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Global Solutions for a Semilinear Heat Equation in the Exterior Domain of a Compact Set

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1 Introduction

We consider the Cauchy-Dirichlet problem for a semilinear heat equation,

$$\begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $p > 1$, $N \geq 3$, Ω is a smooth domain in $\mathbf{R}^N$, and $\phi \in L^\infty(\Omega)$. The problem (1.1) has been studied in many papers since the pioneering work due to Fujita [7], and it is well known that, for the case $\Omega = \mathbf{R}^N$,

- if $1 < p \leq 1 + 2/N$ and $\phi \not\equiv 0$ in Ω, then the solution u of (1.1) blows up at some time $T > 0$, that is,
$$\limsup_{t \to T-0} \|u(t)\|_{L^\infty(\Omega)} = \infty;$$
- if $p > 1 + 2/N$, then there exists a positive solution globally in time for some initial data ϕ.

These conclusions also hold for the case where Ω is the exterior domain of a compact set (see [1] and [21]). In this paper we assume that

(1.2) Ω is the exterior $C^{2,\alpha}$ domain of a compact set for some $\alpha \in (0,1)$,

(1.3) $p > 1 + 2/N, \quad (N-2)p < N + 2,$

(1.4) $\phi \in X := \left\{ f \in L^\infty(\Omega) \cap L^2(\Omega, e^{|x|^2/4}dx) \,:\, f \geq 0 \text{ in } \Omega \right\},$

and study the large time behavior of global in time solution u of (1.1). In particular, we give in Theorem 1.1 a sufficient condition for the solution u to behave like

$$(1.5)\qquad \|u(t)\|_{L^\infty(\Omega)} = O(t^{-1/(p-1)}) \quad \text{as} \quad t \to \infty,$$

and obtain in Theorem 1.2 and in Corollary 1.1 a classification of the decay rate of such a solution.

The large time behavior of global in time solutions of (1.1) has been studied in many papers and by various methods. It seems impossible to give a complete list of references for studies of this direction. We here only cite [15], [17], [18], [23], [26], and a survey [24], which includes a considerable list of references on this topic. Among others, in [17], Kavian studied the large time behavior of the global in time solution u of (1.1) for the case $\Omega = \mathbf{R}^N$ under the conditions (1.3) and (1.4). He put

$$(1.6)\qquad v(y,s) = (1+t)^{1/(p-1)}u(x,t), \quad y = (1+t)^{-1/2}x, \quad s = \log(1+t),$$

and introduced the following energy,

$$(1.7)\qquad E[v](s) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2 \rho dy - \frac{1}{2(p-1)}\int_{\mathbf{R}^N} v^2 \rho dy - \frac{1}{p+1}\int_{\mathbf{R}^N} v^{p+1}\rho dy,$$

where $\rho(y) = \exp(|y|^2/4)$. Then the energy $E[v](s)$ is monotone decreasing in the variable s. By using the energy method together with this monotonicity of the energy $E[v](s)$, he proved that

$$(1.8)\qquad \sup_{s>0} \|v(s)\|_{L^\infty(\mathbf{R}^N)} < \infty, \quad \text{that is,} \quad \|u(t)\|_{L^\infty(\mathbf{R}^N)} = O(t^{-1/(p-1)}) \quad \text{as} \quad t \to \infty.$$

Furthermore, in [18], Kawanago gave a classification of the large time behavior of the global in time solutions of (1.1) under the same conditions as in [17] by using the blow-up argument together with the energy method, see e.g. [10, 17]. In particular, he proved that, for any $\varphi \in X \setminus \{0\}$, there exists a positive constant λ_φ such that

$$(1.9)\qquad \begin{cases} \text{(a)} & \text{if } 0 < \lambda < \lambda_\varphi, \text{ then the solution } u \text{ of (1.1) exists globally in time and} \\ & \|u(t)\|_{L^\infty(\mathbf{R}^N)} \asymp t^{-\frac{N}{2}} \text{ as } t \to \infty; \\ \text{(b)} & \text{if } \lambda = \lambda_\varphi, \text{ then the solution } u \text{ of (1.1) exists globally in time and} \\ & \|u(t)\|_{L^\infty(\mathbf{R}^N)} \asymp t^{-\frac{1}{p-1}} \text{ as } t \to \infty; \\ \text{(c)} & \text{if } \lambda > \lambda_\varphi, \text{ then the solution } u \text{ of (1.1) does not exist globally in time,} \\ & \text{and blows-up at some time } T_M > 0, \text{ that is, } \limsup_{t \to T_M - 0} \|u(t)\|_{L^\infty(\mathbf{R}^N)} = \infty. \end{cases}$$

On the other hand, for any uniformly $C^{2,\alpha}$ smooth domain Ω in $\mathbf{R}^N$ with $0 < \alpha < 1$, Takaichi in [26] considered the problem (1.1) under the condition (1.3), and proved that the global solution u of (1.1) satisfies the inequality

$$(1.10)\qquad \sup_{t>0} \|u(t)\|_{L^\infty(\Omega)} \le C,$$

where C is a constant depending only on N, p, Ω, $\|\phi\|_{L^\infty(\Omega)}$, and $\|\phi\|_{L^2(\Omega)}$. Unfortunately, in this case, it seems difficult to prove the estimate like (1.8) and the classification like (1.9) by applying the arguments in [17] and [18] directly, since the energy associated with the rescaled solution v is not necessarily monotone decreasing in the variable s even when Ω is the exterior domain of a compact set.

In this paper we study the large time behavior of global in time solutions of (1.1) when Ω is the exterior domain of a compact set. In order to state our main results, we need to prepare some notation. For any nonnegative functions $f(t)$ and $g(t)$ in $(0,\infty)$, we say $f(t) \asymp g(t)$ as $t \to \infty$ if there exists a positive constant C such that $C^{-1}f(t) \le g(t) \le Cf(t)$ for all sufficiently large t. Let

$$\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}, \qquad |||\cdot||| := \|\cdot\|_\infty + \|\cdot\|_{L^2(\Omega, e^{|x|^2/4}dx)},$$

where $q \in [1,\infty]$. Then X is a closed cone of the Banach space with the norm $|||\cdot|||$. We denote by $S(t)\phi$ the solution of (1.1), and put

$$\begin{aligned} G &:= \{\phi \in X \,:\, S(t)\phi \text{ exists globally in time}\},\\ H &:= \{\phi \in G \,:\, \|S(t)\phi\|_\infty \asymp t^{-N/2} \text{ as } t \to \infty\} \cup \{0\},\\ K &:= \{\phi \in G \,:\, \|S(t)\phi\|_\infty \asymp t^{-1/(p-1)} \text{ as } t \to \infty\}. \end{aligned}$$

Now we are ready to give the main results of this paper. The first theorem gives a sufficient condition for the solution of (1.1) to behave like (1.5).

Theorem 1.1 *Let $N \ge 3$ and u be a global in time solution of* (1.1) *under the conditions* (1.2)–(1.4). *If there exist a positive constant δ and a point $x_0 \in \Omega$ such that*

$$\limsup_{t\to\infty} t^\delta u(x_0,t) < \infty, \tag{1.11}$$

then there exists a constant C such that

$$\|u(t)\|_\infty \le C(1+t)^{-1/(p-1)}, \qquad t > 0. \tag{1.12}$$

Put

$$M := \Big\{\phi \in G \,:\, \|S(t)\phi\|_\infty = O(t^{-1/(p-1)}) \text{ as } t \to \infty\Big\}. \tag{1.13}$$

Then Theorem 1.1 yields

$$M = \{\phi \in G \,:\, S(t)\phi \text{ satisfies (1.11) for some } x_0 \in \Omega \text{ and } \delta > 0\}.$$

At this stage, we have no precise information concerning the relationship among M, K, and H. The following theorem clarifies this point:

Theorem 1.2 *Let $N \ge 3$ and assume the conditions* (1.2)–(1.4). *Then there holds the following:*

(i) *$M = K \cup H$;*
(ii) *H is an unbounded convex open cone with vertex at 0 in X and $H = \mathrm{Int}\, M$;*
(iii) *if $\phi \in K$, then*

$$\lambda\phi \in H \quad \text{if} \quad 0 < \lambda < 1, \qquad \lambda\phi \notin M \quad \text{if} \quad \lambda > 1. \tag{1.14}$$

Combining these theorems with the estimate (1.10) (see also Proposition 2.1), we have

Corollary 1.1 *Let $N \geq 3$, $\phi \in G$, and u be a global in time solution of* (1.1) *under the conditions* (1.2)-(1.4). *Then there holds either*

(i) $\|u(t)\|_\infty \asymp t^{-N/2}$ *as* $t \to \infty$,

(ii) $\|u(t)\|_\infty \asymp t^{-1/(p-1)}$ *as* $t \to \infty$, *or*

(iii) $\sup_{t>0} \|u(t)\|_\infty < \infty$ *and* $\sup_{t>0} t^\delta u(x,t) = \infty$ *for any* $x \in \Omega$ *and* $\delta > 0$.

Remark 1.1 *We cannot exclude the case* (iii) *of Corollary* 1.1 *in general. In fact, a global in time solution $S(t)\phi$ which tends to a positive stationary solution of* (1.1) *as $t \to \infty$ is an example which satisfies the condition* (iii). *Cazenave-Lions proved in* [4] *that, for some $\phi \in G$, such a solution actually exists if Ω is a bounded domain. As for the nonexistence of nontrivial stationary solutions for* (1.1) *in unbounded domains, see e.g.* [2], [3], *and* [6] *and references therein.*

If Ω is the exterior domain of a starshaped compact set, then we can obtain more precise result on the relationship among M, K, and H.

Theorem 1.3 *Let $N \geq 3$ and Ω be an exterior $C^{2,\alpha}$ domain of a starshaped compact set in $\mathbf{R}^N$ for $\alpha \in (0,1)$. Assume the condition* (1.3). *Then $G = M$ and G is a closed convex set in X. Furthermore there holds the following:*

(i) *H is an unbounded convex open cone with vertex at 0 in X;*

(ii) *$G = K \cup H$, $\partial G = K$, and $\mathrm{Int}\, G = H$;*

(iii) *for any $\phi \in X \setminus \{0\}$, there exists a constant $\lambda_\phi \in (0,\infty)$ such that*

$$\lambda\phi \in H \quad \text{if} \quad 0 < \lambda < \lambda_\phi, \qquad \lambda\phi \in K \quad \text{if} \quad \lambda = \lambda_\phi, \qquad \lambda\phi \notin G \quad \text{if} \quad \lambda > \lambda_\phi.$$

Furthermore the unit sphere S in X and ∂G are homeomorphic by the map $S \ni \phi \to \lambda_\phi \phi \in \partial G$.

Remark 1.2 *Suppose that Ω is an exterior domain of a starshaped compact set and that $u(\in H^2_{\mathrm{loc}}(\Omega) \cap L^{p+1}(\Omega))$ is a stationary solution of* (1.1). *Then we have the Pohožaev identity (see* [22] *and see also* [27, *Theorem* B.3]);

$$\frac{1}{2}\int_{\partial\Omega} (x \cdot \nu)|\nabla u|^2 d\sigma = \left(\frac{N}{p+1} - \frac{N-2}{2}\right) \|\nabla u\|_2^2, \tag{1.15}$$

where ν is the outer unit normal vector to $\partial\Omega$. Since $x \cdot \nu \leq 0$ on $\partial\Omega$ and $p+1 < 2N/(N-2)$, (1.15) *yields $u = 0$. Thus there exist no positive stationary solutions (in $H^2_{\mathrm{loc}}(\Omega) \cap L^{p+1}(\Omega)$) of* (1.1) *in this case. On the other hand, under the same assumption on Ω, by Theorem* 1.3, *we see that $G = K \cup H$. These facts suggest that if* (1.1) *(with an exterior Ω) admits no positive stationary solutions (in $H^2_{\mathrm{loc}}(\Omega) \cap L^{p+1}(\Omega)$), then $G = K \cup H$, that is, there exist no global in time solutions satisfying Corollary* 1.1-(iii).

Now let us explain the idea for the proof of the results above. Let $\phi \in G$ and $\kappa \in (0, 1/(p-1)]$. Put

$$z(y,s) = (1+t)^{\kappa}[S(t)\phi](x), \quad y = (1+t)^{-1/2}x, \quad s = \log(1+t), \tag{1.16}$$

and

$$\Omega(s) := e^{-s/2}\Omega, \qquad W := \bigcup_{s>0}(\Omega(s)\times\{s\}), \qquad \partial W := \bigcup_{s>0}(\partial\Omega(s)\times\{s\}).$$

Then z satisfies

$$\begin{cases} \partial_s z = \dfrac{1}{\rho}\mathrm{div}\,(\rho\nabla_y z) + \kappa z + e^{Ks}z^p & \text{in} \quad W, \\ z = 0 & \text{on} \quad \partial W, \\ z(y,0) = \phi(y) \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{1.17}$$

where $K = -\kappa(p-1)+1(\geq 0)$ and $\rho(y) = e^{|y|^2/4}$. Multiplying z to (1.17) and integrating over the domain $\Omega(s)$, we have the energy inequality

$$\frac{d}{ds}F_\kappa(s) \leq -\int_{\Omega(s)} |\partial_s z|^2 \rho dy \tag{1.18}$$

(see Lemma 2.1). Here F_κ is the modified energy defined by

$$F_\kappa(s) := E_\kappa(s) + \frac{1}{4}\Lambda_\kappa(s) \tag{1.19}$$

with

$$E_\kappa(s) := \frac{1}{2}\int_{\Omega(s)}|\nabla z|^2\rho dy - \frac{\kappa}{2}\int_{\Omega(s)} z^2\rho dy - \frac{e^{Ks}}{p+1}\int_{\Omega(s)} z^{p+1}\rho dy, \tag{1.20}$$

$$\Lambda_\kappa(s) := \int_s^\infty \int_{\partial\Omega(s)} (y\cdot\nu(s))_+ |\partial_{\nu(s)}z(\tau)|^2 \rho d\sigma d\tau, \tag{1.21}$$

where $\nu(s)$ is the outer unit normal vector to $\partial\Omega(s)$ and $_+$ denotes the nonnegative part. Observe that $F_\kappa(s)$ is monotone decreasing in the variable s by virtue of (1.18). On the other hand, with the aid of (1.11) and the interior and the boundary Harnack inequalities for parabolic equations, we can prove

$$\Lambda_\kappa(s) < \infty, \qquad s > 0, \tag{1.22}$$

for some $\kappa \in (0, 1/(p-1)]$ (see Lemma 3.2). Then, by combining the decreasing property of $F_\kappa(s)$ and bounds (1.22) together with the energy method as in [17], we obtain estimates of $\|z(s)\|_{L^2(\Omega(s),\rho dy)}$ and $\|\partial_s z(s)\|_{L^2(\Omega(s),\rho dy)}$ (see Lemma 2.2). By these estimates together with the blow-up argument which is a modification of that in [16] and [10] (see Lemma 3.1 and Remark 3.1), we have a priori bounds for $\|z(s)\|_\infty$, which lead to

$$\|u(t)\|_\infty = O(\max\{t^{-\beta\delta}, t^{-1/(p-1)}\}) \quad \text{as} \quad t\to\infty$$

for some $\beta > 1$ (see Lemma 3.2). Repeating this argument n-times, we obtain

$$\|u(t)\|_\infty = O(\max\{t^{-\beta^n\delta}, t^{-1/(p-1)}\}) = O(t^{-1/(p-1)}) \quad \text{as} \quad t \to \infty$$

for large n, which completes the proof of Theorem 1.1. Furthermore, if the solution u satisfies the asymptotics (1.12) of Theorem 1.1, then we can show that $\Lambda_\kappa(s) < \infty$ with $\kappa = 1/(p-1)$ for $s > 0$. This enables us to define the energy $F_\kappa(s)$ with $\kappa = 1/(p-1)$. By taking advantage of the monotonicity of the energy $F_\kappa(s)$ with $\kappa = 1/(p-1)$, we can apply the similar argument as in [18] with some modifications, and prove Theorems 1.2 and 1.3.

In the rest of this paper we give only the proof of Theorem 1.1. In Section 2 we introduce preliminary facts and give global bounds of the approximate solutions by using the energy $F_\kappa(s)$. In Section 3 we improve the arguments in [10] and [16], and prove Theorem 1.1 by using the global bounds obtained in Section 2.

2 Global bounds for the global in time solutions

In this section we give some global bounds of the global in time solutions of (1.1). We first recall the result of [26], which gives L^∞-global bounds of solutions of (1.1).

Proposition 2.1 *Let Ω be a uniformly $C^{2,\alpha}$ smooth domain Ω in $\mathbf{R}^N$ for some $\alpha \in (0,1)$. Let $\phi \in L^2(\Omega) \cap L^\infty(\Omega)$ and u be a global in time solution of* (1.1) *under the condition* (1.3)*. Then there exists a constant C such that*

$$\sup_{t>0} \|u(t)\|_\infty \le C, \tag{2.1}$$

where C depends only on N, Ω, p, $\|\phi\|_\infty$, and $\|\phi\|_2$.

Next we assume the boundedness of $\Lambda_\kappa(s)$ for some $\kappa \in (0, 1/(p-1)]$, and prove the monotonicity of the energy $F_\kappa(s)$.

Lemma 2.1 *Assume the conditions* (1.2)-(1.4) *and $\phi \in G$. Let $\kappa \in (0, 1/(p-1)]$ and z be a function defined by* (1.16)*. If $\Lambda_\kappa(s_0) < \infty$ for some $s_0 > 0$, then there holds*

$$\frac{d}{ds} F_\kappa(s) \le -\int_{\Omega(s)} |(\partial_s z)(y,s)|^2 \rho dy \le 0, \qquad s \ge s_0. \tag{2.2}$$

In particular,

$$F_\kappa(s) - F_\kappa(s_0) \le -\int_{s_0}^s \int_{\Omega(\tau)} |(\partial_\tau z)(y,\tau)|^2 \rho dy d\tau \le 0, \qquad s \ge s_0. \tag{2.3}$$

Proof. Since

$$\partial_s z = \frac{y}{2} \cdot \nabla z = \frac{y \cdot \nu}{2} \partial_\nu z \qquad \text{on} \qquad \partial W,$$

we have

$$\frac{d}{ds}\int_{\Omega(s)}|\nabla z|^2\rho dy = -\frac{d}{ds}\int_{\Omega(s)} z\,\mathrm{div}\,(\rho\nabla z)dy$$
$$= -\int_{\Omega(s)}\partial_s z\,\mathrm{div}\,(\rho\nabla z)dy - \int_{\Omega(s)} z\,\mathrm{div}\,(\rho\nabla\partial_s z)dy$$
$$= -\int_{\Omega(s)}\partial_s z\,\mathrm{div}\,(\rho\nabla z)dy + \int_{\Omega(s)}\nabla z\cdot\nabla\partial_s z\rho dy$$
$$= \frac{1}{2}\int_{\partial\Omega(s)}(y\cdot\nu)|\partial_\nu z|^2\rho d\sigma - 2\int_{\Omega(s)}\partial_s z\,\mathrm{div}\,(\rho\nabla z)dy.$$

Then, by $K\geq 0$, (1.17), and (1.20), we have

$$\begin{aligned}\frac{d}{ds}E_\kappa(s) &\leq \frac{1}{2}\frac{d}{ds}\int_{\Omega(s)}|\nabla z|^2\rho dy - \kappa\int_{\Omega(s)} z\partial_s z\rho dy - e^{Ks}\int_{\Omega(s)} z^p\partial_s z\rho dy\\ &\leq \frac{1}{4}\int_{\partial\Omega(s)}(y\cdot\nu)|\partial_\nu z|^2\rho d\sigma - \int_{\Omega(s)}|\partial_s z|^2\rho dy\\ &\leq -\frac{1}{4}\frac{d}{ds}\Lambda_\kappa(s) - \int_{\Omega(s)}|\partial_s z|^2\rho dy\end{aligned}$$

for all $s\geq s_0$. This inequality together with (1.19) implies the inequalities (2.2) and (2.3), and the proof of Lemma 2.1 is complete. □

Then we obtain global bounds for the function z by using the monotonicity of $F_\kappa(s)$:

Lemma 2.2 *Assume the same conditions as in Lemma* 2.1. *Then there holds*

$$F_\kappa(s) > 0, \qquad s\geq s_0. \tag{2.4}$$

Furthermore there exists a constant C such that

$$\sup_{s\geq s_0}\int_{\Omega(s)}|z(s)|^2\rho dy \leq CF_\kappa(s_0) < \infty, \tag{2.5}$$

$$\int_{s_0}^\infty\int_{\Omega(s)}|(\partial_s z)(y,s)|^2\rho dyds \leq CF_\kappa(s_0) < \infty. \tag{2.6}$$

Proof. Put

$$f(s) = \frac{1}{2}\int_0^s \|z(\tau)\|^2_{L^2(\Omega(\tau),\rho dy)}d\tau.$$

We apply Proposition 2.3 in [5] to the zero extension of z, and have

$$\int_{\Omega(s)}|\nabla z(s)|^2\rho dy \geq \frac{N}{2}\int_{\Omega(s)}|z(s)|^2\rho dy, \quad s>0.$$

By Lemma 2.1 and (1.17), we obtain

$$
\begin{aligned}
f'(s) &= \frac{1}{2}\|z(s)\|^2_{L^2(\Omega(s),\rho dy)} = \frac{1}{2}\int_{\Omega(s)} |z|^2 \rho dy,\\
f''(s) &= \int_{\Omega(s)} z\partial_s z \rho dy = \int_{\Omega(s)} \left(-|\nabla z|^2 + \kappa z^2\right)\rho dy + e^{Ks}\int_{\Omega(s)} z^{p+1}\rho dy\\
&= -(p+1)E_\kappa(s) + \frac{p-1}{2}\int_{\Omega(s)} \left[|\nabla z|^2 - \kappa|z|^2\right]\rho dy\\
&\geq -(p+1)F_\kappa(s) + \frac{p-1}{2}\left(\frac{N}{2} - \frac{1}{p-1}\right) f'(s),
\end{aligned}
$$

for all $s \geq s_0$. Then we can apply the same arguments as in [17, Lemma 2.3, Proposition 3.1], and obtain (2.4)–(2.6). □

By following (1.6), we introduce a function

$$
w(y,s) = (1+t)^{1/(p-1)}u(x,t), \quad y = (1+t)^{-1/2}x, \quad s = \log(1+t). \tag{2.7}
$$

Then w satisfies

$$
\begin{cases}
\partial_s w = \dfrac{1}{\rho}\operatorname{div}(\rho\nabla_y w) + \dfrac{1}{p-1}w + w^p & \text{in} \quad W,\\
w = 0 & \text{on} \quad \partial W,\\
w(y,0) = \phi(y) \geq 0 & \text{in} \quad \Omega.
\end{cases} \tag{2.8}
$$

Since $w(y,s) = e^{\kappa' s}z(y,s)$ with $\kappa' = -\kappa + 1/(p-1) \geq 0$, Lemma 2.2 yields;

Lemma 2.3 *Assume the same conditions as in Lemma 2.1. Let w be a function defined by (2.7). Then there exists a constant C such that*

$$
\int_{\Omega(s)} |w(s)|^2 \rho dy \leq Ce^{2\kappa' s}F_\kappa(s_0), \tag{2.9}
$$

$$
\int_{s_0}^{s}\int_{\Omega(s)} |(\partial_s w)(y,s)|^2 \rho dy d\tau \leq Ce^{2\kappa' s}F_\kappa(s_0), \tag{2.10}
$$

for all $s \geq s_0$, where $\kappa' = -\kappa + 1/(p-1) \geq 0$.

3 Proof of Theorem 1.1

In this section we obtain L^∞ estimates of the global in time solution of (1.1) satisfying (1.11), and prove Theorem 1.1. We first prove the following lemma, which is proved by the modification of the arguments in [10] and [16] (see also Remark 3.1). In what follows, we write $\|\cdot\| = \|\cdot\|_{L^2(\Omega(s),\rho dy)}$ (see (1.19) and (1.21)) for simplicity.

Lemma 3.1 *Assume the conditions* (1.2)–(1.4) *and* $\phi \in G$. *Let* w *be a function defined by* (2.7). *Let* $0 \le s_0 < s_1 \le S$ *be numbers satisfying*

$$\sup_{s_1<s<S} \|w\|_{L^\infty(\Omega(s)\times\{s\})} = \sup_{s_0<s<S} \|w\|_{L^\infty(\Omega(s)\times\{s\})}. \tag{3.1}$$

Assume that there exists a constant $l > 1$ *such that*

$$\int_{s_0}^{S} \|\partial_s w\|_2^2 ds \le l < \infty, \tag{3.2}$$

$$\sup_{s_0<s<S} \|w(s)\|^2 \le l < \infty. \tag{3.3}$$

Then there exists a constant A, *independent of* w, S, *and* l, *which satisfies*

$$\sup_{s_0<s<S} \|w\|_{L^\infty(\Omega(s)\times\{s\})} \le Al^\alpha, \tag{3.4}$$

where $\alpha = 2/(\sigma(p-1))$ *and* $\sigma = 4p/(p-1) - (N+2) > 0$.

Proof. The proof is by contradiction. We assume that there exist sequences $\{w_n\}$ of solutions of (2.8), $\{l_n\} \subset (1,\infty)$, and $\{S_n\} \subset (s_1,\infty)$ such that

$$\int_{s_0}^{S_n} \|\partial_s w_n\|_2^2 ds \le l_n, \tag{3.5}$$

$$\sup_{s_0<s<S_n} \|w_n(s)\|^2 \le l_n, \tag{3.6}$$

$$\sup_{s_1<s<S_n} \|w_n\|_{L^\infty(\Omega(s)\times\{s\})} = \sup_{s_0<s<S_n} \|w_n\|_{L^\infty(\Omega(s)\times\{s\})}, \tag{3.7}$$

$$\lim_{n\to\infty} l_n^{-\alpha} \sup_{s_0<s<S_n} \|w_n\|_{L^\infty(\Omega(s)\times\{s\})} = \infty. \tag{3.8}$$

Now take $(y_n, s_n) \subset \bigcup_{s_1<s<S_n} (\Omega(s) \times \{s\})$ with

$$w_n(y_n, s_n) \ge \frac{1}{2} \sup_{s_0<s<S_n} \|w_n\|_{L^\infty(\Omega(s)\times\{s\})}. \tag{3.9}$$

Let λ_n be a constant such that

$$\lambda_n^{2/(p-1)} w_n(y_n, s_n) = 1. \tag{3.10}$$

Then, by (3.8)–(3.10), we have

$$\lim_{n\to\infty} l_n^{\alpha(p-1)} \lambda_n^2 = 0. \tag{3.11}$$

It is easily observed from (3.11) and $l_n > 1$ that

$$\lim_{n\to\infty} \lambda_n = 0. \tag{3.12}$$

Put $d_n = \mathrm{dist}(y_n, \partial\Omega(s_n))$. From now on, we consider the following three cases,

$$
\begin{aligned}
&(A) \qquad \sup_{n\geq 1} |\lambda_n^{1/2} y_n| < \infty \quad \text{and} \quad \sup_{n\geq 1} |d_n/\lambda_n| = \infty,\\
&(B) \qquad \sup_{n\geq 1} |\lambda_n^{1/2} y_n| < \infty \quad \text{and} \quad \sup_{n\geq 1} |d_n/\lambda_n| < \infty,\\
&(C) \qquad \sup_{n\geq 1} |\lambda_n^{1/2} y_n| = \infty.
\end{aligned}
$$

<u>**Case (A)**</u> Taking a subsequence if necessary, we can assume, without loss of generality, that

$$
\lim_{n\to\infty} |d_n/\lambda_n| = \infty. \tag{3.13}
$$

Put

$$
\tilde{w}_n(y,s) = \lambda_n^{2/(p-1)} w_n(y_n + \lambda_n y, s_n + \lambda_n^2 s) \quad \text{for} \quad (y,s) \in Q_n,
$$

where

$$
Q_n = \bigcup_{s\in I_n} (\Omega_n(s) \times \{s\}), \quad \Omega_n(s) = \lambda_n^{-1}(\Omega(s) - y_n), \quad I_n = (-(s_n - s_0)/\lambda_n^2, (S_n - s_n)/\lambda_n^2).
$$

Then, by (3.9) and (3.10), we have

$$
\tilde{w}_n(0,0) = 1, \tag{3.14}
$$

$$
\|\tilde{w}_n\|_{L^\infty(Q_n)} = \lambda_n^{2/(p-1)} \sup_{s_0<s<S_n} \|w_n(s)\|_{L^\infty(\Omega(s))} \leq 2. \tag{3.15}
$$

Furthermore $\tilde{w}_n$ satisfies

$$
\partial_s \tilde{w}_n = \Delta \tilde{w}_n + \lambda_n \frac{y_n + \lambda_n y}{2} \cdot \nabla_y \tilde{w}_n + \frac{\lambda_n^2}{p-1} \tilde{w}_n + \tilde{w}_n^p \quad \text{in} \quad Q_n. \tag{3.16}
$$

Let K be a compact set on $\mathbf{R}^N \times (-\infty, 0]$. Since $s_n - s_0 \geq s_1 - s_0 > 0$, by (3.12) and (3.13), we see that

$$
K \subset Q_n
$$

for sufficiently large n. Then, by (A), (3.12), and (3.15), we can apply the interior Schauder estimates to $\tilde{w}_n$, and see that there exists a constant $\beta \in (0,1)$ such that

$$
\sup_{n\in\mathbf{N}} \|\tilde{w}_n\|_{C^{2+\beta,1+\beta/2}(K)} < \infty.
$$

Therefore, by the Ascoli-Arzelá theorem, the diagonal argument, and (3.14), we see that there exist a subsequence $\{\tilde{w}_n'\}$ of $\{\tilde{w}_n\}$ and a nonnegative function $\tilde{w}$ in $\mathbf{R}^N \times (-\infty, 0]$ such that

$$
\lim_{n\to\infty} \|\tilde{w}_n' - \tilde{w}\|_{C^{2+\beta,1+\beta/2}(K)} = 0 \tag{3.17}
$$

for any compact subset K of $\mathbf{R}^N \times (-\infty, 0]$ and

$$\tilde{w}(0,0) = 1. \tag{3.18}$$

Furthermore, by (3.5) and (3.11), we have

$$\int_{-\lambda_n^{-2}(s_n - s_0)}^{0} \int_{\Omega_n(s)} |\partial_s \tilde{w}_n|^2 dyds = \lambda_n^\sigma \int_{s_0}^{s_n} \int_{\Omega(s)} |\partial_s w_n|^2 dyds$$
$$\leq \lambda_n^\sigma \int_{s_0}^{S_n} \|\partial_s w_n(s)\|^2 ds \leq l_n \lambda_n^\sigma = o(l_n^{1-\alpha\sigma(p-1)/2}) \to 0$$

as $n \to \infty$, and see that

$$(\partial_s \tilde{w})(y,s) = 0 \quad \text{in} \quad \mathbf{R}^N \times (-\infty, 0]. \tag{3.19}$$

Therefore $\tilde{w}$ is independent of the variable s, and $\tilde{w} = \tilde{w}(y)$ satisfies

$$\tilde{w} \geq 0 \quad \text{and} \quad \Delta \tilde{w} + \tilde{w}^p = 0 \quad \text{in} \quad \mathbf{R}^N$$

in view of (A), (3.12), (3.16), (3.17), and (3.19). Then the nonexistence result in [8] yields $\tilde{w} \equiv 0$ in $\mathbf{R}^N$, which contradicts (3.18).

Case (B) Taking a subsequence if necessary, we can assume, without loss of generality, that d_n/λ_n converges as $n \to \infty$. Let $\tilde{y}_n \in \partial\Omega(s_n)$ be such that $d_n = |y_n - \tilde{y}_n|$ and R_n be an orthonormal transformation in $\mathbf{R}^N$ that maps $-e_N = (0, \cdots, 0, -1)$ onto the outer normal vector to $\partial\Omega(s_n)$ at $\tilde{y}_n$. Put

$$\hat{w}_n(y,s) = \lambda_n^{2/(p-1)} w_n(y_n + \lambda_n R_n y, s_n + \lambda_n^2 s)$$

for $(y,s) \in \hat{Q}_n$, where

$$\hat{Q}_n = \bigcup_{s \in I_n} (\hat{\Omega}_n(s) \times \{s\}), \qquad \hat{\Omega}_n(s) = \lambda_n^{-1} R_n^{-1} (\Omega(s) - y_n).$$

Then $\hat{w}_n$ satisfies

$$\partial_s \hat{w}_n = \Delta \hat{w}_n + \lambda_n \frac{y_n + \lambda_n R_n y}{2} \cdot R_n \nabla_y \hat{w}_n + \frac{\lambda_n^2}{p-1} \hat{w}_n + \hat{w}_n^p \quad \text{in} \quad \hat{Q}_n. \tag{3.20}$$

Furthermore, taking a subsequence if necessary, we see that $\hat{\Omega}_n(s)$ approaches (locally) the half space

$$H = \{y = (y', y_N) : y' \in \mathbf{R}^{N-1},\ y_N > -d\},$$

as $n \to \infty$, where $d = \lim_{n\to\infty} d_n/\lambda_n$. By the interior and the boundary Schauder estimates, we see that there exists a constant $\beta \in (0,1)$ such that

$$\sup_{n \in \mathbf{N}} \|\hat{w}_n\|_{C^{2+\beta, 1+\beta/2}(\hat{Q}_n \cap K)} < \infty$$

for any compact set K on $H \times (-\infty, 0]$. Therefore, by the similar argument as in the case (A), we see that there exists a nonnegative function $\hat{w}$ in $H \times (-\infty, 0]$ such that

$$\hat{w}(0,0) = 1,$$
$$0 = \partial_s \hat{w} = \Delta \hat{w} + \hat{w}^p \quad \text{in} \quad H \times (-\infty, 0], \quad \hat{w} = 0 \quad \text{on} \quad \partial H \times (-\infty, 0].$$

These relations together with the nonexistence result in [9] yields the same contradiction as in the case (A).

Case (C) Taking a subsequence if necessary, we can assume that

$$(3.21) \qquad |\lambda_n^{1/2} y_n| \geq 1, \qquad n = 1, 2, \ldots.$$

Put

$$W_n(y,s) = w_n\left(y + e^{-\frac{s-s_n}{2}} y_n, s\right)$$

for $y \in \Omega(s) - e^{-\frac{s-s_n}{2}} y_n$ and $s > 0$. Then W_n is also a global in time solution of (2.8) such that

$$W_n(0, s_n) = w_n(y_n, s_n).$$

Similarly to the case (A), putting

$$\tilde{W}_n(y,s) = \lambda_n^{2/(p-1)} W_n(\lambda_n y, s_n + \lambda_n^2 s) \quad \text{for} \quad (y,s) \in Q_n,$$

we obtain

$$(3.22) \qquad \partial_s \tilde{W}_n = \Delta \tilde{W}_n + \lambda_n^2 \frac{y}{2} \cdot \nabla_y \tilde{W}_n + \frac{\lambda_n^2}{p-1} \tilde{W}_n + \tilde{W}_n^p \quad \text{in} \quad Q_n.$$

Furthermore there hold (3.12)–(3.15) with $\tilde{w}_n$ replaced by $\tilde{W}_n$. Then, by the same argument as in the case (A), we see that there exist a subsequence $\{\tilde{W}_n'\}$ of $\{\tilde{W}_n\}$, a function $\tilde{W}$, and a constant $\alpha \in (0,1)$ such that

$$(3.23) \qquad \lim_{n \to \infty} \|\tilde{W}_n' - \tilde{W}\|_{C^{2+\alpha, 1+\alpha/2}(K)} = 0$$

for any compact subset K of $\mathbf{R}^N \times (-\infty, 0]$ and

$$(3.24) \qquad \tilde{W}(0,0) = 1.$$

On the other hand, (C), (3.6), (3.12), and (3.21) imply that, for any $R > 0$, there exists a constant C such that

$$(3.25) \qquad \int_{-\lambda_n^{-2}(s_n - s_0)}^{0} \int_{B(0,R)} |\tilde{W}_n|^2 dyds = \lambda_n^{\sigma'} \int_{s_0}^{s_n} \int_{B(0,\lambda_n R)} |W_n|^2 dyds$$
$$= \lambda_n^{\sigma'} \int_{s_0}^{s_n} \int_{B(e^{-(s-s_n)/2} y_n, \lambda_n R)} |w_n|^2 dyds$$
$$\leq \lambda_n^{\sigma'} e^{-|y_n|^2/C} \int_{s_0}^{s_n} \int_{B(e^{-(s-s_n)/2} y_n, \lambda_n R)} |w_n|^2 \rho(y) dyds$$
$$\leq \lambda_n^{\sigma'} e^{-|y_n|^2/C} \sup_{s_0 < s < S_n} \|w_n(s)\|^2 \leq l_n \lambda_n^{\sigma'} e^{-1/C\lambda_n},$$

where $\sigma' = 4/(p-1) - (N+2)$. By using (3.11) (and (3.12)), we obtain

$$\lim_{n\to\infty} l_n \lambda_n^{\sigma'} e^{-1/C\lambda_n} = 0. \tag{3.26}$$

Therefore, by (3.23), (3.25), and (3.26), we see that

$$\tilde{W} = 0 \quad \text{in} \quad \mathbf{R}^N \times (-\infty, 0]. \tag{3.27}$$

This contradicts (3.24). Thus the proof of Lemma 3.1 is complete. □

Remark 3.1 *Lemma* 3.1 *for* $\Omega = \mathbf{R}^N$ *with* Al^α *replaced by some constant* C *has been already given in* [18, *Lemma* 3], *without the assumption* (3.3). *However, in* [18], *the author did not give the proof of* (3.3) *explicitly, and as is pointed out in* [16], *it seems that he didn't consider the case where* $\lambda_n^2 y_n \cdot \to \infty$ *as* $n \to \infty$ *for the equation* (3.16). *In our proof of Lemma* 3.1, *we exclude this possibility by using the assumption* (3.3) *(see case (C)). Also, the similar lemma to Lemma* 3.1 *with* Al^α *replaced by some constant* C *is given in* [16] *for the study of the large time behavior of solutions of the heat equation with a nonlinear boundary condition, but the assumption* (3.3) *is replaced by a different assumption, which is not suited for our case.*

Next we give upper bounds of the global in time solutions of (1.1) under the assumption (1.11), by using the interior and the boundary Harnack inequalities and the gradient estimates for the parabolic equations.

Lemma 3.2 *Assume the conditions* (1.2)–(1.4) *and* $\phi \in G$. *Let* u *be a solution of* (1.1) *satisfying* (1.11). *Then there holds the following:*
(i) *if* $\kappa < \delta + (N-2)/4$, *then* $\Lambda_\kappa(s) < \infty$ *for any* $s > 0$;
(ii) *if*

$$\delta + \frac{N-2}{4} \le \frac{1}{p-1}, \tag{3.28}$$

then, for any $1 < \beta < 4/[-(N-2)p + N + 2]$, *it holds that* $\beta\delta < 1/(p-1)$ *and there exists a constant* C_1, *depending on* β *and* δ, *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le C_1(1+t)^{-\beta\delta} \tag{3.29}$$

for all $t > 0$;
(iii) *if*

$$\delta + \frac{N-2}{4} > \frac{1}{p-1}, \tag{3.30}$$

then there exists a constant C_2 *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le C_2(1+t)^{-1/(p-1)} \tag{3.31}$$

for all $t > 0$.

Proof. By (2.1), we see that u is a nonnegative solution of

$$\partial_t u = \Delta u + V(x,t)u \quad \text{in} \quad \Omega \times (0,\infty), \qquad u = 0 \quad \text{in} \quad \partial\Omega \times (0,\infty),$$

with $V(x,t) = u(x,t)^{p-1} \in L^\infty(\Omega \times (0,\infty))$. Let $R > 0$ and $\tau > 0$. Then, by using the same arguments as in [13] and [20], we can prove that there exists a constant C_1 such that

$$u(x,t) \le C_1 u(x_0, t+\tau), \qquad x \in \Omega \cap B(0,R),\ t \in (\tau,\infty). \tag{3.32}$$

In fact, we construct a chain of parabolic cylinders, which connects (x,t) with $(x_0, t+\tau)$, and then can prove the inequality (3.32) by the use of the interior and the boundary Harnack inequalities for parabolic equations (for the boundary Harnack inequality, for example, see [12] and [25]). The inequality (3.32) together with (1.11) implies that

$$u(x,t) \le C_2(1+t)^{-\delta}, \qquad x \in \Omega \cap B(0,R),\ t \in (\tau,\infty),$$

for some constant C_2. Then we apply the gradient estimates for parabolic equations to u (see e.g. [19, Section 5, Chapter V]), and obtain

$$|(\nabla u)(x,t)| \le C_3(1+t)^{-\delta}, \qquad (x,t) \in \partial\Omega \times (2\tau,\infty), \tag{3.33}$$

for some constant C_3. This implies that

$$|(\nabla_y z)(y,s)| \le C_3 e^{(\kappa-\delta+1/2)s}, \qquad (y,s) \in \partial\Omega(s) \times (s_\tau,\infty), \tag{3.34}$$

for any $\kappa \in (0, 1/(p-1)]$, where $s_\tau = \log(1+2\tau)$. Then, by $N \ge 3$, (1.21), and (3.34), we can find a constant C_4 such that

$$\Lambda_\kappa(s) \le C_3^2 \int_s^\infty \int_{\partial\Omega(s)} |y| e^{2(\kappa-\delta+\frac{1}{2})s} \rho \, d\sigma d\tau \le C_4 \int_0^\infty e^{-\frac{N}{2}s + 2(\kappa-\delta+\frac{1}{2})s} d\tau \tag{3.35}$$

for all $s \ge s_\tau$. Therefore, if $\kappa < \delta + (N-2)/4$, then $\Lambda_\kappa(s) < \infty$ for $s \ge s_\tau$. By the arbitrariness of τ, we have the conclusion of the statement (i).

Next we assume (3.28), and prove the statement (ii). The inequality $\beta\delta < 1/(p-1)$ easily follows from (3.28) and the assumption on β. We will prove the inequality (3.29). Put

$$\beta' = \frac{4}{-(N-2)p + N + 2} \ (> 1).$$

Let β and δ' be numbers satisfying $1 < \beta < \beta'$, $0 < \delta' < \delta$, and $\delta'\beta' = \delta\beta$. Also put $\kappa = \delta' + (N-2)/4$. Then we have

$$0 < \kappa < \delta + \frac{N-2}{4} \le \frac{1}{p-1}.$$

By Lemma 3.2-(i), we can define the energy $F_\kappa(s)$ for $s > 0$. By Lemma 2.3, for any $s_0 > 0$, we obtain

$$\int_{\Omega(s)} |w(s)|^2 \rho \, dy + \int_{s_0}^s \int_{\Omega(s)} |(\partial_s w)(y,s)|^2 \rho \, dy d\tau \le C_5 e^{2\kappa' s} F_\kappa(s_0), \quad s \ge s_0 > 0,$$

for some constant C_5, where $\kappa' = -\kappa + 1/(p-1) > 0$. Then Lemma 3.1 and (2.1) yield the existence of the constant C_6 satisfying

$$\|w(s)\|_\infty \le \max\{\sup_{s_0\le\tau\le s_0+1}\|w(\tau)\|_\infty, \sup_{s_0+1\le\tau\le s}\|w(\tau)\|_\infty\} \le C_6 e^{2\alpha\kappa' s}$$

for all $s > s_0$, where α is the constant given in Lemma 3.1. This implies that

$$\|u(t)\|_\infty \le C_6(1+t)^{2\alpha\kappa' - \frac{1}{p-1}}$$

for all $t > t_0 := e^{s_0} - 1$. Then, since

$$\begin{aligned} 2\alpha\kappa' - \frac{1}{p-1} &= 2\cdot\frac{2}{\sigma(p-1)}\left(-\kappa + \frac{1}{p-1}\right) - \frac{1}{p-1} \\ &= \beta'\left(-\delta' - \frac{N-2}{4} + \frac{1}{p-1}\right) - \frac{1}{p-1} \\ &= -\beta'\delta' + \beta'\left(-\frac{N-2}{4} + \frac{1}{p-1}\right) - \frac{1}{p-1} \\ &= -\beta'\delta' = -\beta\delta, \end{aligned}$$

we have

$$\|u(t)\|_\infty \le C_6(1+t)^{-\beta\delta} \tag{3.36}$$

for all $t > t_0$. Therefore, by (2.1) and (3.36), we have the conclusion of the statement (ii). If δ satisfies (3.30), by Lemma 3.2-(i), we can define $F_\kappa(s)$ with $\kappa = 1/(p-1)$ for $s > 0$. Then, by repeating the similar argument as above with κ and κ' replaced by $1/(p-1)$ and 0, respectively, we can prove the statement (iii); thus the proof of Lemma 3.2 is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume (1.11). If

$$\delta + \frac{N-2}{4} > \frac{1}{p-1},$$

then, by Lemma 3.2-(iii), we have the inequality (1.12). If not, take $\beta \in (1, 4/[-(N-2)p + N + 2])$ and take a smallest natural number n satisfying

$$\beta^{n-1}\delta + \frac{N-2}{4} \le \frac{1}{p-1}, \qquad \beta^n\delta + \frac{N-2}{4} > \frac{1}{p-1}. \tag{3.37}$$

Since $\delta + (N-2)/4 \le 1/(p-1)$, in view of Lemma 3.2-(ii), we have

$$\|u(t)\|_\infty \le C_1(1+t)^{-\beta\delta}, \quad t > 0, \tag{3.38}$$

for some constant C_1, in particular, $\limsup_{t\to\infty} t^{\beta\delta}u(x_0,t) < \infty$. Repeating this argument n-times, we see that $\limsup_{t\to\infty} t^{\beta^n\delta}u(x_0,t) < \infty$. This relation together with (3.37) implies that the assumption of Lemma 3.2-(iii) with δ replaced by $\beta^n\delta$ is satisfied. Hence we have the inequality (1.12), and the proof of Theorem 1.1 is complete. □

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