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THE PHRAGMÉN-LINDELÖF THEOREM FOR FULLY NONLINEAR ELLIPTIC SYSTEMS WITH UNBOUNDED INGREDIENTS

KAZUSHIGE NAKAGAWA

ABSTRACT. The Phragmén-Lindelöf theorem is established for L^{p}-viscosity solutions of fully nonlinear second order elliptic partial differential weak coupled systems with unbounded coefficients and inhomogeneous terms.

1. INTRODUCTION

In this paper, we study fully nonlinear second order uniformly elliptic partial differential systems;

$$F_{k}(x, u_{1}, \ldots, u_{m}, Du_{k}, D^{2}u_{k})=f_{k}(x) \quad \text{in } \Omega, \quad k\in\{1, \ldots, m\} \tag{1.1}$$

where $F_{k}:\Omega\times \mathbb{R}^{m}\times \mathbb{R}^{n}\times S^{n}\rightarrow \mathbb{R}$ and $f_{k}\in L^{p}(\Omega)$ $(k=1, \ldots, m)$ are given functions. Here Ω denotes a bounded open domain in $\mathbb{R}^{n}$ and S^{n} is the set of $n\times n$ symmetric matrics with the standard ordering. We want prove the Aleksandrov-Bakleman-Pucci (ABP for short) maximum principle for L^{p}-viscosity subsolutions of (1.1).

We make the following hypothesis about F_{k}. We first assume that F_{k} is uniformly elliptic, i. e.

$$\mathcal{P}^{-}(X-Y)\leq F_{k}(x, r_{1}, \ldots, r_{m}, \xi, X)-F_{k}(x, r_{1}, \ldots, r_{m}, \xi, Y)\leq \mathcal{P}^{+}(X-Y) \tag{1.2}$$

for $x\in\Omega$, $(r_{1}, \ldots, r_{m})\in \mathbb{R}^{m}$, $\xi\in \mathbb{R}^{n}$ and $X, Y\in S^{n}$, where $\mathcal{P}^{\pm}(\cdot)$ the Pucci extremal operator defined as

$$\mathcal{P}^{-}(X)=\min\{-\text{trace}(AX) : \lambda I\leq A\leq\Lambda I,\ A\in S^{n}\} \tag{1.3}$$

for fixed uniform ellipticity constants $0<\lambda\leq\Lambda$. The other Pucci extremal operator $\mathcal{P}^{+}(X)$ is defined by $\mathcal{P}^{+}(X)=-\mathcal{P}^{-}(-X)$. Without loss of generality, we may assume that

$$F_{k}(x, 0, \ldots, 0,0, O)=0 \quad \text{in } \Omega, \quad \text{for } k=1, \ldots, m \tag{1.4}$$

by taking $F_{k}(x, r_{1}, \ldots, r_{m}, \xi, X)-F_{k}(x, 0, \ldots, 0,0, O)$ and $f_{k}(x)-F_{k}(x, 0, \ldots, 0,0, O)$ in place of F_{k} and f_{k}. Finally, we assume that there exist functions $\mu_{k}\in L^{q}(\Omega)$, and $c_{k}(x, r_{1}, \ldots, r_{m})$ for $k=1, \ldots, m$ such that

$$|F_{k}(x, r_{1}, \ldots, r_{m}, \xi, O)|\leq\mu_{k}(x)|\xi|+c_{k}(x, r_{1}, \ldots, r_{m}) \tag{1.5}$$

for $x\in\Omega, (r_{1}, \ldots, r_{m})\in \mathbb{R}^{m}$ and $\xi\in \mathbb{R}^{n}$. Here, functions $c_{k}(x, r_{1}, \ldots, r_{m})$ are Lipshitz continuous in $(r_{1}, \ldots, r_{m})\in \mathbb{R}^{m}$ and uniformly in $x\in\Omega\backslash \mathcal{N}$ for some Lebesgue null set $\mathcal{N}\subset\Omega$ with

Lipshitz constant ν in the sense of ℓ^1-norms of $D_r c_k$ ($r = (r_1, \ldots, r_m)$). Under these assumption, it is essential to consider Pucci extremal systems having the form;

$$\mathcal{P}^-(D^2u_k) - \mu_k(x)|Du_k| - c_k(x, u_1, \ldots, u_m) = f_k(x) \quad \text{in } \Omega, \tag{1.6}$$

for subsolutions of (1.1), and

$$\mathcal{P}^+(D^2u_k) + \mu_k(x)|Du_k| + c_k(x, u_1, \ldots, u_m) = f_k(x) \quad \text{in } \Omega, \tag{1.7}$$

for supersolutions of (1.1). Therefore, it is enough to show several properties for subsolutions of

$$\mathcal{P}^-(D^2u_k) - \mu_k(x)|Du_k| - c_k(x, u_1, \ldots, u_m) = f_k(x) \quad \text{in } \Omega, \quad k = 1, \ldots, m. \tag{1.8}$$

This paper is organized as follows. In Section 2, we introduce the notation and some preliminary results. In Section 3, we establish the ABP maximum principle in bounded domain and weak Hrnack inequality. In Section 4, we establish the Phragmén-Lindelöf theorem for nonlinear weak coupled elliptic systems with unbounded coefficients. Finally, Section 5 and 6 , we give a proof of Phragmén-Lindelöf theorem and ABP type estimates for unbounded domains.

2. Preliminaries

For measurable sets $U \subset \mathbb{R}^n$, we denote by $L^p_+(U)$ the set of all nonnegative functions in $L^p(U)$ for $1 \le p \le \infty$. We will often write $\|\cdot\|_p$ ($1 \le p \le \infty$) instead of $\|\cdot\|_{L^p(U)}$ if there is no confusion. We will use the standard notations from [15].

First of all, we recall the definition of L^p-viscosity solutions of

$$G(x, u(x), D\phi(x), D^2\phi(x)) = 0 \quad \text{in } \Omega. \tag{2.1}$$

Definition 2.1. We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (2.1) if

$$ess \liminf_{x \to x_0} \{G(x, u(x), D\phi(x), D^2\phi(x))\} \le 0$$
$$\left(\text{resp.,} \quad ess \limsup_{x \to x_0} \{G(x, u(x), D\phi(x), D^2\phi(x))\} \ge 0\right)$$

whenever $\phi \in W^{2,p}_{\text{loc}}(\Omega)$ and $x_0 \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$. A function $u \in C(\Omega)$ is called an L^p-viscosity solution of (2.1) if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution of (2.1).

We will say L^p-subsolution (resp., -supersolution) for L^p-viscosity subsolution (resp., supersolution) for simplicity. We will also say that u is an L^p-solution of

$$G(x, u, Du, D^2u) \le 0,$$
$$(\text{resp.}, G(x, u, Du, D^2u) \ge 0),$$

if it is an L^p-subsolution (resp., -supersolution) of (2.1).

We will use this abbreviation also for L^p-strong sub- and supersolutions below.

DEFINITION 2.2. We call $u \in C(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ an L^p-strong subsolution (resp., supersolution) of (2.1) if u satisfies

$$G(x, u(x), Du(x), D^2u(x)) \le 0 \quad \text{a.e. in } \Omega,$$
$$(\text{resp., } G(x, u(x), Du(x), D^2u(x)) \ge 0 \quad \text{a.e. in } \Omega).$$

REMARK 2.3. If u is an L^p-subsolution (resp., L^p-supersolution) of (2.1), then it is also an L^q-subsolution (resp., L^q-supersolution) of (2.1) provided $q \ge p$. However, if u is an L^p-strong subsolution (resp., supersolution) of (2.1), then it is also an L^q-strong subsolution (resp., supersolution) of (2.1) provided $p \ge q$.

It is known (e.g. [5, 14]) that there exists $p_0 = p_0(n, \lambda, \Lambda)$ satisfying $n/2 \le p_0 < n$ such that for $p > p_0$, there is a constant $C = C(n, p, \lambda, \Lambda)$ such that if for $f \in L^p(\Omega)$, $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ is an L^p-strong subsolution of

$$\mathcal{P}^-(D^2u) \le f(x) \quad \text{in } \Omega \tag{2.2}$$

such that $u = 0$ on $\partial\Omega$, and

$$-C\|f^-\|_p \le u \le C\|f^+\|_p \quad \text{in } \Omega.$$

Moreover, for each $\Omega' \Subset \Omega$, there is $C' = C'(n, p, \lambda, \Lambda, \mathrm{dist}(\Omega', \partial\Omega)) > 0$ such that

$$\|u\|_{W^{2,p}(\Omega')} \le C'\|f\|_p.$$

Throughout this paper we suume

$$p_0 < p \le q, \quad n < q \tag{2.3}$$

DEFINITION 2.4 (viscosity solution for systems). We call the function $u = (u_1, \dots, u_m) \in C(\Omega, \mathbb{R}^m)$ is an L^p-viscosity subsolution of (1.1) provided the equation

$$\mathcal{P}^-(D^2u) - \mu_k(x)|Du_k| \le c_k(x, u) + f_k(x)$$

is satisfied in the viscosity sense for each $k \in \{1, \dots, m\}$.

3. ABP MAXIMUM PRINCIPLE AND WEAK HARNACK INEQUALITY

We assume that system (1.1) is quasi-monotone (or cooperative) in the following sense; for any $u, v \in \mathbb{R}^m$ with $u \ge v$ component-wise and any $k = 1, \dots, m$, we have

$$c_k(x, u) \ge c_k(x, v) \quad \text{for a.e.} x \in \Omega. \tag{3.1}$$

when $u_k = v_k$.

To consider the this problem, we assume also the one of following condition. For each $j \in \{1, \dots, n\}$,

$$\sum_{k=1}^{n} \frac{\partial c_j}{\partial u_k}(x, u) \le 0 \quad \text{a. e. in } \Omega \times \mathbb{R}^m \tag{3.2}$$

or

$$(3.3) \qquad \langle \overline{M}\xi, \xi\rangle \le 0 \quad \text{for all } \xi \in \mathbb{R}^m,$$

where the matrix $\overline{M} = (\overline{m})_{j,k=1}^m$ is defind by

$$(3.4) \qquad \overline{m}_{jk} := \operatorname*{ess.sup}_{\Omega\times\mathbf{R}^m} \frac{\partial c_j}{\partial u_k}(x,u) \qquad (\overline{M}_{jk} \le \nu < \infty).$$

LEMMA 3.1 (c.f Busca-Sirakov). *Assume* (3.1) *and either* (3.2) *or* (3.3). *Then, there is a matrix* $M = (m_{jk}) \in L^\infty(\Omega \times \mathbb{R}^m, M_m(\mathbb{R}))$ *such that*

$$c(x,u) = M(x,u)u$$

satisfying

$$m_{k\ell}(x,u) \ge 0 \qquad \text{for } k \neq \ell, \text{ a.e. } x \in \Omega, u \in \mathbb{R}^m.$$

In addition,

$$\sum_{\ell=1}^m m_{k\ell}(x,u) \le 0 \qquad \text{for all } k = 1,\dots,m$$

in case (3.2), *and*

$$m_{jk}(x,u) \le \overline{m}_{jk} \qquad \text{for all } j,k = 1,\dots,m$$

in case (3.3) *holds.*

THEOREM 3.2 (c.f. [2]). *Assume* (1.4)–(1.5) *and* (3.1). *Let* $u \in C(\overline{\Omega}, \mathbb{R}^m)$ *be an* L^p*-viscosity subsolutions of* (1.1). *Assume also one of* (3.2)–(3.3). *Then the following ABP type inequality holds,*

$$(3.5) \qquad \sup_{\Omega} \bigvee_{k=1}^m u_k \le C\left(\sup_{\partial\Omega} \bigvee_{k=1}^m u_k + \|\bigvee_{k=1}^m f_k\|_{L^p(\Omega)}\right)$$

for some positive constant $C = C(n,p,q,\lambda,\Lambda,\|\mu\|_q, \text{diam }\Omega)$.

Fix $R > 0$ and $z \in \mathbb{R}^n$. Let T, $T' \subset B_R(z)$ be domains such that

$$\overline{T} \subset T', \quad \text{and} \quad \theta_0 \le \frac{|T|}{|T'|} \le 1 \quad \text{for some } \theta_0 > 0.$$

When we apply our weak Harnack inequality below, our choice of T and T' always satisfies the above condition.

For a given domain $A \subset \mathbb{R}^n$ and a function $v \in C(A)$, we define $v^-_{T',A}$ on $T' \cup A$ by

$$v^-_{T',A}(x) = \begin{cases} \min\{v(x), m\} & \text{if } x \in A, \\ m & \text{if } x \in T' \setminus A, \end{cases}$$

where

$$m = \liminf_{x\to T'\cap\partial A} v(x).$$

Next, we recall the boundary weak Harnack inequality when systems have unbounded coefficients and inhomogeneous terms.

LEMMA 3.3 (c.f. [18, Theorem 6.1]). *Assume either* (3.2) *or* (3.3). *Let* T, T', A *be as above. Assume that* $T \cap A \neq \emptyset$ *and* $T' \setminus A \neq \emptyset$. *Then, there exist constants* $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) > 0$, $r = r(n, \lambda, \Lambda, p, q) > 0$ *and* $C_0 = C_0(n, \lambda, \Lambda, p, q) > 0$ *satisfying the following property: if* $f_k \in L^p_+(T' \cap A)$ *(k=1,...,m), a nonnegative* L^p*-viscosity solution* $w \in C(T' \cap A; \mathbb{R}^m)$ *of*

$$\mathcal{P}^+(D^2 w_k) + \mu_k(x)|Dw_k| + c_k(x, w) \geq -f_k(x) \quad \text{in } T' \cap A \quad (k = 1, \dots, m),$$

and

$$\|\mu\|_{L^n(T' \cap A)} \leq \varepsilon_0, \tag{3.6}$$

then it follows that

$$\left(\frac{1}{|T|}\int_T (\bar{w}^-_{T',A})^r \, dx\right)^{1/r} \leq C_0\left(\inf_T \bar{w}^-_{T',A} + R\|f\|_{L^n(T' \cap A)}\right)$$

provided that $q > n$ *and* $q \geq p \geq n$, *and*

$$\left(\frac{1}{|T|}\int_T (\bar{w}^-_{T',A})^r dx\right)^{1/r} \leq C_0\left(\inf_T \bar{w}^-_{T',A} + R^{2-\frac{n}{p}}\|f\|_{L^p(T' \cap A)} \sum_{k=0}^{M} R^{(1-\frac{n}{q})k}\|\mu\|^k_{L^q(T' \cap A)}\right)$$

provided that $q > n > p > p_0$, *where* $\bar{w} = \vee_k w_k$ *and* $M = M(n, p, q)$ *is an positive integer.*

4. PHRAGMÉN-LINDELÖF THEOREM

In this section, first we establish the local and global ABP type estimates on L^p-viscosity subsolutions for (1.1). To this end, we recall the notations concerning the shape of domains from [9].

DEFINITION 4.1 (Local geometric condition). Let $\sigma, \tau \in (0, 1)$. We call $y \in \Omega$ a local weak$G$ point in Ω if there exist $R = R_y > 0$ and $z = z_y \in \mathbb{R}^n$ such that

$$y \in B_R(z), \quad \text{and } |B_R(z) \backslash \Omega_y| \geq \sigma |B_R(z)|, \tag{4.1}$$

where Ω_y is the connected component of $B_{R/\tau}(z) \cap \Omega$ containing y.

For $\sigma, \tau \in (0, 1)$, and $R_0 > 0$, $\eta \geq 0$, we call $y \in \Omega$ a weak$G$ point in Ω if y is a $G_{\sigma,\tau}$ point in Ω with $R = R_y > 0$ and $z = z_y$ satisfying

$$R \leq R_0 + \eta|y|. \tag{4.2}$$

REMARK 4.2. We will write B_y for $B_{\frac{R_y}{\tau}}(z_y)$, where $R_y > 0$ and $z_y \in \mathbb{R}^n$ are from Definition 4.1.

DEFINITION 4.3 (Global geometric condition). We call Ω a weak$G$ domain if all point $y \in \Omega$ is a weak$G$ point in Ω.

We refer the reader to [24] and [9] for examples of domains Ω satisfying weak$G$. We also refer to [1] for a generalization.

We first present pointwise estimate on L^p-viscosity subsolutions of (1.1), which is often refered as the Krylov-Safonov growth lemma.

Let $y \in \Omega$ be a weak$G$ point. It is possible to apply the boundary weak Harnack inequality in B_y if $\|\mu\|_{L^n(B_y\cap\Omega)} \le \varepsilon_0$ where $\varepsilon_0 > 0$ be a constant from Lemma 3.3.

On the other hand, if $\|\mu\|_{L^n(B_y\cap\Omega)} > \varepsilon_0$, we divide B_y into small pieces such that we can apply the boundary weak Harnack inequality for each pieces which called Cabré's covering arguments.

But, this argument does not work immediately because of unboundedness of radius $\{R_y\}_{y\in\Omega}$ when $\eta > 0$ since we need the uniform estimates in $y \in \Omega$.

To avoid this difficulty, we assume a uniform integrability of μ; for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{R>1}\int_E R^n\mu_k(Rx)^n\,dx < \varepsilon \qquad \text{for } E \subset A_{ab}, |E| < \delta. \tag{4.3}$$

where $A_{ab} = \{0 < a < |x| < b < \infty\}$.

REMARK 4.4. Of cause, if $R_y \le R_0$ then we can apply Cabré's cvering argument.

LEMMA 4.5. *Assume that*

$$F_k(x, r, \xi, X) \le F_k(x, r, \xi, Y) \qquad (k = 1, \dots, m) \tag{4.4}$$

for $(x, r, \xi, X, Y) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^n \times S^n \times S^n$ *provided* $X \le Y$, *there is* $\mu_k \in L^q(\Omega)$ *such that*

$$F_k(x, r, \xi, X) \ge \mathcal{P}^-(X) - \mu_k(x)|\xi| - c_k(x, r) \qquad (k = 1, \dots, m) \tag{4.5}$$

for $(x, r, \xi, X) \in \Omega\times\mathbb{R}^m\times\mathbb{R}^n\times S^n$. *Assume also for* $\eta > 0$ *and* $y \in \Omega$ *be a weakG point with radius* $R = R_y > 0$ *and center* $z = z_y \in \mathbb{R}^n$. *Let* $w \in C(\Omega; \mathbb{R}^m)$ *be an* L^p*-viscosity subsolution of* (1.1) *with* $f_k \in L^p(\Omega)$ *for* $k = 1, \dots, m$. *There exist a positive constant* $\kappa = \kappa(n, \lambda, \Lambda, \sigma, \tau, \eta, R_0) \in (0, 1)$ *and* $\varepsilon = \varepsilon(n, \sigma, \eta) > 0$ *satisfies following properties:*

(i) *Asuume that* $R_y \le R_0$ *and* (3.2). *If* $p \ge n$, *then*

$$\bar{w}(y) \le \kappa \sup_{B_y\cap\Omega} \bar{w}^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} \bar{w}^+ + R_0\|f\|_{L^n(B_y\cap\Omega)}, \tag{4.6}$$

and if $p_0 < p < n$,

$$\bar{w}(y) \le \kappa \sup_{B_y\cap\Omega} \bar{w}^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} \bar{w}^+ + R_0^{2-n/p}\|f\|_{L^n(B_y\cap\Omega)} \sum_{k=0}^{M_0} R_0^{(1-n/q)k}\|\mu\|^k_{L^q(B_y\cap\Omega)}. \tag{4.7}$$

(ii) *Asuume that* $R_y \le R_0$ *and* (3.3). *If* $p \ge n$, *then*

$$\tilde{w}(y) \le \kappa \sup_{B_y\cap\Omega} \tilde{w}^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} \tilde{w}^+ + R_0\|f\|_{L^n(B_y\cap\Omega)}, \tag{4.8}$$

and if $p_0 < p < n$,

$$\tilde{w}(y) \le \kappa \sup_{B_y\cap\Omega} \tilde{w}^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} \tilde{w}^+ + R_0^{2-n/p}\|f\|_{L^n(B_y\cap\Omega)} \sum_{k=0}^{M_0} R_0^{(1-n/q)k}\|\mu\|^k_{L^q(B_y\cap\Omega)}. \tag{4.9}$$

(iii) *Asuume that* (4.3), $R_y > R_0$ *and* (3.2). *If* $p \geq n$, *then*

(4.10)
$$\bar{w}(y) \leq \kappa \sup_{B_y \cap \Omega} \bar{w}^+ + (1-\kappa) \limsup_{x \to B_y \cap \partial\Omega} \bar{w}^+ + R\|f\|_{L^n(B_y \cap \Omega \setminus B_{\varepsilon R}(0))},$$

and if $p_0 < p < n$,

(4.11)
$$\bar{w}(y) \leq \kappa \sup_{B_y \cap \Omega} \bar{w}^+ + (1-\kappa) \limsup_{x \to B_y \cap \partial\Omega} \bar{w}^+ + R^{2-n/p}\|f\|_{L^n(B_y \cap \Omega \setminus B_{\varepsilon R}(0))} \sum_{k=0}^{M_0} R^{(1-n/q)k} \|\mu\|^k_{L^q(B_y \cap \Omega \setminus B_{\varepsilon R}(0))}.$$

where $\bar{w}(x) := \vee_k w_k(x)$, $\tilde{w}(x) := \vee_k (w_k^+/\zeta_k \varphi)$ (ζ_k *and* φ *are bounded function apper in the proof) and* M_0 *is the positive integer in Lemma 3.3.*

When Ω be a weak$G$ domain, we derive the following ABP maximum principle for L^p-viscosity subsolutions bounded from abobe of (1.1).

THEOREM 4.6 (ABP maximum principle in unbounded domains). *Assume* (4.4), (4.5) *and* Ω *be a weakG domain. Assume also*

(4.12)
$$\sup_{y \in \Omega, |y| > R_0} R_y \|f\|_{L^n(A_y \cap \Omega)} < \infty \quad \text{if } p \geq n,$$

(4.13)
$$\sup_{y \in \Omega, |y| > R_0} R_y^{2-p/n} \|f\|_{L^n(A_y \cap \Omega)} < \infty \quad \text{if } p_0 < p < n,$$

and $0 < \varepsilon < \min\{1/(1+\eta), (\sigma/4)^{1/n}\}$. *Let* $w \in C(\Omega; \mathbb{R}^m)$ *be an* L^p*-viscosity subsolution bounded from above of* (1.1) *with* $f_k \in L^p(\Omega)$ *for* $k = 1, \ldots, m$. *Then, there exists positive constants*

$$C = C(n, \lambda, \Lambda, m, p, q, \varepsilon, \sigma, \tau, \eta, R_0) > 0$$

satisfying the following properties:
(i)*Assume* (3.2), *if* $p \geq n$

$$\sup_\Omega \bar{w} \leq \limsup_{x \to \partial\Omega} \bar{w} + C\Big(R_0 \sup_{y \in \Omega, |y| \leq R_0} \|f\|_{L^n(B_y \cap \Omega)} + \sup_{y \in \Omega, |y| \leq R_0} R\|f\|_{L^n(B_y \cap \Omega)}\Big)$$

and, if $p_0 < p < n$

$$\sup_\Omega \bar{w} \leq \limsup_{x \to \partial\Omega} \bar{w} + C\Big(R_0^{2-p/n} \sup_{y \in \Omega, |y| \leq R_0} \|f\|_{L^n(A_y \cap \Omega)} \sum_{k=0}^{M_0} R_0^{(1-n/q)k} \|\mu\|^k_{L^q(A_y \cap \Omega)}$$
$$+ \sup_{y \in \Omega, |y| \leq R_0} R\|f\|_{L^n(A_y \cap \Omega)} \sum_{k=0}^{M_0} R^{(1-n/q)k} \|\mu\|^k_{L^q(A_y \cap \Omega)}\Big).$$

(ii)*Assume* (3.3) *and* $\eta = 0$, *if* $p \geq n$

$$\sup_\Omega \bar{w} \leq C\Big(\limsup_{x \to \partial\Omega} \bar{w} + R_0 \sup_{y \in \Omega, |y| \leq R_0} \|f\|_{L^n(B_y \cap \Omega)} + \sup_{y \in \Omega, |y| \leq R_0} R\|f\|_{L^n(B_y \cap \Omega)}\Big)$$

and, if $p_0 < p < n$

$$\sup_{\Omega} \bar{w} \le C\Big(\limsup_{x\to\partial\Omega} \bar{w} + R_0^{2-p/n} \sup_{y\in\Omega, |y|\le R_0} \|f\|_{L^n(A_y\cap\Omega)} \sum_{k=0}^{M_0} R_0^{(1-n/q)k} \|\mu\|^k_{L^q(A_y\cap\Omega)}$$
$$+ \sup_{y\in\Omega, |y|\le R_0} R\|f\|_{L^n(A_y\cap\Omega)} \sum_{k=0}^{M_0} R^{(1-n/q)k} \|\mu\|^k_{L^q(A_y\cap\Omega)}\Big).$$

where $A_y = B_y \backslash B_{\epsilon R_y}(0)$.

PROOF. Taking the supremum over $y \in \Omega$ with the estimates in Lemma 4.5, we conclude the proof. □

THEOREM 4.7. *Assume* (4.4) *and* (4.5).*Let* $w \in C(\Omega : \mathbb{R}^m)$ *is an* L^p*-viscosity subsolution of*

$$F_k(x, w, Dw_k, D^2 w_k) \le 0 \quad \text{in } \Omega, k = 1, \dots, m \tag{4.14}$$

such that

$$\limsup_{x\to\partial\Omega} (\vee_{k=1}^m w_k) \le 0.$$

There exist a positive constant $\beta > 0$ *such that*
(case 1) if Ω *be a* G *domain , either* (3.2) *or* (3.3) *holds and*

$$(\vee_{k=1}^m w_k)^+ = o(e^{\beta|x|}) \quad \text{as } |x| \to \infty, \tag{4.15}$$

(case 2) if Ω *be a weakG domain,* (3.2) *and* (3.1) *holds and*

$$(\vee_{k=1}^m w_k)^+ = o(|x|^{\beta}) \quad \text{as } |x| \to \infty, \tag{4.16}$$

then $\vee_{k=1}^m w_k \le 0$ *in* Ω.

5. PROOF OF PHRAGMÉN-LINDELÖF THEOREM

We will only consider G domain. Let $\phi : [0, \infty) \to \mathbb{R}$ be a non-decreasing function. Setting $\Phi(x) = \phi(|x|)$, if we define $u(x) = w(x)/\Phi(x)$, then w is bounded from above. Since ϕr is a positive non-decreasing function of r, we have

$$\mathcal{P}^-(D^2\Phi(x)) = -\frac{(n-1)\Lambda}{|x|}\phi' - \lambda\phi.$$

Therefore, u is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 u_k) - \gamma(x)|Du_k| - \frac{1}{\phi}c_k(x, \phi u_k) \le g(x) u_k^+(x) \quad k = 1, \dots, m$$

where

$$\gamma(x) := 2\Lambda\frac{\phi'}{\phi} + \mu(x)$$

and

$$g(x) := \lambda\frac{\phi''}{\phi} + (\Lambda\frac{n-1}{|x|} + \mu(x))\frac{\phi'}{\phi}.$$

By Lemma 3.1, we linialized the zero order term c_k in this system. Then u is an L^p-viscosity subsolutions of

$$\mathcal{P}^-(D^2u_k) - \gamma(x)|Du_k| - \frac{1}{\phi}\sum_{\ell}^{m} m_{k\ell}(x, \phi u)u_\ell \le g(x)u_k^+(x), \tag{5.1}$$

for any $k = 1, \dots, m$.

Since $m_{k\ell}(x, \phi u(x))u_\ell \le m_{k\ell}(x, \phi u(x))u_\ell^+$ for $(k \neq \ell)$ from (??), the functions $v = u_k, 0$ are L^p-viscosity solutions of

$$\mathcal{P}^-(D^2u_k) - \gamma(x)|Du_k| - \frac{1}{\phi}m_{kk}(x, \phi u(x))v \le g(x)u_k^+(x) + \frac{1}{\phi}\sum_{k\neq\ell} m_{k\ell}(x, \phi u(x))u_\ell^+. \tag{5.2}$$

So maximum of two functions $u_k^+ = \max\{u_k, 0\}$ be an L^p-viscosity solutions of

$$\mathcal{P}^-(D^2u_k) - \gamma(x)|Du_k| - \frac{1}{\phi}\sum_{\ell}^{m} m_{k\ell}(x, \phi u(x))u_\ell^+ \le g(x)u_k^+(x). \tag{5.3}$$

for $k = 1, \dots, m$.

Set $\phi(r) = e^{\beta(1+r^2)^{1/2}}$ with $\beta \in [0, \beta_0]$ to be chosen in sequal. Applying the ABP maximum principle to (6.1), if $p \ge n$,

$$\sup_{\Omega} \bar{u} \le CR_0 \sup_{y\in\Omega} \|g\bar{u}^+\|_{L^n(B_y\cap\Omega)} \le CR_0\beta K_0 \sup_{\Omega} \bar{u}^+$$

for some positive constant K_0. Here $\bar{u} = \vee_k u_k$. Taking $\beta_0 > 0$ small enough, we have $\bar{u} \le 0$ in Ω, which implies $\vee_k w_k \le 0$, which conclude the proof.

6. Proof of ABP estimate in unbounded domain

In this paper, we will only consider (3.2). Using the same arguments of proof of Phragmén-Lindelöf theorem, we can check that the function $u = (u_1, \dots, u_m)$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2u_k) - \mu(x)|Du_k| - \sum_{\ell}^{m} m_{k\ell}(x, u(x))u_\ell^+ \le f_k^+(x) \quad \text{in } \Omega \tag{6.1}$$

for $k = 1, \dots, m$.

Idea of proof is the function $v(x) \equiv \vee_{k=1}^m u_k(x)$ satisfying a fully nonlinear elliptic equation.

Claim Under (3.2), the function $\bar{w}$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2\bar{w}) - \mu(x)|D\bar{w}| \le (\vee_{k=1}^m f_k(x)) = f(x) \quad \text{in } \Omega.$$

Proof of Claim. Assume contrary, there exists $\theta > 0$, open ball $B_S(x_0) \subset \mathbb{R}^n$ with radius $S > 0$ and a test function $\psi \in W^{2,p}(B_{2S}(x_0))$ with $0 = (\bar{w} - \psi)(x_0) \ge (\bar{w} - \psi)(x)$ $(x \in B_S(x_0))$ such that

$$\mathcal{P}^-(D^2u_k) - \mu(x)|Du_k| \ge f(x) + 2\theta > 0 \quad \text{in } B_S(x_0). \tag{6.2}$$

Fixed k with $u_k^+(x_0) = v(x_0)$, then we see that

$$0 = (u_k^+ - \psi)(x_0) \ge (u_k^+ - \psi)(x) \quad (x \in B_S(x_0)).$$

If $\psi(x_0) = 0$, then the point x_0 is a local minimum point of ψ. By strong maximum principle of Pucci extremal equation, we obtain $\psi \equiv 0$ in $B_S(x_0)$. Which contradicts (6.2).

If not $\psi(x_0) = 0$, i.e. $u_k(x_0) = \psi(x_0) > 0$, then there exists radius $r > 0$ such that

$$u_k > 0 \quad \text{and } u_k > u_j - \frac{\theta}{\nu} \quad \text{in } B_r(x_0).$$

$$\begin{aligned}\mathcal{P}^-(D^2\psi) - \mu(x)|D\psi| &\geq f + 2\theta \\ &\geq f_k^+ + 2\theta \\ &\geq \frac{1}{\phi}\left(\sum_{\ell=1}^m m_{k\ell}\right) u_k + f_k^+ + 2\theta \\ &\geq \frac{1}{\phi}\sum_{\ell=1}^m m_{k\ell}u_\ell^+ + f_k^+ + \theta,\end{aligned}$$

where we use following estimates;

$$\sum_{i\neq j} m_{ij}(x,u) = \sum_{i\neq j}\int_0^1 \frac{\partial c_i}{\partial u_j}(x,su)\,ds \leq \int_0^1 \sum_{i\neq j}\left|\frac{\partial c_i}{\partial u_j}(x,su)\right| ds \leq \nu \quad \text{for } i = 1,\ldots,m.$$

On the other hand, function u_k is also an L^p-viscosity subsolution of (6.1), which is contradiction.

Here we prove the point wise estimates. It is enough to show the assertion when $0 = \hat{C} := \limsup_{B_y\cap\partial\Omega} \bar{w}^+(x)$. In fact, after having established the assertion when $\hat{C} = 0$, we may apply the result to $\bar{w} - \hat{C}$ to prove the assertion in general case.

Case 1: $R_y \leq (1+\eta)R_0$ or $|y| \leq R_0$

In this case, $B_y = B_{R_y/\tau}(z_y)$ is bounded. The functions $\bar{w}$ and $\tilde{w}$ satisfies

$$\mathcal{P}^-(D^2\bar{w}) - \mu(x)|D\bar{w}| \leq f^+(x) \quad \text{in } B_y,$$

in case (3.2) and

$$\mathcal{P}^-(D^2\tilde{w}) - (\gamma + \mu(x))|D\tilde{w}| \leq f^+(x) \quad \text{in } B_y,$$

in case (3.3) in the L^p-viscosity sense for some positive constant γ. We can use the standard covering arguement by Cabré. Setting $T = B_{R_y}(z_y), T' = B_y$ and $A = \Omega_y$, we have

$$|T\backslash A| = |B_{R_y}(z_y)\backslash\Omega_y| \geq \sigma|B_{R_y}(z_y)| \geq \frac{\sigma}{2}|T|.$$

We shall only give a proofs when $\|\mu\|_{L^n(T'\cap A)} \leq \varepsilon_0$ in case (3.2), or $\|\gamma+\mu\|_{L^n(T'\cap A)} \leq \varepsilon_0$ in case (3.3). Let $w = \bar{w}$ or $\tilde{w}$. For any $r > 0$, we see that

$$\begin{aligned}\left(\frac{\sigma}{2}\right)^{\frac{1}{r}} C_w &\leq \left(\frac{|T\backslash A|}{|T|}\right)^{\frac{1}{r}} C_w \\ &\leq \left(\frac{1}{|T|}\int_{T\backslash A} m^r\,dx\right)^{\frac{1}{r}} \\ &\leq \left(\frac{1}{|T|}\int_T (v_{T',A}^-)^r\,dx\right)^{\frac{1}{r}},\end{aligned}$$

where $m = \liminf_{x \to T' \cap \partial A} v(v)$.

Since $y \in A$, we have

$$\inf_T v^-_{T',A} \leq v(y) = C_w - w(y).$$

Hence, taking $r > 0$ for the constant from weak Harnack inequarity, we have

$$\left(\frac{\sigma}{2}\right)^{\frac{1}{r}} C_w \leq C_0 \left(\inf_T v^-_{T',A} + R\|f\|_{L^n(T' \cap A)}\right)$$
$$\leq C_0 \left(C_w - w(y) + R\|f\|_{L^n(T' \cap \Omega)}\right).$$

Therefore, we conclude that the case (i) holds for $\kappa = 1 - (\sigma/2)^{\frac{1}{r}} \min\{C_0^{-1}, 1\}$.

Case 2 : $R_y > (1+\eta)R_0$ and $|y| > R_0$

Under the assumption (4.3), we can show it as the same argument case (i) similarly.

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MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY
6-3, AOBA, ARAMAKI, AOBA-KU, SENDAI 980-8578, JAPAN
E-mail address: knakagawa@math.tohoku.ac.jp