Topological Reflection Theorems †

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Abstract

We survey some known reflection theorems on topological properties like metrizability, compactness or paracompactness which are assertions equivalent to the Fodor-type Reflection Principle (FRP) over ZFC or drivable from some modification of FRP. We also present slight improvements of two theorems in [10].

1   Introduction and summary of known results

In this note, we survey some known reflection theorems on topological properties like metrizability, compactness or paracompactness. We are mainly interested in reflection theorems which are equivalent to the Fodor-type Reflection Principle (FRP, see below) or drivable from a modification of FRP.

In Section 2, we give a construction of a topological space which serves as a generic counter example of the non reflection under the failure of FRP.

Sections 3 and 4 contain slight improvements of two results in sections 4 and 5 of [10].

Fodor-type Reflection Principle (FRP) is the assertion that the following principle FRP(λ) holds for all regular $\lambda > \aleph_1$.

FRP(λ): For any stationary $S \subseteq E_\omega^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$ and mapping $g : S \to [\lambda]^{\leq \aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ such that

(1.1) $\text{cf}(I) = \omega_1$;

(1.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(1.3) for any regressive $f : S \cap I \to \lambda$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

The principle FRP is introduced in [9] and further studied in [8], [10], [11], [12] and [13]. FRP is a consequence of the reflection principle known as RP (see e.g. [15]) which is a weakening of Fleissner's Axiom R ([7]). In contrast to RP which implies $2^{\aleph_0} \leq \aleph_2$, FRP impose almost no restriction on the size of the continuum since it can be shown that FRP is preserved under c.c.c. generic extension ([9]). On the other hand, many mathematical reflection theorems previously known to be consequences of Axiom R can be proved already under FRP.

For later use in Section 4, let us recall the definition of Axiom R. Axiom R is the assertion that the following property AR(κ) holds for all cardinals $> \aleph_1$:

AR(κ): For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, there is $I \in \mathcal{T}$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$

where $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or *tight and unbounded* in Fleissner's terminology in [7]) if

(1.4) $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and

(1.5) for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha < \omega_1} I_\alpha \in \mathcal{T}$.

In [13], it is proved that FRP is also a consequence of Rado's Conjecture (RC) (for Rado's Conjecture see e.g. [20])
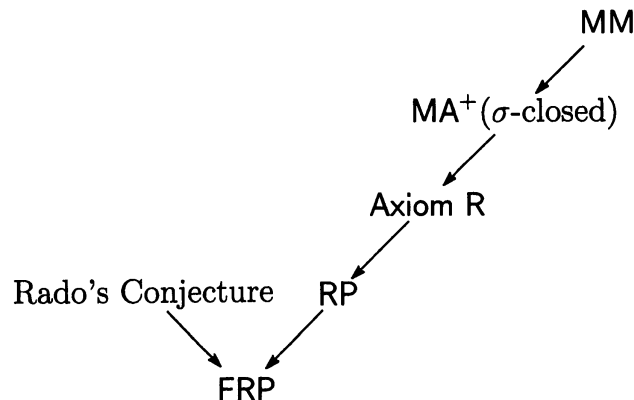
Using the characterization of FRP in [12] which is cited here as Theorem 1.1 and Corollary 1.2 below, we can even prove that most of the mathematical reflection theorems provable under FRP are actually equivalent to FRP over ZFC:

Theorem 1.1 (S.F., H. Sakai, L. Soukup and T. Usuba [12]). *Suppose that* FRP *does not hold and*

(1.6) $\lambda^* = \min\{\mu \; : \; \mu \text{ is a regular cardinal with } \neg\text{FRP}(\mu)\}.$

Then ADS$^-(\lambda^*)$ *holds.*

Here ADS$^-(\lambda)$ for a regular cardinal λ is the following weakening of the square principle:

ADS$^-(\lambda)$ There is a stationary $S \subseteq E^\lambda_\omega$ and a ladder system $g : S \to [\lambda]^{\aleph_0}$ such that g is almost essentially disjoint

where

(1.7) $g : S \to [\lambda]^{\aleph_0}$ for $S \subseteq E^\lambda_\omega$ is a *ladders system* if $g(\alpha)$ is a cofinal subset of α of order type ω for all $\alpha \in S$;

(1.8)   A ladder system $g : S \to [\lambda]^{\aleph_0}$ for $S \subseteq E^\lambda_\omega$ is *essentially disjoint* if there is a regressive $f : S \to \lambda$ such that $\{g(\alpha) \setminus f(\alpha) \; : \; \alpha \in S\}$ is pairwise disjoint; and

(1.9)   A ladder system $g : S \to [\lambda]^{\aleph_0}$ for $S \subseteq E^\lambda_\omega$ is *almost essentially disjoint* if, for all $\gamma < \lambda$, the ladder system $g \restriction S \cap \gamma$ is essentially disjoint.

Corollary 1.2 (S.F., H. Sakai, L. Soukup and T. Usuba [12]). FRP *is equivalent to the assertion that* ADS$^-(\lambda)$ *does not hold for all regular* λ.

Here is a list of mathematical assertions proved to be equivalent to FRP over ZFC. For the notions used in the assertions in the following Theorem 1.3, see the respective papers where the equivalence is shown:

Theorem 1.3. *Each of the following assertions is equivalent to* FRP *over* ZFC:

(A)   ([9],[12])   *For every locally separable countably tight topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf, then X itself is also meta-Lindelöf.*

(B)   ([9],[12])   *For every locally countably compact topological space X with, if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable, then X itself is also metrizable.*

(B') ([10]) *If X is a regular locally countably compact space such that every subspace of X of cardinality $\leq \aleph_1$ has a point countable base, then X is metrizable.*

(C) ([8]) *For every T_1-space X with point countable base, if all subspaces of X of cardinality $\leq \aleph_1$ are left-separated then X itself is also left-separated.*

(C') ([8]) *For every metrizable space X, if all subspaces of X of cardinality $\leq \aleph_1$ are left-separated then X itself is also left-separated.*

(D) ([11]) *for any Boolean algebra B, B is openly generated if and only if there are club many projective subalgebras of B of cardinality $\aleph_1$.*

(E) ([12]) *For every countably tight topological space X of local density $\leq \aleph_1$, if X is $\leq \aleph_1$-cwH, then every closed discrete subsets of X are simultaneously separated.*

(E') ([12]) *For every locally countable, first countable topological space X, if X is $\leq \aleph_1$-cwH, then every closed discrete subsets of X are simultaneously separated.*

2 Construction of a topological space coding the failure of FRP

The implications "(X) $\Rightarrow$ FRP" for X = A, B, B', C, C', E, E' in Theorem 4.1 are immediate consequences of the following Lemma.
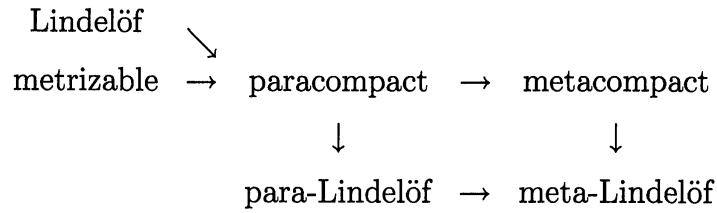
Note that an almost essentially disjoint ladder system $g : S \to [\lambda]^{\aleph_0}$ on $S \subseteq E^\lambda_\omega$ can be easily modified to satisfy:

(2.1) $g(\alpha)$ consists of successor ordinals for all $\alpha \in S$.

Recall that, a topological space X is:

(2.2) *paracompact* if any open covering of X has a locally finite open refinement;

(2.3) *para-Lindelöf* if any open covering of X has a locally countable open refinement;

(2.4) X is *metacompact* if any open covering of X has a point finite open refinement;

(2.5) *meta-Lindelöf* if any open covering of X has a point countable open refinement;

A.H. Stone's theorem states that a metrizable space is paracompact. Morita's theorem states that a Lindelöf space is paracompact.

$$\begin{array}{ccccc} \text{Lindelöf} & & & & \\ & \searrow & & & \\ \text{metrizable} & \to & \text{paracompact} & \to & \text{metacompact} \\ & & \downarrow & & \downarrow \\ & & \text{para-Lindelöf} & \to & \text{meta-Lindelöf} \end{array}$$

In [10], I overlooked (6) in the following Lemma 2.1 and failed to formulate the assertions given as Corollary 3.7 and Proposition 4.5. I would like to thank Toshimichi Usuba for pointing it out for me.

Lemma 2.1. *Suppose that* $\neg\mathsf{ADS}^-(\lambda)$ *for a regular* $\lambda > \aleph_1$ *and let* $S \subseteq E^\lambda_\omega$ *be stationary set with an almost essentially disjoint ladder system* $g : S \to [\lambda]^{\aleph_0}$ *on it satisfying* (2.1).

Let $X = (X, \mathcal{O})$ *be the topological space with* $X = Succ(\lambda) \,\dot\cup\, S$ *where* $Succ(\lambda) = \{\alpha + 1 : \alpha \in \lambda\}$ *and such that* $\mathcal{O}$ *is generated from*

(2.6) $\mathcal{B} = \{\{\alpha\} : \alpha \in Succ(\lambda)\} \cup \{\{\alpha\} \cup g(\alpha) \setminus \gamma : \alpha \in S, \gamma < \alpha\}$

Then we have

(1) X *is a normal space.*

(2) X *is locally countable and locally compact.*

(3) X *is not meta-Lindelöf (and hence it is not metrizable).*

(4) $X \cap \gamma$ *is metrizable for all* $\gamma < \lambda$.

(5) X *is* $< \lambda$-*collectionwise Hausdorff but not collectionwise Hausdorff.*

(6) *For any uncountable* $Y \subseteq X$, $L(Y) = |Y|$ *where* $L(Y)$ *denotes the Lindelöf number of* Y.

(7) *For any infinite* $Y \subseteq X$, $|Y| = |\overline{Y}|$.

Proof. (1): Suppose that $A_i \subseteq X$, $i \in 2$ are disjoint closed subsets of X. This means that, for each $i \in 2$ and $\bar{i} \in 2 \setminus \{i\}$,

(2.7) if $g(\alpha) \cap A_i$ for some $\alpha \in S$ is unbounded in α then $\alpha \in A_i$, and

(2.8) if $\alpha \in A_i \cap S$ then $g(\alpha) \cap A_{\bar{i}}$ is bounded in α.

For each $i \in 2$, $\bar{i} \in 2 \setminus \{i\}$ and $\alpha \in A_i \cap S$, let $f(\alpha) \in \alpha$ be such that $(g(\alpha) \setminus f(\alpha)) \cap A_{\bar{i}} = \emptyset$. This is possible by (2.8).

For $i \in 2$, let $O_i = A_i \cup \{g(\alpha) \setminus f(\alpha) : \alpha \in A_i \cap S\}$. Then O_i, $i \in 2$ are disjoint open sets separating A_i, $i \in 2$.

(2): This is clear since all $\alpha \in Succ(\lambda)$ are isolated and the open countable subspace $g(\alpha)$ of X for $\alpha \in S$ is isomorphic to ω by the definition of $\mathcal{B}$.

(3): It is enough to show that there is no point countable open refinement of the open covering $\mathcal{U}_0 = \{\{\beta\} : \beta \in Succ(\lambda)\} \cup \{\{\alpha\} \cup g(\alpha) : \alpha \in S\}$ of X.

Suppose that $\mathcal{U}$ is an arbitrary open refinement of $\mathcal{U}_0$. For each $\alpha \in S$, let $f(\alpha) \in \alpha$ be such that $f(\alpha) \in g(\alpha)$ and $\{\alpha\} \cup g(\alpha) \setminus f(\alpha) \subseteq U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Since $f : S \to \lambda$ is regressive, there are $\beta^* \in \lambda$ and stationary $T \subseteq S$ such that $f \upharpoonright T$ takes the constant value β^* by Fodor's Lemma. β^* is contained in all U_α, $\alpha \in T$ and, by the definition of $\mathcal{U}_0$, U_α, $\alpha \in T$ are pairwise distinct. This shows that $\mathcal{U}$ is not even point $< \lambda$.

(4): Suppose that $\gamma < \lambda$. Without loss of generality, we may assume $\gamma \in Lim(\lambda)$. Since g is almost essentially disjoint, there is a regressive $f : S \cap \gamma \to \gamma$ such that $\{g(\alpha) \setminus f(\alpha) : \alpha \in S \cap \gamma\}$ is pairwise disjoint. It follows that $\mathcal{F} = \{(\{\alpha\} \cup g(\alpha)) \setminus f(\alpha) : \alpha \in S \cap \gamma\}$ is pairwise disjoint as well.

Let $D = Succ(\gamma) \setminus \bigcup \mathcal{F}$ and $\mathcal{U} = \{\{\beta\} : \beta \in D\} \cup \mathcal{F}$. $\mathcal{U}$ is an open partition of $X \cap \gamma$ and each $U \in \mathcal{U}$ is metrizable.

It follows that $X \cap \gamma$ is also metrizable.

(5): For a closed discrete subset D of X of cardinality $< \lambda$, we can find a simultaneous separation similarly to the construction of the open partition in the proof of (4). S is a closed and discrete subset of X but Fodor's Lemma argument similar to the proof of (3) shows that S cannot be simultaneously separated.

(6): It is enough to show $L(Y) \geq |Y|$. Suppose first that $|Y| = \lambda$. Suppose $\mathcal{U} \in [\mathcal{B}]^{<\lambda}$. Then $\sup\{\sup U : U \in \mathcal{U}\} < \lambda$ by regularity of λ. Hence $\mathcal{U}$ cannot be a covering of X. This shows $L(Y) \geq \lambda$.

If $|Y| < \lambda$, then, by the proof of (3), there is an open partition of Y of size $|Y|$. Thus we have again $L(Y) \geq |Y|$.

$\square$ (Lemma 2.1)

3 Reflection of paracompactness in countably tight locally Lindelöf spaces

In this section we prove that the assertion of Theorem 1.6 in Balogh [2] (proved there under Axiom R) is also equivalent to FRP over ZFC (Corollary 3.7).

Lemma 3.1. *For a topological space $X = (X, \mathcal{O})$, if $\mathcal{F} \subset \mathcal{P}(X)$ is locally finite, then we have $\bigcup \{\overline{Y} : Y \in \mathcal{F}\} = \overline{\bigcup \mathcal{F}}$.*

Proof. The inclusion "$\subseteq$" is clear. To show the other inclusion "$\supseteq$", suppose $x \in \overline{\bigcup \mathcal{F}}$. Let $O \in \mathcal{O}$ be such that $x \in O$ and $\mathcal{F}_0 = \{Y \in \mathcal{F} : O \cap Y \neq \emptyset\}$ is finite. Then we have $x \in \overline{\bigcup \mathcal{F}_0} = \bigcup \{\overline{Y} : Y \in \mathcal{F}_0\}$. Thus $x \in \bigcup \{\overline{Y} : Y \in \mathcal{F}\}$.

$\square$ (Lemma 3.1)

Lemma 3.2. *For a topological space $X = (X, \mathcal{O})$, if $\mathcal{F} \subseteq \mathcal{P}(X)$ is locally finite, then $\overline{\mathcal{F}} = \{\overline{Y} : Y \in \mathcal{F}\}$ is also locally finite.*

Proof. For $x \in X$, let $O \in \mathcal{O}$ be such that $x \in O$ and $\mathcal{F}_0 = \{Y \in \mathcal{F} : O \cap Y \neq \emptyset\}$ is finite. For any $y \in O$ if $y \in \overline{Y}$ for some $Y \in \mathcal{F}$ then $O \cap Y \neq \emptyset$, i.e. $Y \in \mathcal{F}_0$. So we have $\{Y \in \mathcal{F} : O \cap \overline{Y} \neq \emptyset\} = \mathcal{F}_0$.

$\square$ (Lemma 3.2)

The following characterization of paracompactness of locally Lindelöf spaces was already mentioned in [2]. In the proof of Theorem 3.6 we actually only use the trivial direction "(a) $\Rightarrow$ (b)" of this characterization. Nevertheless the characterization explains the need to look at open partitions of a given locally Lindelöf space to prove the paracompactness of the space.

Lemma 3.3. *Suppose that X is a locally Lindelöf space[1]. Then the following are equivalent:*

(a) *X can be partitioned into open Lindelöf subspaces.*

(b) *X is paracompact.*

(c) *X is para-Lindelöf.*

Proof. "(a) $\Rightarrow$ (b)": Suppose that X is partitioned into open Lindelöf subspaces. By Morita's theorem each subspace in the partition is paracompact. Hence it follows that the whole space is paracompact as well.

"(b) $\Rightarrow$ (c)" is trivial.

"(c) $\Rightarrow$ (a)": Suppose now that X is a locally Lindelöf para-Lindelöf space. We show that there is a partition of X into clopen Lindelöf subspaces. Let $\mathcal{A} \subseteq \mathcal{O}$ be an open covering of X such that $\overline{Y}$ is Lindelöf for all $Y \in \mathcal{A}$. Let $\mathcal{B}$ be a locally countable open refinement of $\mathcal{A}$. Then elements of $\mathcal{B}' = \{\overline{Y} : Y \in \mathcal{B}\}$ are Lindelöf and $\mathcal{B}'$ is still locally countable by Lemma 3.2.

Claim 3.3.1. *For any $Y \in \mathcal{B}'$, $\{Z \in \mathcal{B}' : Y \cap Z \neq \emptyset\}$ is countable.*

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[1] We assume that a Lindelöf space is a regular space with Lindelöf property. A topological space X is locally Lindelöf if for every $x \in X$ there is an open set $x \in O \subseteq X$ such that $\overline{O}$ is a Lindelöf space in the subspace topology. In particular, a locally Lindel"of space is locally regular.

⊢ Suppose $Y \in \mathcal{B}'$. Let $S = \{Z \in \mathcal{B}' : Y \cap Z \neq \emptyset\}$. For each $y \in Y$, let $O_y \in \mathcal{O}$ be such that $y \in O_y$ and $\{Z \in \mathcal{B}' : O_y \cap Z \neq \emptyset\}$ is countable. Note that we can find such O_y since $\mathcal{B}'$ is locally finite. Since Y is Lindelöf, there is a countable $Y_0 \subseteq Y$ such that $\{O_y : y \in Y_0\}$ is a cover of Y. Then we have $S \subseteq \{Z \in \mathcal{B}' : O_y \cap Z \neq \emptyset \text{ for some } y \in Y_0\}$ and the right side of the inclusion is easily seen to be countable. ⊣ (Claim 3.3.1)

Let $\sim_{\mathcal{B}'}$ be the intersection relation on $\mathcal{B}'$. That is, for $Y, Y' \in \mathcal{B}'$, $Y \sim_{\mathcal{B}'} Y'$ $\Leftrightarrow$ $Y \cap Y' \neq \emptyset$. Let $\approx_{\mathcal{B}'}$ be its transitive closure. Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_{\mathcal{B}'}$. By the claim above, it follows that each $e \in \mathbb{E}$ is countable. Thus $\bigcup e$ is Lindelöf and $\bigcup e$ is closed by Lemma 3.1. Since $\mathcal{P} = \{\bigcup e : e \in \mathbb{E}\}$ is a partition of X, each $\bigcup e$ for $e \in \mathbb{E}$ is also open.

Thus $\mathcal{P}$ is a partition of X into clopen Lindelöf subspaces of X. □ (Lemma 3.3)

A similar proof shows the following:

Lemma 3.4. *For a locally (separable & Lindelöf) space X, the following are equivalent:*

(a) *X has an open partition into Lindelöf spaces;*

(b) *X is paracompact;*

(c) *X is meta-Lindelöf.*

Proof. "(a) $\Rightarrow$ (b)": If $\mathcal{A}$ is an open partition of X into Lindelöf spaces then each $Y \in \mathcal{A}$ is paracompact by Morita's theorem. Hence X is also paracompact.

"(b) $\Rightarrow$ (c)" is trivial.

"(c) $\Rightarrow$ (a)": Suppose that X is meta-Lindelöf. Let $\mathcal{A}$ be an open covering of X consisting of separable Lindelöf subspaces and $\mathcal{A}'$ be its point countable open refinement. Note that elements of $\mathcal{A}'$ are still separable as open subspaces of separable spaces.

Claim 3.4.1. *For each $Y \in \mathcal{A}'$, the set $\{Z \in \mathcal{A}' : Y \cap Z \neq \emptyset\}$ is countable.*

⊢ Let $D \in [Y]^{\aleph_0}$ be a dense subset of Y. Let $\mathcal{B} = \{Z \in \mathcal{A}' : Z \cap D \neq \emptyset\}$. $\mathcal{B}$ is countable, since $\mathcal{A}'$ is point countable. We show that $\mathcal{B} = \{Z \in \mathcal{A}' : Y \cap Z \neq \emptyset\}$. "$\subseteq$" is clear. To show "$\supseteq$", suppose that $Z \in \mathcal{A}'$ is such that $Y \cap Z \neq \emptyset$. Then as a nonempty open subset of Y, $Y \cap Z$ contains some element of D which means that $Z \in \mathcal{B}$. ⊣ (Claim 3.4.1)

Let $\approx_{\mathcal{A}'}$ be the transitive closure of the intersection relation on $\mathcal{A}'$. Then each equivalence class $e \subseteq \mathcal{A}'$ with respect to $\approx_{\mathcal{A}'}$ is countable by Claim 3.4.1. Since $\bigcup e$ is also closed. $\bigcup e = \bigcup\{\overline{Z} : Z \in e\}$. Since each $\overline{Z}$, $Z \in e$ is Lindelöf

as a closed subspace of a Lindelöf space, it follows that $\bigcup e$ is also Lindelöf. Thus $\{\bigcup e : e \in \mathcal{A}'/ \approx_{\mathcal{A}'}\}$ is a partition of X as in (a). $\square$ (Lemma 3.4)

Lemma 3.5 (Proposition 1.1 in Balogh [2]). *If a topological space $X = (X, \mathcal{O})$ is locally Lindelöf, then $\mathcal{B} = \{V \subseteq X : V$ is an open Lindelöf subspace of $X\}$ forms a base of X.*

Proof. Note that a closed subspace of a Lindelöf space is also Lindelöf. Hence, for $x \in X$ and $x \in O \in \mathcal{O}$, there is a $U \in \mathcal{O}$ such that $x \in U \subseteq O$ and $\overline{U}$ is Lindelöf. Since $\overline{U}$ is a Lindelöf space and thus normal, we can construct a sequence $\langle O_i : i \in \omega \rangle$ of open sets such that

(3.1) $x \in O_0 \subseteq \overline{O_0} \subseteq O_1 \subseteq \overline{O_1} \subseteq \cdots \subseteq U.$

Let $O^* = \bigcup_{i \in \omega} O_i$. Then O^* is an open neighborhood of x and $O^* \subseteq O$. O^* is Lindelöf since we can also represent O^* as the countable union of Lindelöf spaces, namely as $O^* = \bigcup_{i \in \omega} \overline{O_i}$. $\square$ (Lemma 3.5)

Z. Balogh [2] proved the following theorem under **Axiom R**.

Theorem 3.6 (FRP). *Suppose that X is locally Lindelöf and countably tight. If every open subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact then X itself is paracompact.*

Proof. A variation of the proof of Theorem 4.3 in S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba [9] will do.

It is enough to prove that the following $(3.2)_\kappa$ holds for all cardinal κ by induction on κ:

$(3.2)_\kappa$ For any countably tight and locally Lindelöf space X with $L(X) \leq \kappa$, if every open subspace of X of Lindelöf degree $\leq \aleph_1$ is paracompact then X itself is also paracompact.

For $\kappa \leq \aleph_1$, $(3.2)_\kappa$ trivially holds. So assume that $\kappa > \aleph_1$ and that $(3.2)_\lambda$ holds for all $\lambda < \kappa$. Let X be as in $(3.2)_\kappa$. We have to show that X is paracompact.

Case 1. κ is regular.

Let $\{L_\alpha : \alpha < \kappa\}$ be a cover of X consisting of Lindelöf subspaces of X. By Lemma 3.5, we may assume that each L_α is open. For $\beta < \kappa$, let $X_\beta = \bigcup \{L_\alpha : \alpha < \beta\}$. By $L(X) = \kappa$, we have $X \neq X_\beta$ for every $\beta < \kappa$. We may also assume that the continuously increasing sequence $\langle X_\beta : \beta < \kappa \rangle$ of open set in X is strictly increasing.

Let $S = \{\alpha < \kappa : X_\alpha \neq \overline{X_\alpha}\}$.

Claim 3.6.1. *S is non-stationary in* κ.

⊢ We prove first the following weakening of the claim:

Subclaim 3.6.1.1. $S \cap E_\omega^\kappa$ *is non-stationary in* κ.

⊢ For a contradiction, suppose that $S \cap E_\omega^\kappa$ were stationary. For each $\alpha \in S \cap E_\omega^\kappa$, let $p_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $h(\alpha) \in \kappa$ be such that $p_\alpha \in L_{h(\alpha)}$. Since X is countably tight, there is $c_\alpha \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup_{\beta \in c_\alpha} L_\beta}$.

Now, by FRP, there is $I \in [\kappa]^{\aleph_1}$ such that

(3.3) $\mathrm{cf}(I) = \omega_1$;

(3.4) $h(\alpha) \in I$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$;

(3.5) $c_\alpha \subseteq I$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$;

(3.6) if $f : S \cap E_\omega^\kappa \cap I \to \kappa$ is such that $f(\alpha) \in c_\alpha$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1}(\{\xi^*\})) = \sup(I)$.

Let $Y = \bigcup_{\beta \in I} L_\beta$. Note that, by (3.4), $p_\alpha \in Y$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$.

By $|I| = \aleph_1$ and since each L_β is open Lindelöf subspace of X, it follows that Y is open and $L(Y) \le \aleph_1$. Hence, by the assumption on X, Y is a paracompact subspace of X. Thus the open cover $\mathcal{L} = \{L_\beta : \beta \in I\}$ of Y has a locally finite open refinement $\mathcal{E}$. Since each L_β ($\beta \in I$) is Lindelöf, it follows that, for each $\beta \in I$,

(3.7) $\{E \in \mathcal{E} : E \cap L_\beta \ne \emptyset\}$ is countable.

This can be seen as follows: Since $\mathcal{E}$ is locally finite, for each $p \in L_\beta$, there is an open set O_p such that $p \in O_p$ and $\{E \in \mathcal{E} : E \cap O_p \ne \emptyset\}$ is finite. Since L_β is open, we may choose O_p to be a subset of L_β. Since L_β is Lindelöf and $\{O_p : p \in L_\beta\}$ is an open cover of L_β, there is a countable $A \subseteq L_\beta$ such that $\{O_p : p \in A\}$ already covers L_β. We have $\{E \in \mathcal{E} : E \cap L_\beta \ne \emptyset\} = \{E \in \mathcal{E} : E \cap O_p \ne \emptyset$ for some $p \in A\}$. But the right-side of the equation is easily seen to be countable.

Now, for each $\alpha \in S \cap E_\omega^\kappa \cap I$, let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$. Since $p_\alpha \in \overline{\bigcup\{L_\beta : \beta \in c_\alpha\}}$, there is $f(\alpha) \in c_\alpha$ such that $E_\alpha \cap L_{f(\alpha)} \ne \emptyset$. Thus, by (3.6), there is a $\xi^* \in I$ such that $\sup(f^{-1\prime\prime}\{\xi^*\}) = \sup(I)$. By (3.7), we have $E \subseteq X_\eta$ for all $E \in \mathcal{E}$ such that $E \cap L_{\xi^*} \ne \emptyset$ for some large enough $\eta \in S \cap E_\omega^\kappa \cap I$ with $f(\eta) = \xi^*$. But, since $\emptyset \ne E_\eta \cap L_{f(\eta)} = E_\eta \cap L_{\xi^*}$ we have $p_\eta \in E_\eta \subseteq X_\eta$. This is a contradiction to the choice of p_η.   ⊣ (Subclaim 3.6.1.1)

Let C be a club subset of κ consisting of limit ordinals such that $S \cap E_\omega^\kappa \cap C = \emptyset$ and let

(3.8) $D = \{\alpha \in C \ : \ \alpha \setminus S \text{ is cofinal in } \alpha\}.$

Clearly D is also a club subset of κ. So the following subclaim proves the claim.

Subclaim 3.6.1.2. $S \cap D = \emptyset.$

$\vdash$ For $\alpha \in D \cap E_\omega^\kappa$, we have $\alpha \notin S$ by $D \subseteq C$.

For $\alpha \in D \cap E_{>\omega}^\kappa$, suppose $p \in \overline{X_\alpha}$. By the countable tightness of X there is $\beta < \alpha$ such that $p \in \overline{X_\beta}$. By (3.8), we may assume that $\beta \in E_\omega^\kappa \setminus S$. Thus we have $p \in \overline{X_\beta} = X_\beta \subseteq X_\alpha$. This shows that $X_\alpha = \overline{X_\alpha}$ and hence $\alpha \notin S$.

$\dashv$ (Subclaim 3.6.1.2)

$\dashv$ (Claim 3.6.1)

Now let D be a club subset of κ such that $D \cap S = \emptyset$ and let $\langle \xi_\alpha \ : \ \alpha < \kappa \rangle$ be an increasing enumeration of $D \cup \{0\}$. Let $Y_\alpha = X_{\xi_{\alpha+1}} \setminus X_{\xi_\alpha}$ for $\alpha < \kappa$. Then $\{Y_\alpha \ : \ \alpha < \kappa\}$ is a partition of X into clopen subspaces. Since each Y_α is the union of $< \kappa$ many Lindelöf spaces, namely $L_\delta \setminus X_{\xi_\alpha}$, $\xi_\alpha \leq \delta < \xi_{\alpha+1}$, we have $L(Y_\alpha) < \kappa$. It follows from the induction hypothesis that each Y_α is paracompact. Hence X itself is also paracompact.

Case 2. κ is singular.

Similarly to Case 1., let $\{L_\alpha \ : \ \alpha < \kappa\}$ be a cover of X consisting of open Lindelöf subspaces of X. Let $\langle \kappa_i \ : \ i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ. For $i < \mathrm{cf}(\kappa)$, let $X_i = \bigcup\{L_\alpha \ : \ \alpha < \kappa_i\}$. By the induction hypothesis, there is a locally finite open refinement $\mathcal{C}_i$ of the open cover $\{L_\alpha \ : \ \alpha < \kappa_i\}$ of X_i for each $i < \mathrm{cf}(\kappa)$. Let $\mathcal{C} = \bigcup_{i < \mathrm{cf}(\kappa)} \mathcal{C}_i$.

Let $\sim_\mathcal{C}$ be the intersection relation on $\mathcal{C}$ and $\approx_\mathcal{C}$ be its transitive closure. Since each $\mathcal{C}_i$ is locally finite and each $C \in \mathcal{C}_i$ is Lindelöf, we have $|\{C' \in \mathcal{C} \ : \ C \approx_\mathcal{C} C'\}| \leq \mathrm{cf}(\kappa) < \kappa$ for all $C \in \mathcal{C}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_\mathcal{C}$. Then, each $e \in \mathbb{E}$ has cardinality $\leq \mathrm{cf}(\kappa)$.

$\mathcal{P} = \{\bigcup e \ : \ e \in \mathbb{E}\}$ is a partition of X into clopen subspaces. Since each $Y \in \mathcal{P}$ is the union of $\leq \mathrm{cf}(\kappa)$ many Lindelöf subspaces, we have $L(Y) \leq \mathrm{cf}(\kappa) < \kappa$. It follows that each $Y \in \mathcal{P}$ is paracompact by the induction hypothesis and hence X is also paracompact. $\square$ (Theorem 3.6)

Corollary 3.7. *The assertion of* Theorem 3.6 *is equivalent to* FRP.

Proof. Theorem 3.6 shows that the assertion follows from FRP.

Suppose that FRP fails. Then by Theorem 1.1 we can build a topological space X of regular cardinality $\lambda > \aleph_1$ as in Lemma 2.1. By Lemma 2.1, (2), X is

locally Lindelöf and countably tight. Every open subspace Y of X with $L(Y) \leq \aleph_1$ has cardinality $\leq \aleph_1 < \lambda$ by Lemma 2.1, (6). Hence, by Lemma 2.1, (4) and Morita's Theorem, it is paracompact. However X is not paracompact by Lemma 2.1, (3). Thus the assertion of Theorem 3.6 does not hold. $\quad\square$ (Corollary 3.7)

4 Axiom R-like extension of FRP and a stronger reflection property of paracompactness

Similarly to the extension of RP to Axiom R, $\mathsf{FRP}(\kappa)$ for a regular cardinal $\kappa \geq \aleph_2$ can be enhanced with the additional requirement that the reflection point I be an element of a given ω_1-club family $\subseteq [\kappa]^{\aleph_1}$:

$\mathsf{FRP}^R(\kappa)$: For any ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, stationary $S \subseteq E_\omega^\kappa$ and mapping $g : S \to [\kappa]^{\leq \aleph_0}$ there is $I \in \mathcal{T}$ such that

> (4.1)   for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1\,\prime\prime}\{\xi^*\}$ is stationary in $sup(I)$.

Similarly to FRP, let FRP^R be the axiom asserting that $\mathsf{FRP}^R(\kappa)$ holds for all regular $\kappa \geq \aleph_2$.

Note that the constraints (1.1) and (1.2) on I in $\mathsf{FRP}(\kappa)$ can be also realized by thinning out of the ω_1-club family $\mathcal{C}$ in $\mathsf{FRP}^R(\kappa)$. Thus $\mathsf{FRP}^R(\kappa)$ implies $\mathsf{FRP}(\kappa)$ for all regular $\kappa \geq \aleph_2$. The proof of the implication "$\mathsf{RP}(\kappa) \Rightarrow \mathsf{FRP}(\kappa)$" in [9] can be slightly modified to show the implication "$\mathsf{AR}(\kappa) \Rightarrow \mathsf{FRP}^R(\kappa)$".

Lemma 4.1. *For a regular cardinal $\kappa \geq \aleph_2$, $\mathsf{FRP}^R(\kappa)$ is equivalent to the following $\mathsf{FRP}_\bullet^R(\kappa)$:*

$\mathsf{FRP}_\bullet^R(\kappa)$: *For any ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, stationary $S \subseteq E_\omega^\kappa$ and mapping $g : S \to [\kappa]^{\leq \aleph_0}$ there is a continuously increasing sequence $\langle I_\xi : \xi < \omega_1 \rangle$ of countable subsets of κ such that*

> (4.2) $\langle \sup(I_\xi) : \xi < \omega_1 \rangle$ *is strictly increasing;*
>
> (4.3)   *each I_ξ is closed with respect to g;*
>
> (4.4) $\sup(I_\xi) \in I_{\xi+1}$;
>
> (4.5) $\bigcup_{\xi < \omega_1} I_\xi \in \mathcal{T}$ *and*
>
> (4.6) $\{\xi < \omega_1 : \sup(I_\xi) \in S \text{ and } g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ *is stationary in ω_1.*

Proof. First, assume $\mathsf{FRP}^R(\kappa)$. Let $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ be ω_1-club, $S \subseteq E^\kappa_\omega$ be stationary and $g : S \to [\kappa]^{\aleph_0}$. Without loss of generality, we may assume that $g(\alpha) \cap \alpha \neq \emptyset$ for all $\alpha \in S$. Without loss of generality, we may assume that all elements of $\mathcal{T}$ have cofinality ω_1.

Let $I \in \mathcal{T}$ be as in the definition of $\mathsf{FRP}^R(\kappa)$ for these S and g. Then, by (1.2), there is a filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I, that is, a continuously increasing sequence $\langle I_\xi : \xi < \omega_1 \rangle$ of subsets of I of cardinality $< |I|$ with $I = \bigcup_{\xi < \omega_1} I_\xi$, satisfying (4.2), (4.3) and (4.4).

We show that $\langle I_\xi : \xi < \omega_1 \rangle$ satisfies (4.6) as well. Suppose not. Then $\{\xi < \omega_1 : \sup(I_\xi) \notin S$ or $g(\sup(I_\xi)) \cap \sup(I_\xi) \nsubseteq I_\xi\}$ includes a club set $\subseteq \omega_1$. It follows that $S \cap I \setminus S_0$ is non stationary in $\sup(I)$, where

$$S_0 = \{\alpha \in S \cap I : \alpha = \sup(I_\xi) \text{ for some } \xi < \omega_1 \text{ and } g(\alpha) \cap \alpha \nsubseteq I_\xi\}.$$

Let $f : S \cap I \to I$ be defined by

$$(4.7) \quad f(\alpha) = \begin{cases} \min((g(\alpha) \cap \alpha) \setminus I_\xi) & \text{if } \alpha \in S_0 \text{ and } \alpha = \sup(I_\xi); \\ \min(g(\alpha)) & \text{otherwise.} \end{cases}$$

Then f is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. By the choice of I, there is an $\alpha^* \in I$ such that $f^{-1}{}''\{\alpha^*\}$ is stationary in $\sup(I)$. In particular, $S_0 \cap f^{-1}{}''\{\alpha^*\}$ is stationary in $\sup(I)$. Let $\xi^* \in \omega_1$ be such that $\alpha^* \in I_{\xi^*}$ and let $\beta \in S_0 \cap f^{-1}{}''\{\alpha^*\}$ be such that $\beta > \sup(I_{\xi^*})$. Let $\eta < \omega_1$ be such that $\beta = \sup(I_\eta)$. Then $\alpha^* \in I_{\xi^*} \subseteq I_\eta$. Since $\beta \in S_0$, we have $f(\beta) \notin I_\eta$ by the definition (4.7) of f. It follows that $f(\beta) \neq \alpha^*$. This is a contradiction to the choice of β.

Now, assume $\mathsf{FRP}^R_\bullet(\kappa)$. Suppose that $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ is ω_1-club, $S \subseteq E^\kappa_\omega$ is stationary and $g : S \to [\kappa]^{\aleph_0}$. Let $\langle I_\xi : \xi < \omega_1 \rangle$ be as in the definition of $\mathsf{FRP}^R_\bullet(\kappa)$ and let $I = \bigcup_{\xi < \omega_1} I_\xi$.

We claim that this I satisfies the conditions in the definition of $\mathsf{FRP}(\kappa)$. It is clear that I satisfies (1.1) and (1.2). To see that it also satisfies (1.3), suppose that $f : S \cap I \to \kappa$ is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. Let $S_1 = \{\xi \in \omega_1 : f(\sup(I_\xi)) \in I_\xi\}$. Then we have

$$S_1 \supseteq \{\xi \in \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$$

and thus S_1 is stationary by the choice of I. For each $\xi \in S_1$, let

$$h(\xi) = \min\{\eta < \omega_1 : f(\sup(I_\xi)) \in I_\eta\}.$$

Then the mapping $h : S_1 \to \omega_1$ is regressive. Thus, by Fodor's theorem, there is a stationary $S_2 \subseteq S_1$ such that $h''S_2 = \{\eta^*\}$ for some $\eta^* \in \omega_1$. Since I_{η^*} is

countable, there is a stationary $S_3 \subseteq S_2$ such that, for any $\xi \in S_3$, $f(\sup(I_\xi)) = \alpha^*$ for some fixed $\alpha^* \in I_{\eta^*}$. It follows that $f^{-1}{}''\{\alpha^*\} \supseteq \{\sup(I_\xi) : \xi \in S_3\}$ is stationary in $\sup(I)$.   □ (Lemma 4.1)

Theorem 4.2. *For any regular cardinal* $\kappa > \aleph_1$, $\mathsf{AR}(\kappa)$ *implies* $\mathsf{FRP}^R(\kappa)$.

Proof. By Lemma 4.1, it is enough to show that $\mathsf{AR}(\kappa)$ implies $\mathsf{FRP}^R_\bullet(\kappa)$. Suppose that $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, $S \subseteq E^\kappa_\omega$ is stationary and $g : S \to [\kappa]^{\leq \aleph_0}$. Let

$$(4.8) \quad S_0 = \{a \in [\kappa]^{\aleph_0} : \sup(a) \in S \setminus a \text{ and } g(\sup(a)) \cap \sup(a) \subseteq a\}.$$

Claim 4.2.1. S_0 *is a stationary subset of* $[\kappa]^{\aleph_0}$.

⊢ Suppose that $C \subseteq [\kappa]^{\aleph_0}$ is a club. We show that $C \cap S_0 \neq \emptyset$.

By Kueker's theorem, there is a mapping $s : \kappa^{<\omega} \to \kappa$ such that $C \supseteq C(s) = \{a \in [\kappa]^{\aleph_0} : s''a^{<\omega} \subseteq a\}$. Let $D = \{\alpha < \kappa : s''\alpha^{<\omega} \subseteq \alpha\}$. Since κ is regular, D is a club subset of κ. So there is an $\alpha^* \in S \cap D$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals such that $\alpha^* = \sup_{n \in \omega} \alpha_n$. Let a^* be the closure of $a_0 = \{\alpha_n : n \in \omega\} \cup (g(\alpha^*) \cap \alpha^*)$ with respect to s. Since a_0 is cofinal in α^* and $\alpha^* \in D$, we have $\sup(a^*) = \alpha^*$. Hence $a^* \in S_0$. By the definition of a^*, we also have $a^* \in C(s) \subseteq C$.

⊣ (Claim 4.2.1)

Let $\mathcal{T}_0 = \{X \in \mathcal{T} : \mathrm{cf}(X) = \omega_1 \text{ and } X \text{ is closed with respect to } g\}$. Then $\mathcal{T}_0$ is still ω_1-club. By $\mathsf{AR}(\kappa)$, there is $I \in \mathcal{T}_0$ such that

$$(4.9) \quad \mathrm{cf}(I) = \omega_1;$$

$$(4.10) \quad g(\alpha) \subseteq I \text{ for all } \alpha \in I \cap S;$$

$$(4.11) \quad S_0 \cap [I]^{\aleph_0} \text{ is stationary in } [I]^{\aleph_0}.$$

Let $\langle I_\xi : \xi < \omega_1 \rangle$ be a filtration of I such that each I_ξ is closed with respect to g (this is possible by (4.10)) and $\langle \sup(I_\xi) : \xi < \omega_1 \rangle$ is strictly increasing (possible by (4.9)). Let

$$S_1 = \{\xi < \omega_1 : \xi \text{ is a limit and } I_\xi \in S_0\} \text{ and}$$

$$S_2 = \{\xi < \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi)\}.$$

By the definition (4.8) of S_0, we have $S_2 \supseteq S_1$ and S_1 is a stationary subset of ω_1 by (4.11). Thus S_2 is stationary as well.   □ (Theorem 4.2)

Corollary 4.3. Axiom R *implies* FRP^R.   □

A straightforward modification of Theorem 3.4 in [9] shows also that $\mathsf{FRP}^R(\kappa)$ is preserved in generic extensions by c.c.c. forcing.

Shelah proved that Singular Cardinal Hypothesis (SCH) follows from a weakening of RP ([19]). In [11], we showed that FRP already implies Shelah's Strong Hypothesis (SSH). SSH is a strengthening of SCH and under $2^{\aleph_0} > \aleph_\omega$ which is consistent with FRP (but not with RP), it is strictly stronger than SCH.

One of the assertions equivalent to SSH is the following:

(4.12) $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ for all singular cardinal κ of countable cofinality.

Note that from this, it follows that

(4.13) $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa$ for all cardinal of uncountable cofinality.

Thus (4.13) is a consequence of FRP.

Balogh proved the following theorem under Axiom R (Theorem 1.4 in [2]).

Theorem 4.4. *Assume* FRP^R. *Suppose that X is a countably tight locally Lindelöf space such that*

(4.14) *for all open subspaces Y of X with $L(Y) \leq \aleph_1$, we have $L(\overline{Y}) \leq \aleph_1$ and*

(4.15) *every clopen subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact.*

Then X itself is paracompact.

Proof of Theorem 4.4: The proof is a modification of the proof of Theorem 3.6.

It is enough to prove that the following $(4.16)_\kappa$ holds for all cardinal κ by induction on κ:

$(4.16)_\kappa$ For any countably tight and locally Lindelöf space X with $L(X) = \kappa$, if X satisfies (4.14) and (4.15), then X is paracompact.

For $\kappa \leq \aleph_1$, $(4.16)_\kappa$ trivially holds. So assume that $\kappa > \aleph_1$ and that $(4.16)_\lambda$ holds for all $\lambda < \kappa$. Let X be a countably tight and locally Lindelöf space with $L(X) = \kappa$ such that X satisfies (4.14) and (4.15). We have to show that X is paracompact.

By Lemma 3.5, and since X is locally Lindelöf and $L(X) = \kappa$, there is a cover $\{L_\alpha : \alpha < \kappa\}$ of X consisting of open Lindelöf subspaces.

Let

$$\mathcal{T} = \{I \in [\kappa]^{\aleph_1} : \bigcup_{\alpha \in I} L_\alpha \text{ is a clopen subspace of } X\}.$$

By (4.14) and since X is countably tight, it is easy to see that $\mathcal{T}$ is ω_1-club.

Case 1. κ is regular.

For $\beta < \kappa$, let $X_\beta = \bigcup\{L_\alpha : \alpha < \beta\}$. By induction hypothesis we may also assume that $X \neq X_\beta$ for every $\beta < \kappa$ and that the sequence $\langle X_\beta : \beta < \kappa \rangle$ is strictly increasing.

Let $S = \{\alpha < \kappa : X_\alpha \neq \overline{X_\alpha}\}$.

Claim 4.4.1. S is non-stationary in κ.

⊢ We prove first the following weakening of the claim:

Subclaim 4.4.1.1. $S \cap E_\omega^\kappa$ is non-stationary in κ.

⊢ For a contradiction, suppose that $S \cap E_\omega^\kappa$ were stationary. For each $\alpha \in S \cap E_\omega^\kappa$, let $p_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $h(\alpha) \in \kappa$ be such that $p_\alpha \in L_{h(\alpha)}$. Since X is countably tight, there is $c_\alpha \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup_{\beta \in c_\alpha} L_\beta}$.

Now, by FRP^R, there is $I \in \mathcal{T}$ such that

(4.17) $\mathrm{cf}(I) = \omega_1$;

(4.18) $h(\alpha) \in I$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$;

(4.19) $c_\alpha \subseteq I$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$;

(4.20) if $f : S \cap E_\omega^\kappa \cap I \to \kappa$ is such that $f(\alpha) \in c_\alpha$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1\prime\prime}\{\xi^*\}) = \sup(I)$.

Let $Y = \bigcup_{\beta \in I} L_\beta$. Note that, by (4.18), $p_\alpha \in Y$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$.

By $I \in \mathcal{T}$ and since each L_β is open Lindelöf subspace of X, it follows that Y is clopen and $L(Y) \leq \aleph_1$. Hence, by (4.15), Y is a paracompact subspace of X. The rest of this case can be treated exactly as the Case 1 in the proof of Theorem 3.6. Thus the open cover $\{L_\beta : \beta \in I\}$ of Y has a locally finite open refinement $\mathcal{E}$. Since each L_β $(\beta \in I)$ is Lindelöf, it follows that

(4.21) $\{E \in \mathcal{E} : E \cap L_\beta \neq \emptyset\}$ is countable.

Now, for each $\alpha \in S \cap E_\omega^\kappa \cap I$, let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$. Since $p_\alpha \in \overline{\bigcup\{L_\beta : \beta \in c_\alpha\}}$, there is $f(\alpha) \in c_\alpha$ such that $E_\alpha \cap L_{f(\alpha)} \neq \emptyset$. Thus, by (4.20), there is a $\xi^* \in I$ such that $\sup(f^{-1\prime\prime}\{\xi^*\}) = \sup(I)$. By (4.21), there is $\eta \in S \cap E_\omega^\kappa \cap I$ such that $f(\eta) = \xi^*$ and $E \subseteq X_\eta$ for all $E \in \mathcal{E}$ such that $E \cap L_{\xi^*} \neq \emptyset$. But, since $\emptyset \neq E_\eta \cap L_{f(\eta)} = E_\eta \cap L_{\xi^*}$, we have $p_\eta \in E_\eta \subseteq X_\eta$. This is a contradiction to the choice of p_η.   ⊣ (Subclaim 4.4.1.1)

Let C be a club subset of κ consisting of limit ordinals such that $S \cap E_\omega^\kappa \cap C = \emptyset$ and let

(4.22) $D = \{\alpha \in C \ : \ \alpha \setminus S \text{ is cofinal in } \alpha\}$.

Clearly D is also a club subset of κ. So the following subclaim proves the claim.

Subclaim 4.4.1.2. $S \cap D = \emptyset$.

⊢ For $\alpha \in D \cap E^\kappa_\omega$, we have $\alpha \notin S$ by $D \subseteq C$.

For $\alpha \in D \cap E^\kappa_{>\omega}$, suppose $p \in \overline{X_\alpha}$. By the countable tightness of X, there is $\beta < \alpha$ such that $p \in \overline{X_\beta}$. By (4.22), we may assume that $\beta \in E^\kappa_\omega \setminus S$. Thus we have $p \in \overline{X_\beta} = X_\beta \subseteq X_\alpha$. This shows that $X_\alpha = \overline{X_\alpha}$ and hence $\alpha \notin S$.

$$\dashv \text{ (Subclaim 4.4.1.2)}$$
$$\dashv \text{ (Claim 4.4.1)}$$

Now let D be a club subset of κ such that $D \cap S = \emptyset$ and let $\langle \xi_\alpha \ : \ \alpha < \kappa \rangle$ be an increasing enumeration of $D \cup \{0\}$. Let $Y_\alpha = X_{\xi_{\alpha+1}} \setminus X_{\xi_\alpha}$ for $\alpha < \kappa$. Then $\{Y_\alpha \ : \ \alpha < \kappa\}$ is a partition of X into clopen subspaces. Since each Y_α is the union of $< \kappa$ many Lindelöf spaces, namely $L_\delta \setminus X_{\xi_\alpha}$, $\xi_\alpha \leq \delta < \xi_{\alpha+1}$, we have $L(Y_\alpha) < \kappa$. It follows from the induction hypothesis that each Y_α is paracompact. Hence X itself is also paracompact.

Case 2. κ is singular.

Let θ be a sufficiently large cardinal. Let $\mathcal{L} = \{L_\alpha \ : \ \alpha < \kappa\}$. The singularity of κ is not yet necessary in the following claim:

Claim 4.4.2. *If $M \prec \mathcal{H}(\theta)$ is such that*

(4.23) $\omega_1 \subseteq M$;

(4.24) $X, \mathcal{L} \in M$;

(4.25) M *is ω-bounding,*

then $Z = \bigcup(\mathcal{L} \cap M)$ is a clopen subspace of X.

⊢ Z is an open subspace of X as the union of open subspaces $\mathcal{L} \cap M$. Thus it is enough to show that X is closed. Suppose $x \in \overline{Z}$. By the countable tightness of X, there is $c \in [\mathcal{L} \cap M]^{\aleph_0}$ such that $x \in \overline{\bigcup c}$. By (4.25), there is $c' \in [\mathcal{L} \cap M]^{\aleph_0} \cap M$ such that $c \subseteq c'$. By (4.14) and by the elementarity of M, we have

$$M \models \exists d \in [\mathcal{L}]^{\aleph_1} (\overline{\bigcup c'} \subseteq \bigcup d).$$

Let $d \in [\mathcal{L}]^{\aleph_1} \cap M$ be such that $\overline{\bigcup c'} \subseteq \bigcup d$. By (4.23), we have $d \subseteq M$. Thus there is an $L^* \in d = d \cap M$ such that $x \in L^* \subseteq \bigcup d \subseteq \bigcup(\mathcal{L} \cap M)$.

$$\dashv \text{ (Claim 4.4.2)}$$

Let $\langle M_i : i < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\theta)$ such that, for $i < \mathrm{cf}(\kappa)$,

(4.26) $|M_i| < \kappa$;

(4.27) $\omega_1 \subseteq M_i$;

(4.28) $X, \mathcal{L} \in M_i$;

(4.29) M_i is ω-bounding and

(4.30) $\kappa \subseteq \bigcup_{i<\mathrm{cf}(\kappa)} M_i$.

We can construct such a sequence in particular with the property (4.29) by the assumption on the cardinal arithmetic.

Let $X_i = \bigcup(\mathcal{L} \cap M_i)$ for $i < \mathrm{cf}(\kappa)$. By Claim 4.4.2, each X_i is a clopen subspace of X. Since $L(X_i) \leq |M_i| < \kappa$, each X_i is paracompact by induction hypothesis . Note that we need here the closedness of X_i so that (4.14) holds for X_i.

$\mathcal{L} \cap M_i$ has a locally finite open refinement $\mathcal{C}_i$ for each $i < \mathrm{cf}(\kappa)$. Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$.

Let $\sim_{\mathcal{C}}$ be the intersection relation on $\mathcal{C}$ and $\approx_{\mathcal{C}}$ be its transitive closure. Since each $\mathcal{C}_i$ is locally finite and each $C \in \mathcal{C}_i$ is Lindelöf, $|\{C' \in \mathcal{C}_i : C' \approx_{\mathcal{C}_i} C\}| \leq \aleph_0$ for all $i < \mathrm{cf}(\kappa)$. Hence $|\{C' \in \mathcal{C} : C \approx_{\mathcal{C}} C'\}| \leq \mathrm{cf}(\kappa) < \kappa$ for all $C \in \mathcal{C}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_{\mathcal{C}}$. Then each $e \in \mathbb{E}$ has cardinality $\leq \mathrm{cf}(\kappa)$.

$\mathcal{P} = \{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into clopen subspaces. Since each $Y \in \mathcal{P}$ is the union of $\leq \mathrm{cf}(\kappa)$ many Lindelöf subspaces, we have $L(Y) \leq \mathrm{cf}(\kappa) < \kappa$. It follows that each $Y \in \mathcal{P}$ is paracompact by the induction hypothesis and hence X is also paracompact.   □ (Theorem 4.4)

It is unknown if the assertion of Theorem 4.4 implies FRP^R. But it is easy to see that it implies FRP:

Proposition 4.5. *The assertion of* Theorem 4.4 *implies* FRP.

Proof. Suppose that FRP does not hold. Then we can construct a topological space X as in Lemma 2.1. We have $X \models (4.14)$, by Lemma 2.1, (6) and (7). Also $X \models (4.15)$ by Lemma 2.1, (6) and (4). But X is not paracompact by Lemma 2.1, (3).   □ (Proposition 4.5)

Problem 1. *Is the assertion of* Theorem 4.4 *equivalent with* FRP^R *?*

Though we presently do not know if $\text{FRP}^R(\kappa)$ is equivalent to $\text{FRP}(\kappa)$ for all regular κ, it *is* the case for many κ:

Theorem 4.6. *Suppose that κ is regular and*

(4.31) $\quad cf([\lambda]^{\aleph_0}, \subseteq) < \kappa$ *for all* $\lambda < \kappa$.

Then we have $\text{FRP}^R(\kappa) \Leftrightarrow \text{FRP}(\kappa)$.

Proof. It is enough to show the direction "$\Leftarrow$".

Assume that κ is a regular cardinal $> \aleph_1$ with (4.31) and $\text{FRP}(\kappa)$ holds. Let $S \subseteq E^\kappa_\omega$ be stationary, $g : S \to [\kappa]^{\aleph_0}$ and $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ be ω_1-club. We want to show that there is $I \in \mathcal{T}$ such that I satisfies (4.1).

Let θ be sufficiently large and let $\mathcal{M}^* = \langle \mathcal{H}(\theta), S, g, \mathcal{T}, \ldots, \trianglelefteq, \in \rangle$ and let $\mathcal{M} \prec \mathcal{M}^*$ be the union of the continuously increasing chain $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $\mathcal{M}^*$ such that

(4.32) $\quad |M_\alpha| < \kappa$ for all $\alpha < \kappa$;

(4.33) $\quad M_{\alpha+1}$ is ω-bounding for all $\alpha < \kappa$;

(4.34) $\quad M_\alpha \in M_{\alpha+1}$ for all $\alpha < \kappa$ and

(4.35) $\quad \kappa \subseteq \mathcal{M}$.

Note that (4.33) is possible by (4.31). Let $C = \{ \alpha \in \kappa : \kappa \cap M_\alpha = \alpha \}$. Since C is club in κ, $S_0 = S \cap C$ is stationary. Applying $\text{FRP}(\kappa)$ to S_0 and $g \upharpoonright S_0$ we obtain $I_0 \in [\lambda]^{\aleph_1}$ such that, letting $\alpha_0 = \sup(I_0)$,

(4.36) $\quad \text{cf}(\alpha_0) = \omega_1$;

(4.37) $\quad g(\alpha) \subseteq I_0$ for all $\alpha \in I \cap S_0$;

(4.38) $\quad$ for any regressive $f : S_0 \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S_0 \cap I$, there is $\xi^* < \kappa$ such that $f^{-1\prime\prime}\{\xi^*\}$ is stationary in $sup(I_0)$.

Since $S_0 \cap \alpha_0$ is cofinal in α_0, we have $\alpha_0 \in C$. By (4.36) and (4.33) it follows that

Claim 4.6.1. M_{α_0} *is ω-bounding.*

$\vdash$ Suppose that $x \in [M_{\alpha_0}]^{\aleph_0}$ there is $\alpha < \alpha_0$ such that $x \in [M_\alpha]^{\aleph_0}$. Since $M_\alpha \subseteq M_{\alpha+1}$ and $M_{\alpha+1}$ is ω-bounding there is $y \in [M_{\alpha+1}]^{\alpha_0} \cap M \subseteq [M_{\alpha_0}]^{\aleph_0} \cap M_{\alpha_0}$ such that $x \subseteq y$. $\quad \dashv$ (Claim 4.6.1)

Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a continuously increasing sequence of elementary submodels of M_{α_0} such that

(4.39) $|N_\alpha| = \aleph_0$ for every $\alpha < \omega_1$;

(4.40) there is a countable set $x_\alpha \in N_{\alpha+1}$ such that $N_\alpha \subseteq x_\alpha$ for every $\alpha < \omega_1$ and

(4.41) $I_0 \subseteq \bigcup_{\alpha < \omega_1} N_\alpha$.

The condition (4.40) is realizable by Claim 4.6.1. Let $N = \bigcup_{\alpha < \omega_1} N_\alpha$ and $I = \kappa \cap N$. Then $I_0 \subseteq I$ by (4.41). So $|I| = \aleph_1$ by (4.39). Since $N \subseteq M_{\alpha_0}$, we have $\sup(I) = \alpha_0$.

Thus the following claim implies that this I is as in the definition of $\mathsf{FRP}^R(\kappa)$ for S, g and $\mathcal{T}$.

Claim 4.6.2. $I \in \mathcal{T}$.

⊢ For $\alpha < \omega_1$ there is $A_\alpha \in \mathcal{T} \cap N_{\alpha+1}$ such that

(4.42) $\bigcup(\mathcal{T} \cap N_\alpha) \subseteq A_\alpha$

by (4.40) and elementarity. $\langle A_\alpha : \alpha < \omega_1 \rangle$ is then an increasing sequence in $\mathcal{T}$. Let $A = \bigcup_{\alpha < \omega_1} A_\alpha$. By the ω_1-clubness of $\mathcal{T}$, we have $A \in \mathcal{T}$. By (4.42) and (4.40), we have $I \cap N_\alpha \subseteq A_\alpha \subseteq I$ for all $\alpha < \omega_1$. By (4.41), it follows that $A = I$.   ⊣ (Claim 4.6.2)
                                     ☐ (Theorem 4.6)

Corollary 4.7. FRP *implies* $\mathsf{FRP}^R(\kappa)$ *for all regular cardinals which are not the successor of a singular cardinal of countable cofinality.*

Proof. By Theorem 4.6 and by the fact that FRP implies (4.13).
                                     ☐ (Corollary 4.7)

By the theorem above we have $\mathsf{FRP}^R(\aleph_n) \Leftrightarrow \mathsf{FRP}(\aleph_n)$ for all $n \in \omega \setminus 1$. Thus the test question in this connection would be the following:

Problem 2. *Is* $\mathsf{FRP}^R(\aleph_{\omega+1})$ *equivalent to* $\mathsf{FRP}(\aleph_{\omega+1})$ *?*

References

[1] Z. Balogh, Reflecting point countable families, Proceedings of the American Mathematical Society 131, No.4, (2002), 1289–1296.

[2] Z. Balogh, Locally nice spaces and Axiom R, Topology and its Applications, 125, No.2, (2002), 335–341.

[3] R.E. Beaudoin, Strong analogues of Martin's Axiom imply Axiom R, Journal of Symbolic Logic, 52, No.1, (1987), 216–218.

[4] A. Dow, An introduction to applications of elementary submodels to topology, Topology Proceedings 13, No.1 (1988), 17–72.

[5] A. Dow, Set theory in topology, Ch. 4, 168-197 in *Recent Progress in General Topology*, M. Husek and J. van Mill (editors), Elsevier Science Publishers B.V., Amsterdam (1992).

[6] R. Engelking, *General Topology*, Second edition, Heldermann Verlag, Berlin, (1989).

[7] W. Fleissner, Left-separated spaces with point-countable bases, Transactions of American Mathematical Society, 294, No.2, (1986), 665–677.

[8] S. Fuchino, Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kôkyûroku, No.1619, (2008), 32–42.

[9] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (June 2010), Special Issue dedicated to the Proceedings of the Conference "Advances in Set-Theoretic Topology" (in Honour of Tsugunori Nogura on his 60th Birthday), 1415–1429.

[10] S. Fuchino, Fodor-type Reflection Principle and Balogh's reflection theorems, Combinatorial set theory and forcing theory, Teruyuki Yorioka (ed.), RIMS Kôkyûroku, No.1686, (2010), 41–58.

[11] S. Fuchino and A. Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212 (2011), 261-283.

[12] S. Fuchino, H. Sakai, L. Soukup and T. Usuba, More about the Fodor-type Reflection Principle, submitted.
http://kurt.scitec.kobe-u.ac.jp/ fuchino/papers/moreFRP.pdf

[13] S. Fuchino, H. Sakai, V. Torres Perez and T. Usuba, Rado's Conjecture and the Fodor-type Reflection Principle, in preparation.

[14] I. Juhász, Cardinal functions in topology—ten years later. Second edition. Mathematical Centre Tracts, 123. Mathematisch Centrum, Amsterdam, (1980).

[15] T. Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006).

[16] A. Kanamori, The Higher Infinite, Springer-Verlag (1994/97).

[17] S. Shelah, Proper Forcing. North-Holland, Amsterdam (1980).

[18] S. Shelah, A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals, Israel Journal of Mathematics 21 (1975), 319–349.

[19] S. Shelah, Reflection implies the SCH, Fundamenta Mathematicae 198 (2008), 95–111.

[20] S. Todorčević, On a conjecture of R. Rado, Journal of the London Mathematical Society, Vol.27, (1983), 1–8.