数理解析研究所講究録
第 1833 巻 2013 年 77-85

Some notes on countable T_D-spaces

Matthew de Brecht

National Institute of Information and Communications Technology
Kyoto, Japan

Abstract

We provide three canonical examples of countable perfect T_D-spaces corresponding to the T_D, T_1, and T_2 separation axioms. These three spaces are canonical in the sense that any countable T_D-space is either quasi-Polish or else contains one of these spaces as a subspace. These results provide valuable insight as to why a space can fail to be complete.

Keywords: descriptive set theory, non-Hausdorff space, quasi-Polish spaces

1. Introduction

All topological spaces in this paper are assumed to be countably based and satisfy the T_0 separation axiom, but no further assumptions are made unless explicitly stated.

This paper is a continuation of recent work on developing the descriptive set theory of non-metrizable spaces initiated by V. Selivanov (see [8]). It was recently shown in [2] that a very general class of spaces called quasi-Polish spaces allow a smooth extension of the descriptive set theory of Polish spaces (see [4]) to the non-metrizable case. The class of quasi-Polish spaces contains not only the class of Polish spaces, but also many non-metrizable spaces that occur in fields such as theoretical computer science (e.g., ω-continuous domains with the Scott-topology) and algebraic geometry (e.g., the spectrum of countable Noetherian rings with the Zariski topology).

Given that so many important spaces are known to be quasi-Polish, the following natural question arises: Which spaces are *not* quasi-Polish? It was observed in [2] that a metrizable space is quasi-Polish if and only if it is Polish, so we can use results from classical descriptive set theory to obtain a first answer to this question: a countable metrizable space is *not* quasi-Polish if and only if it contains a homeomorphic copy of the rationals as a subspace. The purpose of these notes is to provide a modest extension of this result to cover the case of countable spaces satisfying the T_D separation axiom.

Email address: `matthew@nict.go.jp` (Matthew de Brecht)

The T_D-axiom is a separation axiom introduced by Aull and Thron [1] which is strictly between the T_1 and T_0 axioms. A subset of a space is *locally-closed* if it is equal to the intersection of an open set with a closed set. A topological space satisfies the T_D separation axiom if and only if every singleton subset is locally closed.

Countable T_D-spaces naturally occur in the field of inductive inference as precisely those spaces that can be identified in the limit relative to some oracle [3]. In these notes, we will show that there are three "canonical" countable perfect T_D-spaces respectively corresponding to the T_D, T_1, and T_2 separation axioms. This result implies that a countable T_D-space is either quasi-Polish or else contains one of these three counter-examples. Thus, these spaces provide important insight into why a space can fail to be complete. We will also prove some other interesting results concerning countable T_D-spaces. For example, we will show that a countable space is T_D if and only if it has a $\mathbf{\Delta}^0_2$-diagonal, and that if $X \subseteq Y$ is a countable T_D subspace, then X will be at most $\mathbf{\Delta}^0_3$ in Y.

2. Borel Hierarchy for non-Hausdorff spaces

It is common for non-Hausdorff spaces to have open sets that are not F_σ (i.e., countable unions of closed sets) and closed sets that are not G_δ (i.e., countable intersections of open sets). The Sierpsinski space, which has $\{\bot, \top\}$ as an underlying set and the singleton $\{\top\}$ open but not closed, is perhaps the simplest example of this phenomenon. This implies that the classical definition of the Borel hierarchy, which defines level $\mathbf{\Sigma}^0_2$ as the F_σ-sets and $\mathbf{\Pi}^0_2$ as the G_δ-sets, is not appropriate in the general setting. The following modification of the Borel hierachy due to Victor Selivanov (see [6, 7, 8]) is the appropriate definition for the more general case.

Definition 1. *Let (X, τ) be a topological space. For each ordinal α ($1 \leq \alpha < \omega_1$) we define $\mathbf{\Sigma}^0_\alpha(X, \tau)$ inductively as follows.*

1. $\mathbf{\Sigma}^0_1(X, \tau) = \tau$.
2. *For $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha(X, \tau)$ is the set of all subsets A of X which can be expressed in the form*

$$A = \bigcup_{i \in \omega} B_i \setminus B'_i,$$

where for each i, B_i and B'_i are in $\mathbf{\Sigma}^0_{\beta_i}(X, \tau)$ for some $\beta_i < \alpha$.

We define $\mathbf{\Pi}^0_\alpha(X, \tau) = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\alpha(X, \tau)\}$ and $\mathbf{\Delta}^0_\alpha(X, \tau) = \mathbf{\Sigma}^0_\alpha(X, \tau) \cap \mathbf{\Pi}^0_\alpha(X, \tau)$. Finally, we define $\mathbf{B}(X, \tau) = \bigcup_{\alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X, \tau)$ to be the Borel *subsets of (X, τ).* □

When the topology is clear from context, we will usually write $\mathbf{\Sigma}^0_\alpha(X)$ instead of $\mathbf{\Sigma}^0_\alpha(X, \tau)$.

The definition above is equivalent to the classical definition of the Borel hierarchy on metrizable spaces, but differs in general. V. Selivanov has investigated this hierarchy in a series of papers, with an emphasis on applications to

ω-continuous domains. D. Scott [5] and his student A. Tang [9, 10] have also investigated some aspects of the hierarchy in $\mathcal{P}(\omega)$ (the power set of the natural numbers with the Scott-topology), using the notation $\mathcal{B}_\sigma$ and $\mathcal{B}_\delta$ to refer to the levels $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$, respectively.

In [2] it was shown that much of the descriptive set theory of Polish spaces can be extended to a very general class of countably based T_0-spaces called quasi-Polish spaces. Quasi-Polish spaces are defined as the countably based spaces which admit a Smyth-complete quasi-metric, but many other characterizations are given in [2]. For the purposes of this paper, we can define a space to be quasi-Polish if and only if it is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $\mathcal{P}(\omega)$. Among other results, it was shown that a subspace of a quasi-Polish space is quasi-Polish if and only if it is $\mathbf{\Pi}^0_2$, and a metrizable space is quasi-Polish if and only if it is Polish.

For any topological space X we define $\Delta_X = \{\langle x,y\rangle \in X\times X \mid x = y\}$ to be the *diagonal* of X. The next theorem provides a useful characterization of countably T_D-spaces in terms of the Borel complexity of the diagonal.

Theorem 2. *The following are equivalent for a countably based space X with countably many points:*

1. *X satisfies the T_D separation axiom,*
2. *Every singleton subset $\{x\}$ of X is in $\mathbf{\Delta}^0_2(X)$,*
3. *Every subset of X is in $\mathbf{\Delta}^0_2(X)$,*
4. *The diagonal of X is in $\mathbf{\Delta}^0_2(X\times X)$.*

Proof: $(1 \Rightarrow 2)$. Easily follows from the definition of the T_D-axiom because locally closed sets are $\mathbf{\Delta}^0_2$.

$(2 \Rightarrow 3)$. If every singleton subset of X is $\mathbf{\Delta}^0_2$, then the countability of X implies that every subset of X is the countable union of $\mathbf{\Delta}^0_2$-sets. Thus for any $S \subseteq X$ both S and the complement of S are $\mathbf{\Sigma}^0_2$, hence S is $\mathbf{\Delta}^0_2$.

$(3 \Rightarrow 4)$. For each $x\in X$, the singleton $\{x\}$ is in $\mathbf{\Sigma}^0_2(X)$ by assumption, hence there are open sets U_x and V_x such that $\{x\} = U_x \setminus V_x$. Then $\Delta_X = \bigcup_{x\in X}\left[(U_x\setminus V_x)\times(U_x\setminus V_x)\right]$ is in $\mathbf{\Sigma}^0_2(X\times X)$. It was shown in [2] that the diagonal of every countably based T_0-space is $\mathbf{\Pi}^0_2$, therefore Δ_X is in $\mathbf{\Delta}^0_2(X\times X)$.

$(4 \Rightarrow 1)$. Assume that $\Delta_X = \bigcup_{i\in\omega} U_i\setminus V_i$ for U_i, V_i open in $X\times X$. Let x be any element of X. Then there is some $i\in\omega$ such that $\langle x,x\rangle \in U_i\setminus V_i$. Let U be an open neighborhood of x such that $\langle x,x\rangle\in U\times U\subseteq U_i$. Fix any $y\in U$ distinct from x. Clearly, $\langle x,y\rangle \in U\times U\subseteq U_i$, hence $\langle x,y\rangle\in V_i$ because $\langle x,y\rangle\notin\Delta_X$. Let V and W be open subsets of X such that $\langle x,y\rangle\in V\times W\subseteq V_i$. Then $x\notin W$ because otherwise we would have the contradiction $\langle x,x\rangle\in V_i$. Therefore, W is a neighborhood of y that does not contain x, hence y is not in the closure of $\{x\}$. It follows that $\{x\} = U\cap Cl(\{x\})$ is locally closed and that X is a T_D-space. □

3. Canonical countable perfect T_D-spaces

A space is *perfect* if and only if every non-empty open subset is infinite. Note that if X is a T_0-space, then X is perfect if and only if there is no $x \in X$ such that the singleton subset $\{x\}$ is open.

It is well known that the space of rationals is the unique (up to homeomorphism) example of a countable perfect metrizable space (see Exercise 7.12 in [4]). Things become more complicated when considering non-metrizable spaces that only satisfy the T_D-axiom. There are in fact infinitely many non-homeomorphic examples of countable perfect T_D-spaces. However, the following three spaces are the "canonical" examples of countable perfect T_D-spaces.

- The space ω defined as the set of natural numbers with the topology generated by the upper intervals $\uparrow n = \{m \in \omega \mid n \leq m\}$ for each $n \in \omega$. This space is T_D but not T_1.
- The space ω_{cof} defined as the set of natural numbers with the cofinite topology (i.e., a subset is closed if and only if it is finite or else the whole space). This space is T_1 but not T_2.
- The space $\mathbb{Q}$ of rational numbers with the topology inherited from the space of real numbers. This space is T_2.

These three spaces are canonical in the following sense, which is the main result of these notes.

Theorem 3. *If X is a non-empty countably based perfect T_D-space with countably many points, then X contains a subspace homeomorphic to either ω , ω_{cof}, or $\mathbb{Q}$.* □

Clearly, none of these spaces contain a copy of the others, so this is the best result possible.

A space which does not contain a non-empty perfect subspace is called *scattered.* In [2] it was shown that a countably based T_0-space is scattered if and only if it is a countable T_D quasi-Polish space. We therefore obtain the following.

Corollary 4. *If X is a countably based T_D-space with countably many points, then X is quasi-Polish if and only if X does not contain a subspace homeomorphic to either ω , ω_{cof}, or $\mathbb{Q}$.* □

In other words, ω , ω_{cof} and $\mathbb{Q}$ are the only "reasons" a countable T_D-space can fail to be quasi-Polish.

The purpose of this section is to prove Theorem 3. For the rest of this section we fix X to be some non-empty countably based perfect T_D-space with countably many points.

Lemma 5. *Either X contains a subspace homeomorphic to ω or else X contains a non-empty perfect T_1-subspace.*

Proof: Let $\sqsubseteq$ be the specialization order on X (i.e., $x \sqsubseteq y$ if and only if x is in the closure of $\{y\}$). Since X is a T_0-space, the specialization order is a partial order. Define $Max(X)$ to be the subset of X of elements that are maximal with respect to the specialization order. It is immediate that $Max(X)$ is a T_1-space.

First assume there is some $x_0 \in X$ such that there is no $y \in Max(X)$ with $x_0 \sqsubseteq y$. Then $x_0 \notin Max(X)$, so there is some $x_1 \neq x_0$ with $x_0 \sqsubseteq x_1$. The assumption on x_0 implies $x_1 \notin Max(X)$, so there is $x_2 \neq x_1$ with $x_0 \sqsubseteq x_1 \sqsubseteq x_2$. Continuing in this way, we produce an infinite sequence $\{x_i\}_{i \in \omega}$ of distinct elements of X with $x_i \sqsubseteq x_j$ whenever $i \leq j$. Clearly $\{x_i\}_{i \in \omega}$, viewed as a subspace of X, is homeomorphic to ω .

So if X does not contain a copy of ω , then every element of X is below some element of $Max(X)$ with respect to the specialization order. This implies, in particular, that $Max(X)$ is non-empty. We show that $Max(X)$ is perfect as a subspace of X. Assume for a contradiction that there is $x \in Max(X)$ and open $V \subseteq X$ such that $\{x\} = V \cap Max(X)$. Since X is a T_D-space, there is open $U \subseteq X$ such that $\{x\} = U \cap Cl(\{x\})$, where $Cl(\cdot)$ is the closure operator on X. Then $W = U \cap V$ is an open subset of X containing x. Fix any $y \in W$. By assumption, there is some $y' \in Max(X)$ such that $y \sqsubseteq y'$. Since W is open, the definition of $\sqsubseteq$ implies that $y' \in W$. Since $\{x\} = W \cap Max(X)$, it follows that $y' = x$ hence $y \sqsubseteq x$. Therefore, $y \in Cl(\{x\})$ which implies $x = y$ because $\{x\} = W \cap Cl(\{x\})$. Since $y \in W$ was arbitrary, $\{x\} = W$ is an open subset of X, which contradicts X being a perfect space. Therefore, $Max(X)$ is a non-empty perfect T_1-subspace of X. □

As a result of the above lemma, it only remains to consider the case where X is a T_1-space.

For any topological space Y, open $U \subseteq Y$, and $y \in Y$, we write $y \lhd U$ if $y \in U$ and for every open V containing y and non-empty open $W \subseteq U$, the intersection $V \cap W$ is non-empty. In other words, $y \lhd U$ if and only if every neighborhood of y is dense in the subspace U. Note that if $y \lhd U$ and $V \subseteq U$ is open and contains y, then $y \lhd V$. We define $D(Y)$ to be the set of all $y \in Y$ such that there is open $U \subseteq Y$ with $y \lhd U$.

Fix a countable basis $\{B_i\}_{i \in \omega}$ of open subsets of X. For $x \in X$ and $n \in \omega$, we define $B(x,n) = \bigcap\{B_i \mid x \in B_i \text{ and } i \leq n\}$. Here we use the convention that the empty intersection equals X, so $B(x,n) = X$ if there is no $i \leq n$ with $x \in B_i$. Note that for any open U containing x, there is $n \in \omega$ with $x \in B(x,n) \subseteq U$.

Lemma 6. *If X is a T_1-space and $D(X)$ has non-empty interior, then X contains a subset homeomorphic to ω_{cof}.*

Proof: Choose any x_0 in the interior of $D(X)$ and let U_0 be an open subset of X with $x_0 \lhd U_0 \subseteq D(X)$. Then U_0 is infinite because X is perfect, so we can choose $x_1 \in U_0$ distinct from x_0 and find open $U_1 \subseteq U_0$ with $x_1 \lhd U_1$.

Let $n \geq 1$ and assume we have defined a sequence $x_0, \ldots, x_n \in X$ and open sets $U_0 \supseteq \cdots \supseteq U_n$ with $x_i \lhd U_i \subseteq D(X)$. We choose $x_{n+1} \in X$ and open $U_{n+1} \subseteq U_n$ with $x_{n+1} \lhd U_{n+1}$ as follows. Define $V_i^n = U_i \cap B(x_i, n)$ for

$0 \leq i \leq n$, and let $V^n = V_0^n \cap \ldots \cap V_n^n$. Since $x_{n-1} \in V_{n-1}^n$ and $V_n^n \subseteq U_{n-1}$ is non-empty, $x_{n-1} \lhd U_{n-1}$ implies $V_{n-1}^n \cap V_n^n$ is non-empty. Continuing this argument inductively shows that V^n is a non-empty open set. Thus V^n is infinite, so there is $x_{n+1} \in V^n$ distinct from x_i for $0 \leq i \leq n$. Since $V^n \subseteq U_n \subseteq D(X)$, there is open $U_{n+1} \subseteq U_n$ with $x_{n+1} \lhd U_{n+1} \subseteq D(X)$.

Let $S = \{x_i \in X \mid i \in \omega\}$ be the subset of X of the elements enumerated in the above construction. We claim that S is homeomorphic to ω_{cof}. S is infinite by construction, and the assumption that X is a T_1-space implies that the subspace topology on S contains the cofinite topology. Therefore, it suffices to show that every non-empty open subset of S is cofinite. Let $U \subseteq S$ be non-empty open, so there is some $i \in \omega$ with $x_i \in U$. Let $m \geq i$ be large enough that $S \cap B(x_i, m) \subseteq U$. By the construction of S, $x_n \in V^n \subseteq B(x_i, n) \subseteq B(x_i, m)$ for all $n \geq m$. It follows that $x_n \in U$ for all $n \geq m$, hence U is a cofinite subset of S. Therefore, $S \subseteq X$ is homeomorphic to ω_{cof}. □

The final case to consider is when X is a T_1-space and $X \setminus D(X)$ is dense in X.

Lemma 7. *If X is a T_1-space and $X \setminus D(X)$ is dense in X, then X contains a subspace homeomorphic to $\mathbb{Q}$.*

Proof: Note that if $x \in X \setminus D(X)$ and U is any open set containing x, then there exists non-empty open sets $V, W \subseteq U$ with $x \in V$ and $V \cap W = \emptyset$.

In the following, we denote the length of a sequence $\sigma \in 2^{<\omega}$ by $|\sigma|$. We associate each $\sigma \in 2^{<\omega}$ with an element $x_\sigma \in X \setminus D(X)$ and open set $U_\sigma \subseteq X$ containing x_σ as follows. For the empty sequence ε choose any $x_\varepsilon \in X \setminus D(X)$ and let $U_\varepsilon = X$.

Next let $\sigma \in 2^{<\omega}$ be given and assume $x_\sigma \in X \setminus D(X)$ and U_σ have been defined. Let $U, V \subseteq B(x_\sigma, |\sigma|) \cap U_\sigma$ be non-empty open sets such that $x_\sigma \in U$ and $U \cap V = \emptyset$. Since V is non-empty and $X \setminus D(X)$ is dense, there exists some $y \in V \setminus D(X)$. Let $x_{\sigma\diamond 0} = x_\sigma$, $U_{\sigma\diamond 0} = U$, $x_{\sigma\diamond 1} = y$, and $U_{\sigma\diamond 0} = V$.

Let $S = \{x_\sigma \mid \sigma \in 2^{<\omega}\}$. A simple inductive argument shows that $U_\sigma \cap S$ is clopen in S for each $\sigma \in 2^{<\omega}$. We show that S is a perfect zero-dimensional T_2-space. Fix any $\sigma \in 2^{<\omega}$ and open $U \subseteq S$ containing x_σ. Let $n \in \omega$ be large enough that $B(x_\sigma, n) \cap S \subseteq U$. We can append a finite number of 0's to the end of σ to obtain a sequence σ' with $|\sigma'| \geq n$ and $x_{\sigma'} = x_\sigma$. Then $x_{\sigma'\diamond 1} \neq x_\sigma$ and $x_{\sigma'\diamond 1} \in B(x_\sigma, n) \cap S \subseteq U$. It follows that $\{x_\sigma\}$ is not open in S, so S is a perfect space. Furthermore, $U_{\sigma'\diamond 0} \cap S$ is a clopen set containing x_σ and contained in U, which implies that S is a zero-dimensional T_2-space.

It follows that S is a non-empty countable perfect metrizable space, hence S is homeomorphic to $\mathbb{Q}$. □

Theorem 3 now follows from the previous three lemmas.

4. Countable $\mathbf{\Delta}_3^0$-spaces

If Y is a countably based T_0-space, then it is immediate that every countable $X \subseteq Y$ is in $\mathbf{\Sigma}_3^0(Y)$. We will show in this section that there exist countable subsets of quasi-Polish spaces which are strictly $\mathbf{\Sigma}^0_3$ (i.e., $\mathbf{\Sigma}^0_3$ but not $\mathbf{\Pi}^0_3$), so this is the best upper bound in general. However, in the special case that $X \subseteq Y$ is both countable and satisfies the T_D-axiom, then X is guaranteed to be in $\mathbf{\Delta}^0_3(Y)$.

Theorem 8. *Assume Y is a countably based T_0-space and $X \subseteq Y$ is countable. If for every non-empty $A \in \mathbf{\Pi}^0_2(X)$ there is a finite non-empty $F \in \mathbf{\Delta}^0_2(A)$, then $X \in \mathbf{\Delta}^0_3(Y)$.*

Proof: Assume $X \subseteq Y$ is countable and for every non-empty $A \in \mathbf{\Pi}^0_2(X)$ there is a finite non-empty $F \in \mathbf{\Delta}^0_2(A)$. For each ordinal α, we inductively define X^α as follows:

- $X^0 = X$,
- $X^{\alpha+1} = X^\alpha \setminus \{x \in X^\alpha \mid \{x\} \text{ is locally closed in } X^\alpha\}$,
- $X^\alpha = \bigcap_{\beta<\alpha} X^\beta$ when α is a limit ordinal.

Since X is countable there is some ordinal $\alpha < \omega_1$ such that $X^\alpha = X^{\alpha+1}$. We define $\ell(X)$ to be the least such ordinal. Using again the fact that X is countable, it is straight forward to show that $X^\alpha \in \mathbf{\Pi}^0_2(X)$ for each $\alpha < \ell(X)$. Thus our assumption on X implies that if X^α is not empty, then there is a finite non-empty $F \in \mathbf{\Delta}^0_2(X^\alpha)$. It follows that $\{x\}$ is locally closed in X^α for each $x \in F$, hence $X^\alpha \neq X^{\alpha+1}$. Therefore, $X^{\ell(X)} = \emptyset$.

The claim is trivial is X is finite, so fix an infinite enumeration $x_0, x_1, \ldots$ of X without repetitions. Since $X^{\ell(X)} = \emptyset$, for each $i \in \omega$ there is a countable ordinal $\alpha_i < \ell(X)$ such that $x_i \in X^{\alpha_i} \setminus X^{\alpha_i+1}$. Choose an open subset U_i of Y such that $Cl(\{x_i\}) \cap U_i \cap X^{\alpha_i} = \{x_i\}$ (here and in the following, $Cl(\cdot)$ is the closure operator for Y).

For each $i \in \omega$, define

$$A_i = (Cl(\{x_i\}) \cap U_i) \setminus \bigcup\{Cl(x_j) \cap U_j \mid j < i \text{ and } \alpha_j = \alpha_i\}.$$

Then $A_i \in \mathbf{\Delta}^0_2(Y)$, $x_i \in A_i$, and $A_i \cap A_j = \emptyset$ whenever $j \neq i$ and $\alpha_j = \alpha_i$.

For each $i \in \omega$, let $\{V^i_j\}_{j\in\omega}$ be a decreasing sequence of open subsets of Y such that $\{x_i\} = Cl(\{x_i\}) \cap \bigcap_{j\in\omega} V^i_j$, and $x_k \notin V^i_j$ whenever $k \leq j$ and $x_i \notin Cl(\{x_k\})$.

Define $W_j = \bigcup_{i\in\omega} A_i \cap V^i_j$. Then $W = \bigcap_{j\in\omega} W_j$ is in $\mathbf{\Pi}^0_3(Y)$, and $X \subseteq W$ is clear from the construction.

Next, let $y \in W$ be fixed. The set of ordinals $\{\alpha_i \mid y \in A_i\}$ is non-empty, so let α be its minimal element. Then the $k \in \omega$ satisfying $\alpha_k = \alpha$ and $y \in A_k$ is uniquely determined.

Assume for a contradiction that there is $j \geq k$ and $i \neq k$ such that $y \in A_i \cap V^i_j$. Then $x_k \in V^i_j$ because V^i_j is an open set containing y and $y \in Cl(\{x_k\})$. Thus, $k \leq j$ together with our definition of V^i_j implies $x_i \in Cl(\{x_k\})$. We also have $x_i \in U_k$ because $y \in U_k$ and $y \in Cl(\{x_i\})$. Since $Cl(\{x_k\}) \cap U_k \cap X^{\alpha_k} =$

$\{x_k\}$, we must have $x_i \notin X^{\alpha_k}$. But then $y \in A_i$ and $\alpha_i < \alpha_k$, contradicting our choice of α.

Since $y \in \bigcap_{j\in\omega} W_j$, it follows that $y \in A_k \cap V_j^k$ for all $j \in \omega$. Our choice of V_j^k implies $y = x_k$. Since $y \in W$ was arbitrary, $W \subseteq X$.

Therefore, $X = W \in \mathbf{\Pi}_3^0(Y)$. As every countable subset of a countably based space is a $\mathbf{\Sigma}_3^0$-set, it follows that $X \in \mathbf{\Delta}_3^0(Y)$. □

Corollary 9. *If Y is a countably based T_0-space and $X \subseteq Y$ is a countable T_D-space, then $X \in \mathbf{\Delta}_3^0(Y)$.* □

The use of transfinite ordinals in the proof of Theorem 8 might seem excessive. However, the following example suggests that it is not avoidable.

Let $\omega^{<n}$ be the set of sequences of natural numbers of length less than n. Give $\omega^{<n}$ the topology generated by subbasic open sets of the form $B_\sigma = \omega^{<n} \setminus \{\sigma' \in \omega^{<n} \mid \sigma \preceq \sigma'\}$, where σ varies over elements of $\omega^{<n}$ and $\preceq$ is the prefix relation. The specialization order on $\omega^{<n}$ is simply $\succeq$. Then $\{\sigma\}$ is locally closed in $\omega^{<n}$ if and only if the length of σ equals $n-1$. Therefore, $\ell(\omega^{<n}) = n$. If we take X to be the disjoint union of the sequence of spaces $\{\omega^{<n}\}_{n\in\omega}$, then $\ell(X) = \omega$.

If Y is quasi-Polish, then the converse of Theorem 8 holds as well. The reader should consult [2] for background on the usage of quasi-metrics in the following proof.

Corollary 10. *Assume Y is quasi-Polish and $X \subseteq Y$ is countable. Then $X \in \mathbf{\Delta}_3^0(Y)$ if and only if for every non-empty $A \in \mathbf{\Pi}_2^0(X)$ there is a finite non-empty $F \in \mathbf{\Delta}_2^0(A)$.*

Proof: For the remaining half of the proof, if $X \in \mathbf{\Delta}_3^0(Y)$, then by Theorem 32 of [2] there is a quasi-metric d on X such that the induced metric space $(X, \widehat{d})$ is Polish. Since X is countable, $(X, \widehat{d})$ is scattered, hence for any $A \subseteq X$ there is $x \in A$ such that $\{x\} \in \mathbf{\Sigma}_1^0(A, \widehat{d})$. It follows that $\{x\}$ is $\mathbf{\Sigma}_2^0$ in the quasi-metric space (A, d), hence $\{x\} \in \mathbf{\Delta}_2^0(A, d)$ because singleton subsets of countably based spaces are $\mathbf{\Pi}_2^0$. □

A simple example of a countable space without non-empty finite $\mathbf{\Delta}_2^0$ subsets is the space $\mathbb{Q}$ of rational numbers with the upper interval topology (i.e., the topology generated by the sets $\uparrow q = \{x \in \mathbb{Q} \mid q \leq x\}$ for $q \in \mathbb{Q}$). Another example is the space $\omega^{<\omega}$ of all finite sequences of natural numbers with the topology generated by open sets of the form $\omega^{<\omega} \setminus \{\sigma' \in \omega^{<\omega} \mid \sigma \preceq \sigma'\}$, with σ varying over elements of $\omega^{<\omega}$. It follows from the results above that both of these spaces will be strictly $\mathbf{\Sigma}_3^0$ whenever they are embedded into a quasi-Polish space.

References

[1] C. E. Aull and W. J. Thron, *Separation axioms between T_0 and T_1*, Indag. Math. **24** (1963), 26–37.

[2] M. de Brecht, *Quasi-Polish spaces*, Annals of Pure and Applied Logic **164** (2013), no. 3, 356 – 381.

[3] M. de Brecht and A. Yamamoto, *Topological properties of concept spaces (full version)*, Information and Computation **208** (2010), no. 4, 327 – 340.

[4] A. Kechris, *Classical descriptive set theory*, Springer-Verlag, 1995.

[5] D. S. Scott, *Data types as lattices*, SIAM Journal on Computing **5** (1976), 522–587.

[6] V. Selivanov, *Index sets in the hyperarithmetical hierarchy*, Siberian Mathematical Journal **25** (1984), 474–488.

[7] ———, *Difference hierarchy in ϕ-spaces*, Algebra and Logic **43** (2004), 238–248.

[8] ———, *Towards a descriptive set theory for domain-like structures*, Theoretical Computer Science **365** (2006), 258–282.

[9] A. Tang, *Chain properties in $P(\omega)$*, Theoretical Computer Science **9** (1979), 153–172.

[10] ———, *Wadge reducibility and Hausdorff difference hierarchy in $P(\omega)$*, Lecture Notes in Mathematics **871** (1981), 360–371.