数理解析研究所講究録
第 1845 巻 2013 年 13-19

Scattering theory from a geometric view point

筑波大学大学院数理物質科学研究科　伊藤 健一 (Kenichi ITO)
Graduate School of Pure and Applied Sciences,
University of Tsukuba

This article is based on the author's recent joint works with Erik Skibsted [IS1, IS2].

1 Assumptions

Let (M, g) be a connected and complete Riemannian manifold, and we consider the Schrödinger operator

$$H = H_0 + V; \quad H_0 = -\frac{1}{2}\triangle$$

on the Hilbert space $\mathcal{H} = L^2(M) = L^2(M, (\det g)^{1/2}dx)$. The Laplace-Beltrami operator $-\triangle$ is defined in local coordinates by

$$-\triangle = p_i^* g^{ij} p_j = (\det g)^{-1/2} p_i (\det g)^{1/2} g^{ij} p_j,$$

where

$$p_i = -\mathrm{i}\partial_i, \qquad g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad \det g = \det(g_{ij}), \qquad (g^{ij}) = (g_{ij})^{-1}.$$

Under the following Conditions 1.1–1.4 H is essentially self-adjoint on $C_c^\infty(M)$. We will denote the self-adjoint extension also by H.

Condition 1.1 (End structure). There exists a relatively compact open set $O \Subset M$ with smooth boundary ∂O such that the exponential map restricted to outward normal vectors on ∂O:

$$\exp_O := \exp|_{N^+\partial O} \colon N^+\partial O \to M$$

is diffeomorphic onto $E := M \setminus \overline{O}$.

A component of E is called an *end*, and such M a *manifold with ends*, cf. [K1]. Then there exists a function $r \in C^\infty(M)$ such that

$$r(x) = \mathrm{dist}(x, O), \quad x \in E.$$

Note that r is not uniquely determined on O.

Recall that the geometric Hessian by $\nabla^2 f \in \Gamma(T^*M \otimes T^*M)$ for $f \in C^\infty(M)$ is defined in local coordinates by

$$(\nabla^2 f)_{ij} = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f; \quad \Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}). \tag{1.1}$$

Condition 1.2 (Mourre type condition). There exist $\delta \in (0,1]$ and $r_0 \geq 0$ such that for $x \in E$ with $r(x) \geq r_0$

$$\nabla^2 r^2 \geq (1+\delta)g, \tag{1.2}$$

where the ineuqality is understood as that for quadratic forms on fibers of TM,

Condition 1.3 (Quantum mechanics bound). There exists $\kappa \in (0,1)$ such that

$$|\mathrm{d}\Delta r^2|^2 = g^{ij}(\partial_i \Delta r^2)(\partial_j \Delta r^2) \leq C\langle r\rangle^{-1-\kappa}; \qquad \langle r\rangle = (1+r^2)^{1/2}. \tag{1.3}$$

The quantities in Conditions 1.2 and 1.3 appear in the Morre-type commutator computations: If we define

$$A = \mathrm{i}[H_0, r^2] = \frac{1}{2}\{(\partial_i r^2)g^{ij}p_j + p_i^* g^{ij}(\partial_j r^2)\}, \tag{1.4}$$

then

$$\mathrm{i}[H_0, A] = p_i^*(\nabla^2 r^2)^{ij}p_j + \frac{\mathrm{i}}{4}(\partial_i \Delta r^2)g^{ij}p_j - \frac{\mathrm{i}}{4}p_i^* g^{ij}(\partial_j \Delta r^2).$$

Condition 1.4 (Short-range potential). The potential $V \in L^\infty(M;\mathbb{R})$ satisfies for some $\eta \in (0,1]$

$$|V(x)| \leq C\langle r\rangle^{-1-\eta}. \tag{1.5}$$

2 Free propagator

Set $K(t,x) = r(x)^2/2t$ and let A be as defined by (1.4). We define the free propagator $U(t)\colon \mathcal{H} \to \mathcal{H}$, $t > 0$, by

$$U(t) = \mathrm{e}^{\mathrm{i}K(t,\cdot)}\mathrm{e}^{-\mathrm{i}\frac{\ln t}{2}A}.$$

Note that the function K is a solution to the Hamilton-Jacobi equation

$$\partial_t K = -\frac{1}{2}g^{ij}(\partial_i K)(\partial_j K) \quad \text{on } E. \tag{2.1}$$

In fact, r satisfies the eikonal equation

$$|\nabla r|^2 = g^{ij}(\partial_i r)(\partial_j r) = 1 \quad \text{on } E.$$

On the other hand, $\mathrm{e}^{-\mathrm{i}\frac{\ln t}{2}A}$ is written explicitly by

$$\mathrm{e}^{-\mathrm{i}\frac{\ln t}{2}A}u(x) = \exp\Big(\int_1^t \frac{1}{4s}(-\Delta r^2)(\omega(s,x))\,\mathrm{d}s\Big)u(\omega(t,x)), \tag{2.2}$$

where the flow $\omega = \omega(t,x)$, $(t,x) \in (0,\infty) \times M$, is given by

$$\partial_t \omega^i = -\frac{1}{2t}g^{ij}(\omega)(\partial_j r^2)(\omega), \quad \omega(1,x) = x. \tag{2.3}$$

In fact, if we differentiate $e^{-i\frac{\ln t}{2}A}u$ in t, then we obtain a transport equation and thus (2.2) by solving the equation. By (2.2) we can see that $e^{-i\frac{\ln t}{2}A}$ is the geodesic dilation on $\mathcal{H}$ with respect to r. In fact we note that, using the relation $-\Delta f = g^{ij}(\nabla^2 f)_{ij} = \operatorname{tr}(\nabla^2 f)$,

$$\exp\Big(\int_1^t \frac{1}{4s}(-\Delta r^2)(\omega(s,x))\,\mathrm{d}s\Big) = J(\omega(t,x))^{1/2}\Big(\frac{\det g(\omega(t,x))}{\det g(x)}\Big)^{1/4}, \tag{2.4}$$

and that (2.3) is solved for $(t,x) \in (0,\infty) \times E$ by

$$\omega(t,x) = \exp_O\Big[\frac{1}{t}(\exp_O)^{-1}(x)\Big],$$

and for $(t,x) \in (0,\infty) \times O$ by something different and complicated. The first factor in the right-hand side of (2.4) is the Jacobian for $\omega(t,\cdot)$, and the second is the change of density for $\omega(t,\cdot)$.

In particular, we learn that $U(t)$ is unitary on both

$$\mathcal{H}_{\mathrm{aux}} := L^2(E) \subset \mathcal{H} \quad \text{and} \quad (\mathcal{H}_{\mathrm{aux}})^{\perp} = L^2(O) \subset \mathcal{H}.$$

3 Main results

Theorem 3.1 (Positive eigenvalues, [Do, K2, IS2]). *Suppose Conditions 1.1–1.4. Then the positive eigenvalues of H are absent: $\sigma_{\mathrm{pp}}(H) \cap (0,\infty) = \varnothing$.*

Theorem 3.2 (Wave operator, [IS1]). *Under Conditions 1.1–1.4 there exist the strong limits*

$$\Omega_+ := \operatorname*{s-lim}_{t\to+\infty} e^{itH}U(t)P_{\mathrm{aux}}, \quad \widetilde{\Omega}_+ := \operatorname*{s-lim}_{t\to+\infty} U(t)^* e^{-itH}P_{\mathrm{c}},$$

where P_{aux} is the orthogonal projection onto $\mathcal{H}_{\mathrm{aux}}$, and $P_{\mathrm{c}} = \chi_{(0,\infty)}(H)$. Moreover the wave operator Ω_+ is complete, i.e.

$$\widetilde{\Omega}_+ = \Omega_+^*, \quad \Omega_+^*\Omega_+ = P_{\mathrm{aux}}, \quad \Omega_+\Omega_+^* = P_{\mathrm{c}}.$$

We denoted the characteristic function of $\mathcal{O} \subset \mathbb{R}$ by $\chi_{\mathcal{O}}$. It follows by a standard local compactness argument that the negative spectrum of H (if not empty) consists of eigenvalues of finite multiplicities accumulating at most at zero.

Corollary 3.3 (Intertwining property and spectrum). *One has the intertwining property:*

$$\Omega_+^* H \Omega_+ = \frac{1}{2}r^2 P_{\mathrm{aux}}.$$

In particular, the singular continuous spectrum of H is absent, i.e., $\sigma_{\mathrm{sc}}(H) = \varnothing$, and the continuous spectrum $\sigma_{\mathrm{c}}(H) = [0,\infty)$.

The following corollary implies the existence of "the asymptotic speed". For self-adjoint operators B and B_i, $i = 1, 2, \ldots$, we denote

$$B = \text{s-}C_c(\mathbb{R})\text{-}\lim_{i \to +\infty} B_i,$$

if for any $f \in C_c(\mathbb{R})$ the following equality holds:

$$f(B) = \text{s-}\lim_{i \to +\infty} f(B_i).$$

Corollary 3.4 (Asymptotic observables). *In the continuous subspace $\mathcal{H}_c(H)$ there exists the $*$-representation*

$$\omega_\infty^+ := \text{s-}C_c(M)\text{-}\lim_{t \to +\infty} e^{itH} \omega(t, \cdot) e^{-itH}. \tag{3.1}$$

In particular, the asymptotic speed

$$r(\omega_\infty^+) = \text{s-}C_c(\mathbb{R})\text{-}\lim_{t \to +\infty} e^{itH} \frac{r(\cdot)}{t} e^{-itH}$$

exists as a self-adjoint operator on $\mathcal{H}_c(H)$. This operator is positive with zero kernel.

Moreover, for all $\varphi \in C_c(M)$

$$\varphi(\omega_\infty^+) = \Omega_+ M_\varphi \Omega_+^*, \quad H_c = 2^{-1} r(\omega_\infty^+)^2.$$

Here M_φ denotes the multiplication operator by φ. In local coordinates $\omega(t, \cdot)$ has d (dimension of M) components which we can substitute for any $f \in C_c(M)$, so the limit in (3.1) makes sense.

Remarks 3.5. 1. Theorem 3.1 is generalized under weaker conditions including asymptotically hyperbolic manifolds, [IS2].

2. This type of the free propagator in Theorem 3.2 appeared first in [Y]. For later developments refer to [DeG, CHS, HS].

3. The above results are independent of choice of r on O.

4. As for Theorem 3.1, Conditions 1.2–1.4 are optimal in the sense that we can construct counterexamples to the existence of Ω_+ under the slight relaxation of the conditions allowing either $\delta = 0$ in (1.2), $\kappa = 0$ in (1.3) or $\eta = 0$ in (1.5).

4 Generator of the free propagator

We briefly see why the free propagator $U(t)$ works as a comparable system, and see also the relationship with the previous result on the wave operators on manifolds with ends, [IN], where the radial Laplacian was chosen as the free operator.

Let $G(t)$ be the time-dependent generator of $U(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -\mathrm{i}G(t)U(t).$$

By a formal computation we can see

$$G(t) = -\partial_t K + \frac{1}{2}\{(\partial_i K)g^{ij}(p_j - \partial_j K) + (p_i - \partial_i K)g^{ij}(\partial_j K)\},$$

so that

$$\begin{aligned} H - G(t) &= V + W(t) + \alpha(t); \qquad (4.1)\\ W(t) &= \frac{1}{2}(p_i - \partial_i K)^* g^{ij}(p_j - \partial_j K),\\ \alpha(t) &= \alpha(t,x) = (\partial_t K) + \frac{1}{2}g^{ij}(\partial_i K)(\partial_j K). \end{aligned}$$

The right-hand side of (4.1) is interpred to be *short-range.* In fact the first is so by Condition 1.4; The second term is so from a classical point of view in the sense that for any nontrapped classical trajectory $(x(t), p(t))$

$$0 \le \frac{1}{2}g^{ij}(x(t))\{p_i(t) - \partial_i K(t, x(t))\}\{p_j(t) - \partial_j K(t, x(t))\} \le C\langle t\rangle^{-1-\delta}, \qquad (4.2)$$

cf. the fact that K is a solution to the Hamilton-Jacobi equation; As for the third term this is due to (2.1): For any $N > 0$

$$|\alpha(t,x)| \le C_N t^{-2}\langle r\rangle^{-N}.$$

In the proof of Theorem 3.2 the translation of the classical estimate (4.2) into the quantum mechanics plays an essential role.

We remark that, since

$$G(t) = \frac{1}{2}p_r^* p_r - \frac{1}{2}\Big(p_r - \frac{r}{t}\Big)^*\Big(p_r - \frac{r}{t}\Big) \quad \text{on } E; \quad p_r := (\partial_k r)g^{kl}p_l,$$

which we can see with ease in the *geodesic spherical coordinates*, $G(t)$ differs from the one-dimensional radial Laplacian by a short-range term, cf. [IN]. Note that $r(t)/t$ classically approaches the radial momentum $p_r(t)$, cf. (4.2).

5 Example: Ends of warped-product type

Here we give an example of a manifold that satisfies Conditions 1.1–1.4.

Let $V = 0$, and suppose that there exists a relatively compact open subset $O \Subset M$ such that isometrically the closure $\bar{E} := M\backslash O \cong [0,\infty)\times S$ for some $(d-1)$-dimensional manifold S, and that

$$g = \mathrm{d}r \otimes \mathrm{d}r + f(r)h_{\alpha\beta}(\sigma)\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta; \quad g_{rr} = 1,\ g_{r\alpha} = g_{\alpha r} = 0, \qquad (5.1)$$

where $(r,\sigma) \in [0,\infty) \times S$ denotes local coordinates and the Greek indices run over $2,\dots,d$.

Then Condition 1.1 is automatically satisfied. By (1.1), it follows

$$(\nabla^2 r^2)_{rr} = 2, \quad (\nabla^2 r^2)_{r\alpha} = (\nabla^2 r^2)_{\alpha r} = 0, \quad (\nabla^2 r^2)_{\alpha\beta} = r f' h_{\alpha\beta}.$$

Thus, if we set $f = \mathrm{e}^{2\varphi}$, (1.2) is equivalent to

$$2r\varphi' \geq 1 + \delta, \tag{5.2}$$

and, by $\triangle r^2 = g^{ij}(\nabla^2 r^2)_{ij} = 2 + 2(d-1)r\varphi'$, (1.3) to

$$|(r\varphi')'| \leq C\langle r\rangle^{-(1+\kappa)/2}. \tag{5.3}$$

We see that the inequalities (5.2) and (5.3) allow, for example,

$$f(r) = f_{1,\mu}(r) = r^2\langle r\rangle^{2\mu}, \quad \mu \geq -(1-\delta)/2,$$
$$f(r) = f_{2,\nu}(r) = r^2 \mathrm{e}^{-2}\exp\left(2\langle r\rangle^{\nu}\right), \quad 0 \leq \nu \leq (1-\kappa)/2.$$

Note that the Euclidean space corresponds to $f(r) = f_{1,0}(r) = f_{2,0}(r) = r^2$. We also note that in [IS2] the absence of embedded eigenvalues is discussed for a wider class of manifolds with ends including $f_{1,\mu}$ with $\mu > -1$ and $f_{2,\nu}$ with $0 \leq \nu \leq 1$.

References

[CHS] H. D. Cornean, I. Herbst, E. Skibsted, *Spiraling attractors and quantum dynamics for a class of long-range magnetic fields*, *J. Funct. Anal.* **247** (2007), no. 1, 1–94.

[DeG] J. Dereziński, C. Gérard, *Long-range scattering in the position representation*, J. Math. Phys. **38** no. 8 (1997), 3925–3942.

[Do] H. Donnelly, *Spectrum of the Laplacian on asymptotically Euclidean spaces*, Michigan Math. J. **46** no. 1 (1999), 101–111.

[HS] I. Herbst, E. Skibsted, *Quantum scattering for potentials independent of* $|\mathbf{x}|$*: asymptotic completeness for high and low energies*, Comm. PDE. **29** no. 3–4 (2004), 547-610.

[IN] K. Ito, S. Nakamura, *Time-dependent scattering theory for Schrödinger operators on scattering manifolds*, J. Lond. Math. Soc. **81** no. 3 (2010), 774–792.

[IS1] K. Ito, E. Skibsted, *Scattering theory for Riemannian Laplacians*, Preprint 2011.

[IS2] K. Ito, E. Skibsted, *Absence of positive eigenvalues for Riemannian Laplacians*, Preprint 2011.

[K1] H. Kumura, *On the essential spectrum of the Laplacian on complete manifolds*, J. Math. Soc. Japan **49** no. 1 (1997), 1–14.

[K2] H. Kumura, *The radial curvature of an end that makes eigenvalues vanish in the essential spectrum. II.*, preprint 2009.

[Y] D. Yafaev, *Wave operators for the Schrödinger operator*, Teor. Mat. Fiz.**45** (1980), 224–234.