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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF $\Delta u+g(r)u+h(r)u^{p}=0$ AND ITS APPLICATIONS

横浜国立大学大学院工学研究院 塩路直樹 (NAOKI SHIOJI)
FACULTY OF ENGINEERING, YOKOHAMA NATIONAL UNIVERSITY

防衛大学校情報工学科 渡辺宏太郎 (KOHTARO WATANABE)
DEPARTMENT OF COMPUTER SCIENCE, NATIONAL DEFENSE ACADEMY

1. INTRODUCTION AND MAIN RESULTS

We consider the problem

$$(1.1)\qquad \begin{cases} u_{rr}+\dfrac{n-1}{r}u_{r}+g(r)u+h(r)u^{p}=0, & 0<r<R,\\ u(0)\in(0,\infty), \quad u(R)=0, \end{cases}$$

where $n\geq 2$, $R\in(0,\infty]$, $p\in(1,\infty)$ and $g,h:(0,R)\rightarrow\mathbb{R}$ are appropriate functions. Here, $u(R)=0$ in the case $R=\infty$ means $u(x)\rightarrow 0$ as $|x|\rightarrow\infty$. Such a problem has been studied by many researchers; see [1, 3, 5, 8, 9, 12–18, 20–27, 30, 32–36] and others.

In this note, we introduce a result obtained in [28].

Theorem 1. *Let $0<R\leq\infty$, $n\in\mathbb{R}$ with $n\geq 2$ and $p\in(1,\infty)$. Let $g\in C([0,R))\cap C^{1}((0,R))$ and $h\in C^{2}([0,R))\cap C^{3}((0,R))$ such that h is positive on $[0,R)$. with $R'=0$. Assume the following.*

(i) *In the case of $R<\infty$, $g\in C([0,R])$, $h\in C^{2}([0,R])$ and $h(R)>0$ are also satisfied.*
(ii) *There exists $\kappa\in[0,R]$ such that*

$$G(r)\geq 0 \text{ on } (0,\kappa) \quad\text{and}\quad G(r)\leq 0 \text{ on } (\kappa,R),$$

where

$$\begin{aligned} G(r)=\frac{r^{\frac{2(n-1)(p+1)}{p+3}-3}}{2(p+3)^{3}h(r)^{\frac{2}{p+3}+3}}\Big(&4(n-1)\big[n+2-(n-2)p\big]\big[n-4+(n-2)p\big]h(r)^{3}\\ &+\Big[2(n-1)(p-1)(p+3)^{2}r^{2}h(r)^{3}-4(p+3)^{2}r^{3}h(r)^{2}h_{r}(r)\Big]g(r)\\ &+(p+3)^{3}r^{3}g_{r}(r)h(r)^{3}\\ &+(n-1)\big[(2n-3)p(6-p)+6n-33\big]rh(r)^{2}h_{r}(r)\\ &+3(n-1)(p-1)(p+5)r^{2}h(r)h_{r}(r)^{2}-2(p+4)(p+5)r^{3}h_{r}(r)^{3}\end{aligned}$$

$$- 3(n-1)(p-1)(p+3)r^2h(r)^2h_{rr}(r)$$
$$+ 3(p+3)(p+5)r^3h(r)h_r(r)h_{rr}(r) - (p+3)^2r^3h(r)^2h_{rrr}(r)\Big).$$

(iii) *In the case of $R = \infty$, $G^- \not\equiv 0$ is satisfied.*

Then in the case of $R < \infty$, problem (1.1) has at most one positive solution, and in the case of $R = \infty$, problem (1.1) has at most one positive solution u which satisfies $J(r;u) \to 0$ as $r \to \infty$, where

$$a(r) = r^{\frac{2(n-1)(p+1)}{p+3}} h(r)^{\frac{-2}{p+3}},$$

$$b(r) = \frac{r^{\frac{2(n-1)(p+1)}{p+3}-1}}{(p+3)h(r)^{\frac{p+5}{p+3}}}\big(2(n-1)h(r) + rh_r(r)\big),$$

$$c(r) = \frac{r^{\frac{2(n-1)(p+1)}{p+3}-2}}{(p+3)^2h(r)^{\frac{2(p+4)}{p+3}}}\Big(2(n-1)\big[n+2-(n-2)p\big]h(r)^2 + (p+5)r^2h_r(r)^2$$
$$- (n-1)(p-5)rh(r)h_r(r) - (p+3)r^2h(r)h_{rr}(r)\Big),$$

$$J(r;u) = \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(r)u(r) + \frac{1}{2}c(r)u(r)^2$$
$$+ \frac{1}{2}a(r)g(r)u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1}.$$

Remark 1. In [32, Theorems 2.1 and 2.2], Yanagida obtained a closely related result.

By the theorem above, we can obtain the following; see [13, Theorem 0.1].

Corollary 1 (Kabeya-Tanaka). *Let $n \in \mathbb{N}$ with $n \geq 2$. Let $p > 1$ and $g \in C^2([0,\infty))$ such that $-\infty < \inf_{r\in[0,\infty)} g(r) \leq \sup_{r\in[0,\infty)} g(r) < 0$, and set*

$$L = \frac{2(n-1)[(n-2)p + n - 4]}{(p+3)^2} \quad \text{and} \quad \beta = \frac{2(n-1)(p-1)}{p+3}.$$

Assume that

$$g_r(r)r^3 + \beta g(r)r^2 - (\beta - 2)L < 0 \quad \text{for each } r \geq 0$$

in the case of $n = 2$, and that $p < (n+2)/(n-2)$ and

$$\sup_{r>0}\big(g_{rr}(r)r^2 + (3+\beta)g_r(r)r + 2\beta g(r)\big) < 0$$

in the case of $n \geq 3$. Then the problem

$$\text{(1.2)} \qquad u \in H^1(\mathbb{R}^n), \qquad \Delta u(x) + g(|x|)u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n$$

has a unique positive radial solution.

Next, we consider the problem

$$(1.3)\qquad \begin{cases} u_{rr}(r) + \dfrac{n-1}{r}u_r + g(r)u(r) + h(r)u(r)^p = 0, & R' < r < R, \\ u(R') = 0, \quad u(R) = 0. \end{cases}$$

The uniqueness of a positive solution of such a problem was studied in [4, 6, 7, 10, 11, 19, 24, 29–31].

The following is also obtained in [28].

Theorem 2. *Let* $0 < R' < R \le \infty$, $n \in \mathbb{R}$, $p \in (1,\infty)$, $g \in C([R',R)) \cap C^1((R',R))$, $h \in C^2([R',R)) \cap C^3((R',R))$ *such that* h *is positive on* $[R',R)$. *Let* a, b, c, G *and* J *be the functions given in Theorem 1. Assume the following.*

(i) *In the case of* $R < \infty$, $g \in C([R',R])$, $h \in C^2([R',R])$ *and* $h(R) > 0$ *are also satisfied.*

(ii) *There exists* $\kappa \in [R',R]$ *such that*

$$G(r) \ge 0 \text{ on } (R',\kappa) \quad \text{and} \quad G(r) \le 0 \text{ on } (\kappa, R).$$

Then in the case of $R < \infty$, *problem* (1.3) *has at most one positive solution, and in the case of* $R = \infty$, *problem* (1.3) *has at most one positive solution* u *which satisfies* $J(r;u) \to 0$ *as* $r \to \infty$.

Remark 2. For the case $h(r) \equiv 1$, a similar result is obtained by Felmer-Martínez-Tanaka; see [10, Theorem 1.1].

2. Applications

In this section, we give examples of Theorem 1. First, we give a comment on the scalar field equation

$$\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty.$$

The unique existence of its positive solution was established by Kwong [18]. Since the uniqueness of its positive solution can be derived from Corollary 1, of course, it can be also done by Theorem 1.

Next, we consider the following Brezis-Nirenberg problem.

$$(2.1)\qquad \begin{cases} \Delta_{S^n} u + \lambda u + u^p = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Here, n is a natural number with $n \ge 3$, S^n is the unit sphere in $\mathbb{R}^{n+1}$, Δ_{S^n} is the Laplace-Beltrami operator on S^n, $D = \{X \in S^n : X_{n+1} > \cos\theta_1\}$ with $\theta_1 \in (0,\pi)$,

$1 < p \le (n+2)/(n-2)$ and $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta_{S^n}$ on D with the Dirichlet boundary condition.

Let $P : S^n \setminus \{(0,\dots,0,-1)\} \to \mathbb{R}^n$ be the stereographic projection defined by

$$P(X_1,\dots,X_n,X_{n+1}) = \frac{1}{X_{n+1}+1}(X_1,\dots,X_n) \quad \text{for } X \in S^n \setminus \{(0,\dots,0,-1)\}.$$

Then we can see $P(D) = B_R$, where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ with

$$R = \frac{\sin\theta_1}{1+\cos\theta_1}.$$

Let u be a positive solution of (2.1) and define $v : \overline{B_R} \to \mathbb{R}$ by $u(P^{-1}x) = (1+|x|^2)^{\frac{n-2}{2}} v(x)$ for $x \in \overline{B_R}$. Then we see that v is a positive solution of

$$\begin{cases} \Delta v + \dfrac{n(n-2)+4\lambda}{(1+|x|^2)^2} v + 4(1+|x|^2)^{\frac{(n-2)p-(n+2)}{2}} v^p = 0 & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R. \end{cases}$$

We set

$$g(r) = \frac{n(n-2)+4\lambda}{(1+r^2)^2} \quad \text{and} \quad h(r) = 4(1+r^2)^{\frac{(n-2)p-(n+2)}{2}} \quad \text{for } r \ge 0.$$

We can see that G in Theorem 1 is given by

$$G(r) = \frac{2^{\frac{p-1}{p+3}}(n-1)}{(p+3)^3} r^{\frac{2(n-1)(p+1)}{p+3}-3} (1+r^2)^{\frac{n+2-(n-2)p}{p+3}-3} (1-r^2)(Ar^4 + Br^2 + A),$$

where

$$\begin{aligned} A &= (n-2)^2\left(\frac{n+2}{n-2} - p\right)\left(p + \frac{n-4}{n-2}\right) \\ &= (p+3)[3n^2 - 6n - (n^2 - 4n + 4)p] - 8(n-1)^2, \\ B &= (p+3)[-6n^2 + 12n + (2n^2 + 4\lambda - 4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n-1)^2. \end{aligned}$$

Then we can infer the following. For the details, see [28].

Theorem 3. *Let $n \in \mathbb{N}$ with $n \ge 3$, $1 < p \le (n+2)/(n-2)$ and $\theta_1 \in (0,\pi)$. Assume that one of the following conditions:*

(i) $\theta_1 \in (0, \pi/2]$ *and* $\lambda < \lambda_1$,

(ii) $\theta_1 \in (\pi/2, \pi)$ *and*

$$\frac{6+(6-4n)p}{(p+3)(p-1)} \le \lambda < \lambda_1.$$

Then (2.1) has at most one positive radial solution. Moreover, if $\lambda \ge -n(n-2)/4$ is also satisfied, then (2.1) has at most one positive solution.

Remark 3. It holds that

$$\frac{6+(6-4n)p}{(p+3)(p-1)} \leq -\frac{n(n-2)}{4},$$

and if $p = (n+2)/(n-2)$ then the constants in the both sides in the inequality above coincide.

Remark 4. In the case of $n = 3$, Bandle-Benguria obtained a sharper result. For the details, see [2].

Remark 5. In the case of $R > 1$, we cannot apply Yanagida's uniqueness theorem [32, Theorem 2.1]. Indeed, by his notation, we have

$$G(r; n-2) = \frac{2(4\lambda + n(n-2))r^{n-1}(1-r^2)}{(r^2+1)^3}.$$

So one of his assumptions $G(r; n-2) \leq 0$ on $(0, R)$ is not satisfied even if $\lambda > -n(n-2)/4$.

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