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UNPREDICTABILITY OF QUASI-PERIODIC DYNAMICAL SYSTEMS WITH FREQUENCY OF p-ADIC LIOUVILLE TYPE NUMBERS

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1. Introduction

There are only two types of complete extensions of the rationals, the real numbers and the p-adic numbers. The p-adic numbers were first introduced by Kurt Hensel in 1897 and during almost 100 years they are considered mainly objects of pure mathematics. From 1980's applications of p-adic numbers were started and proposed in mathematical physics and quantum mechanics. Now p-adic analysis has been studied in various fields to investigate extremely complex models, which have chaotic properties.

In this paper we introduce the definitions of p-adic numbers and a special type of Schneider's continued fractions for p-adic numbers ([2], [5]). Then we study a quasi-periodic dynamical system with its frequency, which has a p-adic Liouville type property. We investigate the recurrent properties of its orbits by using these p-adic continued fractions and we estimate the positive gap values of the recurrent dimensions, which measure the unpredictability level of the orbit (cf. [8]).

Our plan of this paper is as follows. In section 2 we give a brief introduction of p-adic numbers. In section 3 we introduce p-adic continued fractions. In Section 4 we treat approximation lattices of p-adic numbers and we define p-adic weak Liouville numbers. In Section 5 we study the quasi-periodic dynamical system and in section 6 we give some numerical results on exponents of p-adic weak Liouville numbers by using an open-source mathematics software SAGE.

2. p-adic numbers

Here we give a brief introduction to p-adic numbers. (For further details, see [1], [3].)

For a prime p the p-adic valuation v_p is the function

$$v_p : \mathbb{Z}\backslash\{0\} \to \mathbb{Z}$$

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defined as follows. For $n \in \mathbb{Z}$, let $v_p(n)$ be the unique non-negative v such that

$$n = p^v \cdot n', \quad \gcd(p, n') = 1.$$

For $x = a/b \in \mathbb{Q}\backslash\{0\}$, define

$$v_p(x) = v_p(a) - v_p(b).$$

Then we can easily show the following properties

$$\begin{aligned} &(1)\ v_p(xy) = v_p(x) + v_p(y) \\ &(2)\ v_p(x+y) \geq \min\{v_p(x), v_p(y)\} \end{aligned}$$

for all $x, y \in \mathbb{Q}\backslash\{0\}$.

For $x \in \mathbb{Q}$ the p-adic absolute value $|x|_p$ is defined as follows.

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then the function $x \to |x|_p$ satisfies the following conditions for all $x, y \in \mathbb{Q}$.

$$\begin{aligned} &\text{(i) } |x|_p = 0 \Longleftrightarrow x = 0 \\ &\text{(ii) } |xy|_p = |x|_p |y|_p \\ &\text{(iii) } |x+y|_p \leq \max\{|x|_p, |y|_p\}. \end{aligned}$$

Since $\max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$, the property (iii), called the strong triangle inequality, implies the triangle inequality

$$\text{(iv) } |x+y|_p \leq |x|_p + |y|_p.$$

The absolute value, which satisfies the condition (i), (ii), (iv), is said to be Archimedean and if (iv) is replaced by the stronger (iii), then the absolute value is said to be non-Archimedean. The property (iii) implies the following isosceles triangle principle.

$$|x|_p \neq |y|_p \Longrightarrow |x+y|_p = \max\{|x|_p, |y|_p\}$$

for all $x, y \in \mathbb{Q}$.

The property (iii) also implies the following equivalent relation for Cauchy sequences in $\mathbb{Q}$:

$$|x_n - x_m|_p \to 0 \text{ as } n, m \to \infty \Longleftrightarrow |x_n - x_{n+1}|_p \to 0 \text{ as } n \to \infty.$$

The completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$ is called the field of p-adic numbers, denoted by $\mathbb{Q}_p$. The p-adic numbers $x, y \in \mathbb{Q}_p$ satisfy the conditions (i), (ii), (iii) . We define the ring of p-adic integers by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

where $x \in \mathbb{Z}_p$ has the expansion in base p

$$x = \sum_{k=0}^{\infty} d_k p^k, \quad d_k \in \{0, 1, .., p-1\}.$$

3. p-ADIC CONTINUED FRACTIONS

First we introduce a special type of Schneider's continued fractions, which was given by Bugeaud in [2] to investigate p-adic extremal numbers.

Let $\{v_n\}$ be a sequence of positive integers. Then we consider the following p-adic continued fraction, which gives a p-adic number ω:

$$\omega = 1 + \cfrac{p^{v_1}}{1 + \cfrac{p^{v_2}}{1 + \cfrac{p^{v_3}}{1 + \ddots}}}.$$

We denote the continued fraction by

$$\omega = 1 + [p^{v_1}/1, p^{v_2}/1, p^{v_3}/1, ...].$$

On the other hand we can obtain the rational approximation ω_n of ω iteratively as follows.

Let $P_{-1} = 1, Q_{-1} = 0, P_0 = 1, Q_0 = 1,$

$$\begin{aligned} P_n &= P_{n-1} + p^{v_n} P_{n-2}, \\ Q_n &= Q_{n-1} + p^{v_n} Q_{n-2}, \quad n = 1, 2, ... \end{aligned}$$

Then we call (P_n, Q_n) the p-adic convergents pair of p-adic number ω. Then we have

$$\omega_n := P_n/Q_n = 1 + [p^{v_1}/1, p^{v_2}/1, ..., p^{v_n}/1]$$

with the following estimates

$$|\omega - P_n/Q_n|_p = p^{-(v_1+v_2+\cdots+v_{n+1})},$$

$$\max\{P_n, Q_n\} < p^{v_1+v_2+\cdots+v_n}.$$

We need the following Lemma for the inequality relation between P_n and Q_n.

Lemma 3.1. *Given $\omega \in \mathbb{Z}_p$, let (P_n, Q_n) be the p-adic convergents pair of ω. Then there exist positive constants c_1, c_2:*

$$c_1 Q_n \leq P_n \leq c_2 Q_n, \quad \forall n.$$

Proof. Using the p-adic continued fraction of ω, we have

$$c_1 = p^{-v_2}, \quad c_2 = p^{v_1}.$$

In fact,

$$P_1 = p^{v_1}, \ Q_1 = 1,$$

$$P_2 = p^{v_1}, \ Q_2 = 1 + p^{v_2}$$

and the induction argument completes the proof. □

4. Approximation Lattices

In this section we introduce approximation lattices of p-adic numbers given by B. de Weger in [11]. Let p be a prime number and $\alpha \in \mathbb{Z}_p$. The ordered pair of rational integers (P, Q) is called a p-adic approximations to α of order m if

$$|P - Q\alpha|_p = p^{-m}.$$

The set

$$\Gamma_m = \{(P, Q) \in \mathbb{Z}^2 : \ |P - Q\alpha|_p \leq p^{-m}\}$$

is called the mth approximation lattice of α.

The approximation lattices satisfy the following properties.

(i) Γ_m is a lattice in $\mathbb{Z}^2$ of rank 2.

(ii) $\mathbb{Z}^2 = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_m \supset \Gamma_{m+1} \supset \cdots$

(iii) A pair of points $\{(P, Q), (R, S)\}$ in Γ_m is a basis of Γ_m if and only if $|PS - QR| = p^m$.

We consider the norm in Γ_m:

$$\|(P, Q)\| = \max\{|P|, |Q|\}.$$

First we choose the minimal non-zero point (X_1, Y_1) in Γ_m and secondly we choose the minimal point (X_2, Y_2) which is independent of (X_1, Y_1). The pair of points $\{(X_1, Y_1), (X_2, Y_2)\}$ is called the norm reduced basis.

We can show the following theorems by applying the results by Weger.

Theorem 4.1. *If $(P, Q) \in \mathbb{Z}^2$ is the first minimal norm point, then it holds*

$$|P - Q\alpha|_p \leq \frac{1}{\|(P, Q)\|^2}.$$

Theorem 4.2. *If $(P, Q) \in \mathbb{Z}^2$ is not the first minimal norm point in any lattice, then it holds*

$$|P - Q\alpha|_p > \frac{1}{2\|(P, Q)\|^2}.$$

Constructing an algorithm to search the minimal norm point in a general lattice is an extremely difficult problem, called SVP (shortest vector problem), which has direct applications to cryptography. Here we can use the algorithm for constructing p-adic continued fractions to find these minimal norm points.

To find these minimal points efficiently we define the following restrictions of the lattices. We denote

$$\Gamma_m^{/c_1} = \{(P, Q) \in \Gamma_m : c_1 Q \leq P\},$$
$$\Gamma_m^{c_2/} = \{(P, Q) \in \Gamma_m : P \leq c_2 Q\},$$
$$\Gamma_m^{c_2/c_1} = \{(P, Q) \in \Gamma_m : c_1 Q \leq P \leq c_2 Q\}$$

for some constants $c_2 > c_1 > 0$.

In section 3 we can iteratively construct the convergents (P_n, Q_n), using p-adic continued fractions of ω:

$$|P_n - Q_n\omega|_p = p^{-(v_1 + \cdots + v_{n+1})},$$

$$\|(P_n, Q_n)\| < p^{v_1+\cdots+v_n}.$$

For a sequence of the first minimal norm points we can choose the subsequence of $\{(P_{m_j}, Q_{m_j})\}$ from the p-adic convergents pairs under some algebraic assumptions on p-adic numbers. In our previous papers ([7],[9]) we introduced a class of irrational numbers, which have well rational approximation properties.

In p-adic numbers case we also say that a p-adic number ω is a weak Liouville number with its order $\gamma > 0$ if there exists a subsequence $\{(P_{m_j}, Q_{m_j})\}$ which satisfies

$$|P_{m_j} - Q_{m_j}\omega|_p \le \frac{1}{\|(P_{m_j}, Q_{m_j})\|^{2+\gamma}}. \tag{4.1}$$

Assume that $v_{n+1} > v_1 + \cdots + v_n + 2c$ for a constant $c : 2p^{-2c} < 1$. Then we have

$$\begin{aligned} |P_n - Q_n\omega|_p &\le p^{-2(v_1+\cdots+v_n+c)} \\ &\le \frac{1}{2\|(P_n, Q_n)\|^2}. \end{aligned} \tag{4.2}$$

It follows from Theorem 4.2 that (P_n, Q_n) is the first minimal norm point in Γ_n.

If we assume that $v_{n+1} > (1+\gamma)(v_1 + \cdots + v_n)$ for $\gamma > 0$, then we have

$$\begin{aligned} |P_n - Q_n\omega|_p &\le p^{-(2+\gamma)(v_1+\cdots+v_n)} \\ &\le \frac{1}{\|(P_n, Q_n)\|^{2+\gamma}}. \end{aligned}$$

5. Quasi-Periodic Orbits

For a given $\omega \in \mathbb{Z}_p$ and $Q \in \mathbb{N}_0$ we denote

$$\|Q\omega\|_p = \min\{|P - Q\omega|_p : c_1 Q \le P \le c_2 Q,\ P \in \mathbb{N}_0\}$$

where c_1, c_2 are the constants in Lemma 3.1. We denote a minimal pair by (P_Q, Q):

$$|P_Q - Q|_p = \min\{|P - Q\omega|_p : c_1 Q \le P \le c_2 Q,\ P \in \mathbb{N}_0\}.$$

We consider the shift mapping f on $\mathbb{Z}_p$ by

$$f(x) = x + \omega, \quad x \in \mathbb{Z}_p.$$

We investigate some recurrent property of the orbit Σ_x given by f,

$$\Sigma_x = \{x, f(x), f^2(x), \cdots, f^n(x), \cdots\}$$

where $f^2(x) = f(f(x)), f^3(x) = f(f(f(x))), \cdots$. We have

$$f^n(x) = x + n\omega$$

and hence

$$\|f^n(x) - x\|_p = \|n\omega\|_p.$$

We estimate the recurrent properties of the q.p. orbits and the gap of recurrent dimensions by using the algebraic property, p-adic weak Liouville property, of the irrational frequency ω.

Definitions of recurrent dimensions:

Define the first ε-recurrent time by

$$M_\omega(\varepsilon) = \min\{m \in \mathbb{N} : \|f^m(x) - x\|_p = \|m\omega\|_p \leq \varepsilon\}.$$

The upper recurrent dimension is defined by

$$\overline{D}_\omega = \limsup_{\varepsilon \to 0} \frac{\log M_\omega(\varepsilon)}{-\log \varepsilon}$$

and the lower recurrent dimension is defined by

$$\underline{D}_\omega = \liminf_{\varepsilon \to 0} \frac{\log M_\omega(\varepsilon)}{-\log \varepsilon}.$$

Then we can define the gaps of recurrent dimensions by $G_\omega = \overline{D}_\omega - \underline{D}_\omega$.

If the gap values G_ω take positive values, we cannot exactly determine or predict the ε-recurrent time of the orbits:

$$M_\omega(\varepsilon) \in [\left(\frac{1}{\varepsilon}\right)^{\underline{D}_\omega}, \left(\frac{1}{\varepsilon}\right)^{\overline{D}_\omega}].$$

Thus we propose the value G_ω as the parameter, which measures the unpredictability level of the orbit (cf. [8]).

We obtain the following theorem.

Theorem 5.1. *Let ω be a weak Liouville number with its order $\gamma > 0$, which satisfies (4.1). Then we have*

$$\underline{D}_\omega \leq \frac{1}{2+\gamma}.$$

Proof. For the sequence $\{(P_{m_j}, Q_{m_j})\}$, which specifies the weak Liouville property of ω;

$$|P_{m_j} - Q_{m_j}\omega|_p \leq \frac{1}{\|(P_{m_j}, Q_{m_j})\|^{2+\gamma}},$$

it follows from Lemma 3.1 that we have

$$c_1 Q_{m_j} \leq \|(P_{m_j}, Q_{m_j})\| \leq c_2 Q_{m_j}.$$

It follows from the definitions that

$$\begin{aligned} \|f^{Q_{m_j}}x - x\|_p &= \|Q_{m_j}\omega\|_p \\ &\leq |P_{m_j} - Q_{m_j}\omega|_p \\ &\leq \frac{1}{\|(P_{m_j}, Q_{m_j})\|^{2+\gamma}}. \end{aligned}$$

Thus we have

$$\|f^{Q_{m_j}}x - x\|_p \leq \frac{1}{c_1^{2+\gamma} Q_{m_j}^{2+\gamma}} := \varepsilon_j.$$

Now we can estimate the lower recurrent dimension as follows.

$$
\begin{aligned}
\underline{D}_\omega &= \liminf_{\varepsilon\to 0} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&= \liminf_{j\to\infty} \inf_{\varepsilon_{j+1}\le\varepsilon\le\varepsilon_j} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&\le \liminf_{j\to\infty} \frac{\log M(\varepsilon_j)}{-\log\varepsilon_j} \\
&\le \lim_{j\to\infty} \frac{\log Q_{m_j}}{-\log c_1 + (2+\gamma)\log Q_{m_j}} \\
&= \frac{1}{2+\gamma}.
\end{aligned}
$$

□

For the upper estimate we assume some conditions on the sparse between (P_{m_j}, Q_{m_j}) and $(P_{m_{j-1}}, Q_{m_{j-1}})$.

Theorem 5.2. *Under the same hypotheses as Theorem 5.1 we assume that there exist a sequence of lattice points* $(P_{l_j}, Q_{l_j}) \in \Gamma_{l_j}^{c_2/c_1}$*:* $Q_{m_{j-1}} < Q_{l_j} < Q_{m_j}$, $\forall j$, *which are not the first minimal points in any lattice, and a sequence of constants* $K_j : 0 < K_j < 1$, *which satisfy*

$$
|P - Q\omega|_p \ge \frac{K_j}{\|(P_{l_j}, Q_{l_j})\|^2}
$$

for every $(P,Q) \in \Gamma_m^{c_2/c_1} : Q_{m_{j-1}} \le Q < Q_{l_j}$. *Then we have*

$$
\overline{D}_\omega \ge \frac{1}{2}.
$$

Consequently, we can estimate the positive gap value

$$
G_\omega \ge \frac{\gamma}{2(2+\gamma)}.
$$

Proof. It follows from Theorem 4.2 that we have

$$
\begin{aligned}
|P_{l_j} - Q_{l_j}\omega|_p &\ge \frac{K_j}{2\|(P_{l_j}, Q_{l_j})\|^2} \\
&\ge \frac{1}{cQ_{l_j}^2} := \varepsilon_j.
\end{aligned}
$$

For $Q \in \mathbb{N} : Q_{m_{j-1}} \le Q < Q_{l_j}$, take the minimal pair (P_Q, Q):

$$
|P_Q - Q\omega|_p = \min\{|P - Q\omega|_p : c_1 Q \le P \le c_2 Q,\ P \in \mathbb{N}_0\}.
$$

Then we have

$$
\begin{aligned}
|P_Q - Q\omega|_p &\ge \frac{K_j}{\|(P_{l_j}, Q_{l_j})\|^2} \\
&\ge \frac{1}{cQ_{l_j}^2} = \varepsilon_j.
\end{aligned}
$$

Now we can estimate the upper recurrent dimension.

$$\begin{aligned}
\overline{D}_\omega &= \limsup_{\varepsilon\to 0} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&= \limsup_{j\to\infty} \sup_{\varepsilon_{j+1}\le\varepsilon\le\varepsilon_j} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&\ge \limsup_{j\to\infty} \frac{\log M(\varepsilon_j)}{-\log\varepsilon_j} \\
&\ge \lim_{j\to\infty} \frac{\log Q_{l_j}}{-\log c + 2\log Q_{l_j}} = \frac{1}{2}.
\end{aligned}$$

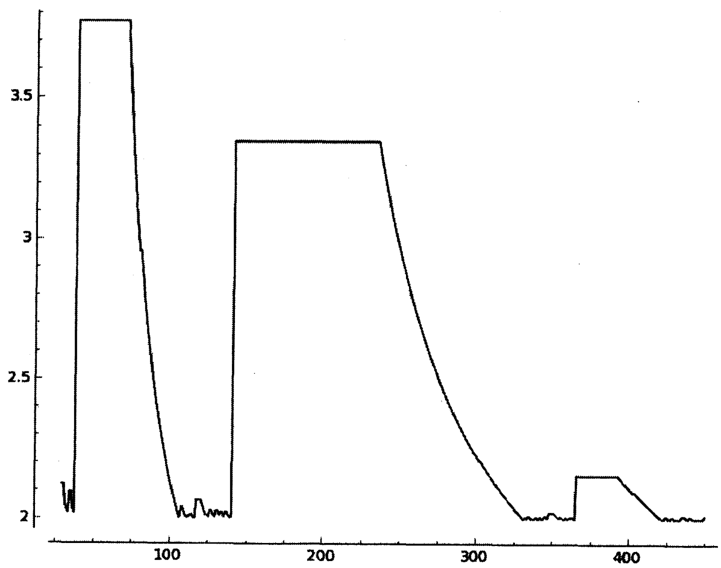
□

6. Numerical results

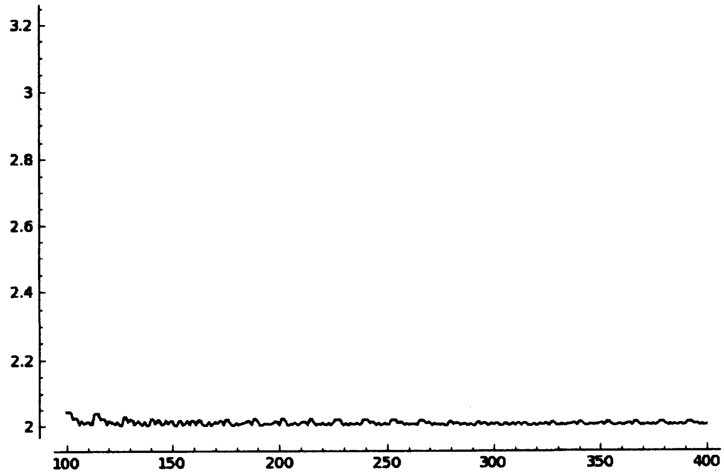
In this section we give some numerical results on exponents of p-adic weak Liouville numbers by using an open-source mathematics software SAGE. Here we apply the algorithm given by the continued fractions in section 3 and the Gaussian algorithm that finds the shortest vector in a two dimensional lattice. (For further details, see [6].) We examine the exponent values $2+\gamma$ of the weak Liouville numbers, which are numerically given by the values $\log p^m / \log \|(P_m, Q_m)\|$ where $\|(P_m, Q_m)\|$ is the norm of the shortest vector obtained by the Gaussian algorithm in each lattice Γ_m. However we have not yet obtained some numerical results which satisfy Hypotheses of Theorem 5.2.

Weak Liouvlle case: $p = 5, m = 30 \sim 450, n = 1 \sim 35$

$$v_n = [1,1,1,1,1,1,12,1,1,1,1,1,1,48,1,1,1,1,1,1,156,1,1,1,1,1,1,480,\\ 1,1,1,1,1,1,1452]$$



Quadratic irrational case: $\sqrt{29} \in \mathbb{Z}_p,\ p = 5, m = 100 \sim 400$ (cf.[6])



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