2]$.

Take C and subtract *Cho* from it. Then, the remainder equals the sum of the short *Koh* and the short *Ko*; $\begin{array}{l} -1\,Cho \\ \underline{+1\,C\quad} \end{array}$ $[C - Cho]$. Multiply it by *Koh*. Then, the product equals[35] the product of *Hei* and the sum of *Koh* and *Ko*; $\begin{array}{l} -1\,KohCho \\ \underline{+1\,KohC\quad} \end{array}$ $[Koh \times C - Koh \times Cho]$. Since it cancels out [the formula] *Left* at the left-hand side, we get

[34] The title of the volume 3 appears again in the original book. So, we follow it.

[35] By the similarity of the right-angled triangles we have $(\text{short}Koh + \text{short}Ko)/Hei = (Koh + Ko)/Koh$.

$$\textbf{the former equation}\quad \begin{array}{c|c|c} +1\,KohCho & +1\,Koh & \\ -1\,KohC & +1\,C & -1 \\ \hline -\,ⓐ & +\,ⓑ & \end{array} \quad \text{in the unknown } Hei.$$

$$[-\alpha + \beta y - y^2 = 0;\ \alpha = Koh \times C - Koh \times Cho,\ \beta = Koh + C].$$

Take Ko; $\underline{+1C}\ \underline{-1}$ $[C-y]$, and multiply it by Koh. Then, the product equals 2 times the area of the triangle; $\underline{+1\,KohC}\ \underline{-1\,Koh}$ $[Koh \times C - Koh \times y]$. Move it to the 1st place.[36]

Take Hei ○ $\underline{\quad}$ $[y]$, and multiply it by 2 times Cho. Then, the product equals 2 times the area of the rectangle; ○ $\underline{+2\,Cho}$ $[2Cho \times y]$. Add 2 times A to it; $\underline{+2A}\ \underline{+2\,Cho}$ $[2A + 2Cho \times y]$. Subtract the formula at the 1st place from the sum. Then, the remainder equals 2 times Gen; $\begin{array}{c|c} -1\,KohC & +1\,Koh \\ +2A & +2\,Cho \\ \hline +\,ⓒ & +\,ⓓ \end{array}$ $[\gamma + \delta y,\ \gamma = 2A - Koh \times C, \delta = 2Cho + Koh]$. Move its square to the 2nd place ; $\underline{+1\gamma^2}\ \underline{+2\gamma\delta}\ \underline{+1\delta^2}$ $[\gamma^2 + 2\gamma\delta y + \delta^2 y^2]$.

Take Ko $\underline{+1C}\ \underline{-1}$ $[C-y]$ and make its square $\underline{+1C^2}\ \underline{-2C}\ \underline{+1}$ $[C^2 - 2Cy + y^2]$. Add Koh squared to it. Then, the sum equals Gen squared; $\begin{array}{c} +1\,Koh^2 \\ +1\,C^2 \\ \hline \end{array}$ $\underline{-2C}\ \underline{+1}$ $[Koh^2 + C^2 - 2Cy + y^2]$. Multiplying it by number 4, the product cancels out the formula at the 2nd place. So, we get

$$\textbf{the latter equation}\quad \begin{array}{c|c|c} +1\,\gamma^2 & & \\ -4\,Koh^2 & +2\ \gamma\delta & +1\ \delta^2 \\ -4\,C^2 & +8\ C & -4 \\ \hline -\,ⓔ & +ⓕ & +ⓖ \end{array} \quad \text{in the unknown } Hei.$$

$$[-\varepsilon + \zeta y + \eta y^2 = 0;\ \varepsilon = \gamma^2 - 4Koh^2 - 4C^2,\ \zeta = 2\gamma\delta + 8C,\ \eta = \delta^2 - 4].$$

Hereafter we refer to "Case of two quadratic equations" in the volume 1.

$$\textbf{The former equation}\quad \begin{array}{ccc} -\alpha & +\beta & -1 \\ \hline a & b & c \end{array} \quad \text{in the unkhown } Hei.$$

$$\textbf{The latter equation}\quad \begin{array}{ccc} -\epsilon & +\zeta & +\eta \\ \hline d & e & f \end{array} \quad \text{in the unknown } Hei.$$

Following the letters given under the segments in the quadratic *Yō-ritsu* [converted expression], we obtain the [converted] *Yō*-diagram as follows:

[36] In the original text the author uses the name of ten calender signs *kō*(甲), *otsu*(乙), *hei*(丙), *tei*(丁), *bo*(戊), $\cdots$, which are rendered by the n-th places by their meaning.

The [converted] *Yō*-diagram: $\left\{\begin{array}{ll} \quad +\beta\varepsilon & \quad -\varepsilon \\ \quad \underline{-\alpha\zeta} & \quad \underline{-\alpha\eta} \\ g: +\theta & h: -\iota \\ \quad -\varepsilon & \quad +\zeta \\ \quad \underline{-\alpha\eta} & \quad \underline{+\beta\eta} \\ i: -\iota & j: +\kappa \end{array}\right.$

The 1st equation in *Hei.*

The 2nd equation in *Hei.*

Now following the letters given under the segments in the quadratic *In-ritsu* [eliminated expression], we obtain the [eliminated] *In*-diagram as follows:

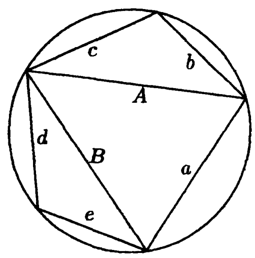
The [eliminated] *In*-diagram: $\left\{\begin{array}{ll} +gj: & +\theta\kappa \\ -hi: & -\iota^2 \end{array}\right.$

The terms with + sign in this diagram are moved to the left-hand side as in the volume 2 and canceled out by the terms with - sign.[37]

3.2 Problems 2 to 6, Formation of equations

(*We omit to translate the subsections 3.2 - 3.6.*)

3.7 Problem 7, Formation of equations



- a is given.[38]
- b is given.
- c is given.
- d is given.
- e is given.
- The diameter of the disk is [supposed to be] given.
  It is the unknown [x] in the answer process.
- The square [x^2] of the diameter of the disk is given.
  It will be denoted by α.

[37]The resultant is given by the equation $\theta\kappa - \iota^2 = 0$ in the original unknown x.
[38]In the original text the author uses *kō*(甲), *otsu*(乙), *hei*(丙), *tei*(丁), *bo*(戊).

- Since it is difficult to find the formulas which cancel out each other at once, we assume that A squared[39] is given.

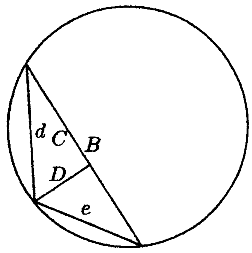


Figure for the first former equation

First, let B squared be a new unknown $\bigcirc\ \underline{\quad\quad}$ by the celestial element method . Add d squared to it and we get $\underline{+1\,d^2}\ \underline{+1\quad}$ from which we subtract e squared. Then, the remainder equals 2 times the product of B and C;

$\begin{array}{ll} -1\,e^2 & \\ \underline{+1\,d^2} & \underline{+1\quad} \end{array}$. Its square $\begin{array}{lll} +1\,e^4 & & \\ -2\,d^2e^2 & -2\,e^2 & \\ \underline{+1\,d^4} & \underline{+2\,d^2} & \underline{+1} \end{array}$

equals 4 times the product of B squared and C squared. Move it to Heaven.

Take 4 times d squared $\underline{+4\,d^2}$ and multiply it by B squared $\bigcirc\ \underline{d^2\,4+}$. From the product we subtract the formula at Heaven and get $\begin{array}{lll} -1\,e^4 & & \\ +2\,d^2e^2 & +2\,e^2 & \\ \underline{-1\,d^4} & \underline{+2\,d^2} & \underline{-1} \end{array}$,

which equals[40] 4 times the product of B squared and D squared. Multiply it by the square of the diameter and we get $\begin{array}{lll} -1\,\alpha e^4 & & \\ +2\,\alpha d^2e^2 & +2\,\alpha e^2 & \\ \underline{-1\,\alpha d^4} & \underline{+2\,\alpha d^2} & \underline{-1\,\alpha} \end{array}$ and move it to Left.

Take d squared and multiply it by e squared $\underline{+1\,d^2e^2}$. Then, the product equals the product of the square of the diameter and D squared. Multiply it by 4 times B squared and we get $\bigcirc\ \underline{+4\,d^2e^2}$ which cancels out the formula at Left. [So, we get]

the first former equation $\begin{array}{lll} -1\,\alpha e^4 & -4\,d^2e^2 & \\ +2\,\alpha d^2e^2 & +2\,\alpha e^2 & \\ \underline{-1\,\alpha d^4} & \underline{+2\,\alpha d^2} & \underline{-1\,\alpha} \\ +1\,\alpha\ ① & +1\ ② & \end{array}$

in the unknown B squared.

Here ① and ② are the formulas which will be moved to the 1st place and the 2nd place respectively in the second latter process.

From the sum of B squared and A squared $\underline{+1\,A^2}\ \underline{+1\quad}$ we subtract a squared. Then, the remainder $\begin{array}{ll} -1\,a^2 & \\ \underline{+1\,A^2} & \underline{+1\quad} \end{array}$ equals 2 times the product of B and E. Its square $\begin{array}{lll} +1\,a^4 & & \\ -2\,a^2A^2 & -2\,a^2 & \\ \underline{+1\,A^4} & \underline{+2\,A^2} & \underline{+1} \end{array}$ equals 4 times the product of B squared and E

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[39] A, B, C, D, E, F, G and H as well as a, b, c, d, e are shown in the figures.

[40] By the similarity of two right-angled triangles we have $d/D = diameter/e$.

squared. Move it to Earth.

Take 4 times A squared $\underline{+4A^2}$ and multiply it by B squared; ○ $\underline{+4A^2}$. From the product we subtract the formula at Earth and get the remainder

$$\begin{array}{lll} -1a^4 & & \\ +2a^2A^2 & +2a^2 & \\ \underline{-1A^4} & \underline{+2A^2} & \underline{-1} \end{array}$$

which equals 4 times the product of B squared and F squared. Multiply it by the square of the diameter and we get

$$\begin{array}{lll} -1\alpha a^4 & & \\ +2\alpha a^2A^2 & +2\alpha a^2 & \\ \underline{-1\alpha A^4} & \underline{+2\alpha A^2} & \underline{-1\alpha} \end{array}$$

which we move to Left 2.

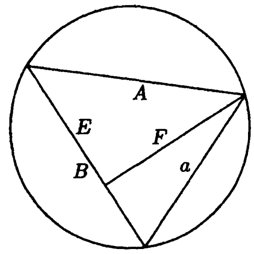

Figure for the first latter equation

Take a squared and multiply it by A squared. Then, we get $\underline{+1a^2A^2}$, which equals the product of the square of the diameter and F squared. Multiply it by 4 times B squared and we get ○ $\underline{+4a^2A^2}$. which cancels out the formula at Left 2 [and we get]

the first latter equation

$$\begin{array}{lll} -1\alpha a^4 & -4a^2A^2 & \\ +2\alpha a^2A^2 & +2\alpha a^2 & \\ \underline{-1\alpha A^4} & \underline{+2\alpha A^2} & \underline{-1\alpha} \\ +1\alpha\ \textcircled{3} & +1\ \textcircled{4} & \end{array}$$

in the unknown B squared.

Here ③ and ④ are the formulas which will be moved to the 3rd place and the 4th place respectively in the second latter process.

Referring to "Case of two quadratic equations" in the volume 1 with the first former and latter equations, we obtain the first [converted] *Yo*-diagram and the first [eliminated] *In*-diagram as follows:

The first *Yo*-diagram

$$\begin{array}{cc} -1\textcircled{2}\textcircled{3} & +1\alpha\textcircled{3} \\ \underline{+1\textcircled{1}\textcircled{4}} & \underline{-1\alpha\textcircled{1}} \\ \textcircled{g} & \textcircled{h} \\ +1\alpha\textcircled{3} & +1\textcircled{4} \\ \underline{-1\alpha\textcircled{1}} & \underline{-1\textcircled{2}} \\ \textcircled{i} & \textcircled{j} \end{array}$$

Making the *Yo*-diagram as usual and dividing all [the coefficients] by α, we obtain this simplified *Yō*-diagram.

The first *In*-diagram

$$\begin{array}{l} +1\ ②^2③ \\ +1\ ①④^2 \\ +2\ \alpha^2①③ \\ -1\ ②③④ \\ -1\ ①②④ \\ -1\ \alpha^2③^2 \\ -1\ \alpha^2①^2 \end{array}$$

The sum of the terms with + sign will be the formula at the 5th place and the sum of the terms with − sign will cancel it out [and they will form the second latter equation].

Second, let A squared be another unknown ○ $\underline{\quad}$. Add b squared to it and we get the formula $\underline{+1\,b^2}\ \ \underline{+1}$ from which we subtract c squared. Then, the remainder equals 2 times the product of A and G; $\begin{array}{ll} -1\,c^2 & \\ \underline{+1\,b^2} & \underline{+1} \end{array}$. Its square

$$\begin{array}{lll} +1\,c^4 & & \\ -2\,b^2c^2 & -2\,c^2 & \\ \underline{+1\,b^4} & \underline{+2\,b^2} & \underline{+1} \end{array}$$

equals 4 times the product of A squared and G squared.

Move it to Top.

Take 4 times b squared $\underline{+4\,b^2}$ and multiply it by A squared. Then, we get ○ $\underline{+4\,b^2}$ from which we subtract the formula at Top. The remainder

$$\begin{array}{lll} -1\,c^4 & & \\ +2\,b^2c^2 & +2\,c^2 & \\ \underline{-1\,b^4} & \underline{+2\,b^2} & \underline{-1} \end{array}$$

equals 4 times the product of A squared and H squared. Multiply it by the square of the diameter and we get

$$\begin{array}{lll} -1\,\alpha c^4 & & \\ +2\,\alpha b^2c^2 & +2\,\alpha c^2 & \\ \underline{-1\,\alpha b^4} & \underline{+2\,\alpha b^2} & \underline{-1\,\alpha} \end{array}$$

which we move to Bottom.

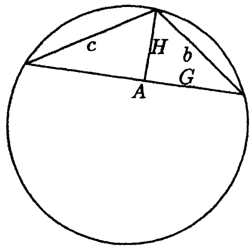

Figure for the second former equation

Take b squared and multiply it by c squared. Then, the product equals the product of the square of the diameter and H squared $\underline{+1\,b^2c^2}$. Multiply it by 4 times A squared and we get ○ $\underline{+4\,b^2c^2}$. It cancels out the formula at Bottom [and we get]

the second former equation

$$\begin{array}{lll} -1\,\alpha c^4 & -4\,b^2c^2 & \\ +2\,\alpha b^2c^2 & +2\,\alpha c^2 & \\ \underline{-1\,\alpha b^4} & \underline{+2\,\alpha b^2} & \underline{-1\quad \alpha} \\ -1\,\textcircled{\beta} & +2\,\textcircled{\gamma} & \text{Named } LC\ [隅] \end{array}$$

in the unknown A squared.

Hereafter, we will concern the process to get the second latter equation. Refer to the first *In*-diagram above.

Taking the 3 sum of -1 times the 4th power of d, 2 times the product of d squared and e squared and -1 times the 4th power of e, we get the number [the formula] $\begin{array}{l} -1\ e^4 \\ +2\,d^2e^2 \\ -1\ d^4 \\ \hline -1\ \delta \end{array}$. Move it to the 1st place.[41]

Taking the 3 sum of 2 times the product of the square of the diameter and d squared, 2 times the product of the square of the diameter and e squared and -4 times the product of d squared and e squared, we get $\begin{array}{l} -4\,d^2e^2 \\ +2\ \alpha e^2 \\ +2\ \alpha d^2 \\ \hline +2\ \varepsilon \end{array}$ which we move to the 2nd place.

Taking the 3 sum of -1 times the 4th power of A, 2 times the product of a squared and A squared and -1 times the 4th power of a, we get $\underline{-1\,a^4}\quad \underline{+2\,a^2}\quad \underline{-1\quad}$ which we move to the 3rd place.

Taking the 3 sum of 2 times the product of square of the diameter and A squared, 2 times the product of the square of the diameter and a squared and -4 times the product of a squared and A squared, we get $\underline{+2\,\alpha a^2}\quad \begin{array}{l} -4\,a^2 \\ +2\,\alpha \\ \hline +2\ \zeta \end{array}$ which we move to the 4th place.

Taking the 3 sum of the product of the square of the formula at the 2nd place and the formula at the 3rd place $\underline{-4\,\varepsilon^2a^4}\quad \underline{+8\,\varepsilon^2a^2}\quad \underline{-4\,\varepsilon^2}$, the product of the formula at the 1st place and the square of the formula at the 4th place $\underline{-4\,\alpha^2\delta a^4}\quad \underline{-8\,\alpha\delta\zeta a^2}\quad \underline{-4\,\delta\zeta^2}$ and 2 times the product of the 4th power of the diameter, the formula at the 1st place and the formula at the 3rd place $\underline{+2\,\alpha^2\delta a^4}\quad \underline{-4\,\alpha^2\delta a^2}\quad \underline{+2\,\alpha^2\delta}$,

we get $\begin{array}{lll} & +8\ \varepsilon^2a^2 & -4\ \varepsilon^2 \\ -4\,\varepsilon^2a^4 & -8\ \alpha\delta\zeta a^2 & -4\ \delta\zeta^2 \\ -2\,\alpha^2\delta a^4 & -4\ \alpha^2\delta a^2 & +2\ \alpha^2\delta \\ \hline \end{array}$ which we move to the 5th place.

Taking the 4 sum of the product of the formulas at the 2nd, the 3rd and the 4th places $\begin{array}{llll} & +8\,\alpha\varepsilon a^4 & -4\,\alpha\varepsilon a^2 & \\ -4\,\alpha\varepsilon a^6 & -4\,\varepsilon\zeta a^4 & +8\,\varepsilon\zeta a^2 & -4\,\varepsilon\zeta \\ \hline \end{array}$, the product of the formulas at the 1st, the 2nd and the 4th places $\underline{-4\,\alpha\delta\varepsilon a^2}\quad \underline{-4\,\delta\varepsilon\zeta}$, the product of the 4th power of the diameter and the square of the formula at the 3rd place $\underline{+1\,\alpha^2a^8}\quad \underline{-4\,\alpha^2a^6}\quad \underline{+6\,\alpha^2a^4}\quad \underline{-4\,\alpha^2a^2}\quad \underline{+1\,\alpha^2}$ and the product of the 4th power of the diameter and the square of the for-

[41]In the original text the author uses the name of animal zodiac *ne*(子), *ushi*(丑), *tora*(寅), *u*(卯), *tatsu*(辰) instead of *kō*(甲), *otsu*(乙), *hei*(丙), *tei*(丁), *bo*(戊). But we keep to use the the n-th places.

mula at the 1st place $\underline{+1\,\alpha^2\delta^2}$, we get

$$
\begin{array}{lllll}
-4\,\alpha\varepsilon a^6 & +8\,\alpha\varepsilon a^4 & & & \\
-4\,\alpha\delta\varepsilon a^2 & -4\,\varepsilon\zeta a^4 & -4\,\alpha\varepsilon a^2 & & \\
+1\,\alpha^2 a^8 & -4\,\delta\varepsilon\zeta & +8\,\varepsilon\zeta a^2 & -4\,\varepsilon\zeta & \\
\underline{+1\,\alpha^2\delta^2} & \underline{-4\,\alpha^2 a^6} & \underline{+6\,\alpha^2 a^4} & \underline{-4\,\alpha^2 a^2} & \underline{+1\,\alpha^2}
\end{array}
$$

which cancels out the formula at the 5th place.[42] [So, we get]

the second latter equation

$$
\begin{array}{lllll}
 & +8\,\varepsilon^2 a^2 & & & \\
-4\,\varepsilon^2 a^4 & -8\,\alpha\delta\zeta a^2 & -4\,\varepsilon^2 & & \\
-2\,\alpha^2\delta a^4 & -4\,\alpha^2\delta a^2 & -4\,\delta\zeta^2 & & \\
+4\,\alpha\varepsilon a^6 & -8\,\alpha\varepsilon a^4 & +2\,\alpha^2\delta & & \\
+4\,\alpha\delta\varepsilon a^2 & +4\,\varepsilon\zeta a^4 & +4\,\alpha\varepsilon a^2 & & \\
-1\,\alpha^2 a^8 & +4\,\delta\varepsilon\zeta & -8\,\varepsilon\zeta a^2 & +4\,\varepsilon\zeta & \\
\underline{-1\,\alpha^2\delta^2} & \underline{+4\,\alpha^2 a^6} & \underline{-6\,\alpha^2 a^4} & \underline{+4\,\alpha^2 a^2} & \underline{-1\,\alpha^2} \\
-1\,\kappa & +4\,\iota & & & \text{Named } LC \text{ [隅]}
\end{array}
$$

in the unknown A squared.

We can obtain the *Yo*-diagram and *In*-diagram for the second former and latter equations by referring to "Case of two quartic equations" in the volume 1. But the sentence in the process becomes complicated. So, for the beginners we choose the following process. It is similar to the way that we obtained the *Yo-ritsu* in the volume 1. You should understand that there are various ways to solve the problem.

The highest degree term in the second former equation is $\underline{-1\,\alpha}$. The highest degree term in the second latter equation is $\underline{-1\,\alpha^2}$. Thus, we multiply the second former equation by α; $\underline{-1\,\alpha\beta}$ $\underline{+2\,\alpha\gamma}$ $\underline{-1\,\alpha^2}$ and subtract the product from the second latter equation [by multiplying the square of A squared] to cancel the highest degree terms. [Then, we get]

the remainder equation

$$
\begin{array}{llll}
 & & +1\,\alpha\beta & \\
 & & -4\,\varepsilon^2 & \\
 & & -4\,\delta\zeta^2 & \\
 & & +2\,\alpha^2\delta & \\
 & & +4\,\alpha\varepsilon a^2 & -2\,\alpha\gamma \\
 & & -8\,\varepsilon\zeta a^2 & +4\,\varepsilon\zeta \\
\underline{-1\,\kappa} & \underline{+4\,\iota} & \underline{-6\,\alpha^2 a^4} & \underline{+4\,\alpha^2 a^2} \\
 & & -1\,\theta & +2\,\eta
\end{array}
$$

in the unknown A squared.

[42]This sum equals that of the terms with − sign in the first *In*-diagram, and the formula at the 5th place equals the sum of the terms with + sign in the same *In*-diagram.

We obtain the [converted] *Yo*-diagram[43] for the remainder and the second former equations by referring "Case of two cubic equation" in the volume 1, and we can find the formulas[44] of λ, μ and ν. By applying Cubic *In-ritsu*, we obtain the formula *Left* at the left-hand side and the formula which cancels out the formula *Left*.[45]

The end of Sanpo-Hakki Volume 3.

4 Sanpo-Hakki Postscript

Sanpo-Hakki is a book written by my student Mr. IZEKI Tomotoki. In recent years the manuscript has been concealed in a box as a jewelry. I looked this and laughed the tasteless tongue. Then I made to cut printing blocks. Ah, the people who aspire this technique get the boats and bridges to go up, high and far. Now the printing is finished and I signed: SHIMADA Naomasa with two seals (指雲, 島田尚政).

On a good morning of June 1690[46],

published[47] by Nagano-hikouemon, 4ken-cho, Kouraibashi-suji, Osaka [大坂高麗橋筋四軒町 長野彦右衛門 開板].

Appendix: Translators' comment on the procedure to solve the problems in the volumes 2 and 3

Seven problems are solved in the volumes 2 and 3. The first six problems can be solved by making two equations in an auxiliary unknown whose coefficients contain the true unknown. Applying the method given in the volume 1 we get a polynomial equation in the true unknown. The problem was thought solved by this, because they knew the **celestial element method** [天元術 tengen-jutsu] which gives a (possibly approximate) solution of the polynomial equation with numerical coefficients if it has a real-valued solution.

[43]The three equations in the unknown $y = A^2$ in the simplified *Yo*-diagram are as follows:

$$\begin{cases} -\lambda & + \ \mu\, y & - 2\eta\beta\, y^2 & = & 0 \\ \mu & - 2\nu\, y & + 4\eta\gamma\, y^2 & = & 0 \\ -\beta & + 2\gamma\, y & - \quad \alpha\, y^2 & = & 0 \end{cases} .$$

[44]They are given in the volume 2.

[45]The desired answer equation in the true unknown x is given in the volume 2.

[46]On a good day of May 1710 in the 2nd version.

[47]By Ikedaya-saburouemon, Gohuku-cho-kado, Shinsaibashi-suji, Osaka and by Naganoya-hikouemon, 4ken-cho, Kouraibashi-suji, Osaka in the 2nd version. One more difference is that the part "edited by IZEKI Jubeejo Tomotoki, a student of SHIMADA Naomasa" at the top of each volume is deleted in the 2nd version. The 2nd version might be a pirated edition.

We get the following table which shows the degrees of two equations in an auxiliary unknown and the degree of the final answer equation in the true unknown for each problem.

Problem	Degrees of two equations	Degree of the final equation
No. 1	2 and 2	8
No. 2	2 and 2	8
No. 3	3 and 3	7
No. 4	2 and 3	8 with a comment[48]
No. 5	3 and 4	7
No. 6	2 and 4	8

The last problem No. 7 is to find the diameter from the length of five edges of a 5-lateral on a circle. If we have a triangle with B, d, e edges on a circle with the diameter x, we see that the following equation holds:

$$x^2(2d^2e^2 - d^4 - e^4) + \{2x^2(d^2 + e^2) - 4d^2e^2\}B^2 - x^2B^4 = 0.$$

This is an equation in B^2 of degree 2 and is the first former equation. Another same type of equation in B^2 of degree 2 with another unknown A^2 in the coefficients is the first latter equation. We get an equation in A^2 of degree 4 as a resultant. Combining this equation with an equation in A^2 of degree 2 with respect to another triangle, we get a final equation in x^2 of degree 7. So, any exercise for the case of quintic[49] equations did not given.

Moreover, when the degrees of two equations are different, it is not difficult to modify the equation with the higher degree to an equation of one degree lower without changing the common solution as the author used for Problem 7 in the volume 3. So, the problems given in the volumes 2 and 3 can be solved by using at highest the case of cubic equations in the volume 1. In this sense even the real exercise for the case of quartic equations did not given. This would give modern readers a little disappointment.

[48]The author wrote that the final equation is of degree 9 but the term of degree 9 is actually zero by our calculation. He commented also that the final equation in another unknown has degree 8.

[49]Taisei-sankei [大成算経] treated the case of quintic equations in volume 19.