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Iterated proper forcing with side conditions

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Abstract

We formulate an iterated proper forcing with side conditions. We start with a measurable cardinal. Then we construct our iterated proper forcing along a sequence of increasing transitive models. The measurable cardinal is turned into the second uncountable cardinal. This is a rendition of one of Neeman's original constructions with two types of models.

Introduction

Neeman introduced a new method of iterating proper forcing in [N]. The new method makes use of models of set theory and is of finite in nature. This note is a rendition of a small fraction of [N]. We start with a measurable cardinal κ. We formulate an iterated forcing along a sequence of transitive models of set theory that cofinally $\in$-increase in H_κ. Hence it looks like an ordinary iterated forcing with small initial segments. However in this new iteration projections are locally done. Each condition has limited initial segments and so limited access to intermediate stages.

There is a similar construction that adds many reals by iterating a class of proper forcing in [AM]. Their iteration takes notions of forcing stronger than just being proper and uses systems of countable elementary substructures with markers so that the iteration is proper. Their conditions have every initial segment and so access to every intermediate stage.

Since these new methods are of finite in nature, reals are to be added or even intended to add. Hence it appears that preserving, say the Continuum Hypothesis, there remain roles of countable support iterated proper forcing. The iterations in this note preserve ω_1 and automatically collapse the cadinals trictly between ω_1 and κ. And κ is turned into new ω_2. Hence, the idea of forming iterated forcing along a sequence of transitive models, as in this note, does not work in contexts of higher analogues of proper forcing, where both ω_1 and ω_2 are intended to be preserved with new ω_3.

Preliminary

The items in this section are in use throughout this note without being metioned. Let κ be a regular uncountable cardinal. Let $H_\kappa = \{x \mid |\mathrm{TC}(x)| < \kappa\}$, where $\mathrm{TC}(x)$ denotes the transitive closure of x. Let $X \prec H_\kappa$ abbreviate that $(X, \in)$ is an elementary substructure of $(H_\kappa, \in)$. Typically X is countable but not restricted in general. Hence we may consider $X = H_\theta \prec H_\kappa$, where θ would be a singular cardinal. Let P be a preorder (i.e. reflexive and transitive) and $P \in X \prec H_\kappa$. Then we denote $X[G] = \{\tau_G \mid \tau \in X$ is a P-name$\}$, where G is any P-generic filter over the ground model V.

Lemma A. Let P be a preorder and κ be a regular uncountable cardinal with $P \in H_\kappa$. Let $P \in X \prec H_\kappa$. Then

(1) For any P-name τ, there exists a P-name $\pi \in H_\kappa$ such that $\Vdash_P$"if $\tau \in (H_\kappa)^{V[G]}$, then $\tau = \pi$".

(2) And so $\Vdash_P$"$H_\kappa[G] = (H_\kappa)^{V[G]}$".

(3) For any $\in$-formula $\phi(y, v)$ and any P-name $\tau \in X$, there exists a P-name $\pi \in X$ such that $\Vdash_P$"$H_\kappa[G] \models$ "$\forall y\, (\phi(y, \tau) \longrightarrow \phi(\pi, \tau))$""

(4) And so $\Vdash_P$"$X \cup \{G\} \subseteq X[G] \prec H_\kappa[G]$".

The following contents may be improved but suffices for this note.

Lemma B. Let P be a preorder and κ be a regular uncountable cardinal. Let θ be a cardinal (may or may not be regular). If θ is singular, then think of $H_\theta = \bigcup\{H_\chi \mid \chi < \theta, \chi$ is regular$\}$. Let $P \in H_\theta \in N \prec H_\kappa$, $P \in N$ and N be countable. Then

(1) $P \in N \cap H_\theta \prec H_\theta$.

(2) $\Vdash_P$ "$H_\theta[G] \in N[G] \prec H_\kappa[G] = (H_\kappa)^{V[G]}$".

(3) $\Vdash_P$ "$N[G] \cap H_\theta[G] \prec H_\theta[G] = (H_\theta)^{V[G]}$".

(4) For any P-name $\tau \in N$, there exists a P-name $\pi \in N \cap H_\theta$ such that $\Vdash_P$ "if $\tau \in (H_\theta)^{V[G]}$, then $\tau = \pi$".

(5) And so $\Vdash_P$ "$N[G] \cap H_\theta[G] = (N \cap H_\theta)[G]$".

Furthermore, let $\dot{Q} \in N \cap H_\theta$ and $\Vdash_P$ "$\dot{Q}$ be a preorder". Let θ be singular such that $\underline{H_\theta \prec H_\kappa}$. Then

(6) $\Vdash_P$ "$\dot{Q} \in N[G] \prec H_\kappa[G]$".

(7) $\Vdash_P$ "$\dot{Q} \in (N \cap H_\theta)[G] = N[G] \cap H_\theta[G] \prec \underline{H_\theta[G] \prec H_\kappa[G]}$".

(8) $\Vdash_P$ "q is $(\dot{Q}, N[G])$-generic iff q is $(\dot{Q}, (N \cap H_\theta)[G])$-generic".

In the following, we state it with two countable elementary substructures for simplification.

Lemma C. Let κ be a regular uncountable cardinal and θ be of uncountable cofinality. Let $H_\theta \in N \in M$, $N \prec H_\kappa$, $M \prec H_\kappa$ and both N and M be countable. Then $N \subset M$ and $N \cap H_\theta \in M \cap H_\theta \in H_\theta$ holds.

The following is related to atomlessness of relevant preorders and hence adding Cohen reals.

Lemma D. Let κ be a measurable cardinal. Let $f : [H_\kappa]^{<\omega} \longrightarrow H_\kappa$. Then

(1) There exists a regular uncountable cardinal $\theta < \kappa$ such that H_θ is closed under f.

(2) For any $X \in H_\kappa$ and $a \in X$, there are two countable sets M_1, M_2 of H_κ such that M_1, M_2 are closed under f, $a \in M_1 \cap M_2$, $M_1 \cap X = M_2 \cap X$ and $M_1 \cap \kappa \neq M_2 \cap \kappa$.

§ 1. Contructions of Iterations

In the rest of this note, we assume that κ is a strongly inaccessible cardinal and denote $K = H_\kappa$. Let

$$T = \{H_\theta \in K \mid H_\theta \prec K,\ [H_\theta]^\omega \subset H_\theta\}$$

We also assume that $\{H_\theta \in T \mid \theta \text{ is regular uncountable}\}$ is $\in$-cofinal in K. We indicate when we further assume that κ is measurable. Let us denote

$$S = \{N \in K \mid N \text{ is countable},\ N \prec K\}.$$

The elements of T and S are called of *transitive type* (or *rank type*) and of *small type* (or *countable type*), respectively. Hence we are concerned with the elementary substructures, of transitive or countable, of K.

We construct a preorder P_κ that is the direct limit of an iterated forcing $\langle P_i \mid i < \kappa\rangle$. This $\langle P_i \mid i < \kappa\rangle$ has associated sequences $\langle K_i \mid i < \kappa\rangle$ and $\langle \dot{Q}_i \mid i < \kappa\rangle$. The sequence $\langle K_i \mid i < \kappa\rangle$ is $\in$-increasing and $\in$-cofinal in T and so in K. The sequence $\langle \dot{Q}_i \mid i < \kappa\rangle$ lists names such that for each $i < \kappa$, we have $\Vdash_{P_i}$ "$\dot{Q}_i$ is proper" and the associated two stage iteration $P_i * \dot{Q}_i$ is formed as a preorder in your favorite manner. We have $P_i \subset K_i$ and $P_i, \dot{Q}_i, P_i * \dot{Q}_i \in K_{i+1} \prec K$ for all $i < \kappa$. The finally constructed $P_\kappa = \bigcup\{P_i \mid i < \kappa\}$ does not satisfy the κ-chain condition. We demand the continuity as in (2) below so that P_κ preserves κ to be a cardinal. By the properness of P_κ, the least uncontable cardinal ω_1 is preserved and every cardinal strictly between ω_1 and κ is collapsed. Hence, we have $\kappa = (\omega_2)^{V[G_\kappa]}$, where G_κ is any P_κ-generic filter over the ground model V.

Lemma 1.1. We may construct sequences $I = \langle (K_i, P_i, \dot{Q}_i, P_i * \dot{Q}_i) \mid i < \kappa\rangle$ such that for each $i < \kappa$

(0) $I \lceil i = \langle (K_j, P_j, \dot{Q}_j, P_j * \dot{Q}_j) \mid j < i\rangle \in K$, $\langle K_j \mid j < i\rangle$ is an $\in$-increasing sequence of elements in T, $\langle P_j \mid j < i\rangle$ is a sequence of preorders and $\langle \dot{Q}_j \mid j < i\rangle$ is a sequence of names such that for each $j < i$, we have $\Vdash_{P_j}$ "$\dot{Q}_j$ is a proper preorder" and the associated two stage iteration $P_j * \dot{Q}_j \equiv \{(p, \tau) \mid p \in P_j,$ $\Vdash_{P_j}$ "$\tau \in \dot{Q}_j$"$\}$ is formed as a preorder in your favorite manner.

(1) If i is 0, a successor ordinal or a limit ordinal with $\mathrm{cf}(i) = \omega$, then $I\lceil i \in K_i \in T$ and θ_i is a regular uncountable cardinal, where $K_i = H_{\theta_i}$.

(2) If i is a limit ordinal with $\mathrm{cf}(i) \geq \omega_1$, then $K_i = \bigcup\{K_j \mid j < i\}$.

(3) (Basic Closure) If $m < i$, then $I\lceil(m+1) = \langle (K_j, P_j, \dot{Q}_j, P_j * \dot{Q}_j) \mid j \leq m \rangle \in K_i$.

(4) P_i is a preorder such that for each element $p \in P_i$, p is a pair and $P_i \subset K_i$ holds.

(5) For $p = (f^p, A^p)$, $p \in P_i$ iff

- A^p is a finite $\in$-chain, $A^p = T^p \cup S^p$, $T^p \subset \{K_j \mid j < i\}$ and $S^p \subset S \cap K_i$.
- (Basic Closure) If $N \in S^p$ and $K_j \in N$ (note that we do not talk about $j \in N$, because N may contain j bigger than i and that K_j may or may not be in T^p), then $I\lceil(j+1) \in N$.
- (Intersection Property) For all $X, Y \in A^p$, $X \cap Y \in A^p$.
- f^p is a finite function with $\mathrm{dom}(f^p) = \{j < i \mid K_j \in A^p\}$.
- (Local Projection) For $j \in \mathrm{dom}(f^p)$, we have $w(p,j) \in P_j$, $(w(p,j), f^p(j)) \in P_j * \dot{Q}_j$ and $\Vdash_{P_j}$ "$f^p(j) \in \dot{Q}_j$", where $w(p,j) = (f^p\lceil j, A^p \cap K_j)$.
- (Generic) For $j \in \mathrm{dom}(f^p)$ and $N \in S^p$ with $K_j \in N$, $\Vdash_{P_j}$ "$f^p(j)$ is $(\dot{Q}_j, N[G_j])$-generic".

(6) For $p, q \in P_i$, $q \leq p$ in P_i iff

- $A^q \supseteq A^p$.
- For $j \in \mathrm{dom}(f^p)$, $w(q,j) \Vdash_{P_j}$ "$f^q(j) \leq f^p(j)$ in $\dot{Q}_j$".

(7) $\Vdash_{P_i}$ "$\dot{Q}_i \in K[G_i]$ is proper" (whose exact choice depend on the purposes) and the associated two stage iteration $P_i * \dot{Q}_i \in K$ formed.

Proof. Here is a recursive construction of K_i, P_i, $\dot{Q}_i$ and $P_i * \dot{Q}_i$. Suppose $i < \kappa$ and we have constructed $I\lceil i = \langle (K_j, P_j, \dot{Q}_j, P_j * \dot{Q}_j) \mid j < i \rangle$.

To construct K_i, pick any sufficiently large $K_i \in T$ as in (1), unless $i = 0$ or $\mathrm{cf}(i) \geq \omega_1$. As far as K_0 is concerned, we are free to choose any $K_0 = H_{\theta_0} \in T$ with a regular uncountable cardinal θ_0. To construct P_i, let $p = (f^p, A^p) \in P_i$, if either the following (I) or (II) holds.

(I) $f^p = \emptyset$ and A^p is a finite $\in$-chain such that

- $A^p \subset S \cap K_i$.
- (Basic Closure) If $N \in A$ and $K_m \in N$, then $I\lceil(m+1) \in N$.

(II) There exist $q \in P_j$, $j < i$, τ and A such that

- $(q, \tau) \in P_j * \dot{Q}_j$ and $\Vdash_{P_j}$ "$\tau \in \dot{Q}_j$".
- A is a finite $\in$-chain such that for all $N \in A$, $K_j \in N \in S \cap K_i$.
- (Basic Closure) If $N \in A$ and $K_m \in N$, then $I\lceil(m+1) \in N$.
- (Weak Intersection Property) For all $N \in A$, $K_j \cap N \in A^q$.
- (Weak Generic) For all $N \in A$, $\Vdash_{P_j}$ "τ is $(\dot{Q}_j, N[G_j])$-generic".
- $f^p = f^q \cup \{(j, \tau)\}$ and $A^p = A^q \cup \{K_j\} \cup A$.

For $q, p \in P_i$, define $q \leq p$ in P_i, if

- $A^q \supseteq A^p$.
- For all $j \in \mathrm{dom}(f^p)$, $w(q,j) \leq w(p,j)$ in P_j and $w(q,j) \Vdash_{P_j}$ "$f^q(j) \leq f^p(j)$ in $\dot{Q}_j$". Namely,

$$(w(q,j), f^q(j)) \leq (w(p,j), f^p(j))$$

in $P_j * \dot{Q}_j$.

To construct $\dot{Q}_i$ and $P_i * \dot{Q}_i$, let $\dot{Q}_i$ and $P_i * \dot{Q}_i$ be any as in (7). This completes the recursive construction of K_i, P_i, $\dot{Q}_i$ and $P_i * \dot{Q}_i$. We have to show that they satisfy the induction hypotheses (0) through (7). Since it is quite routine, we make remarks rather than putting details.

For (0),(1) and (2): Due to the assumption on κ and T, these are satisfied. Notice that there exists a lot of freedom in choosing K_i for $i = 0$, successors or limits with countable cofinality. Also notice that θ_i has no choice other than specified and may not be regular, when $\mathrm{cf}(i) \geq \omega_1$ but $K_i \in T$ holds.

For (3): If i is a limit with $\mathrm{cf}(i) \geq \omega_1$ and $m < i$, then $I\lceil(m+1) \in K_{m+1} \in K_i$ by induction.

For (4): We have to wait for (5). But $P_i \subset K_i \in K_{i+1} \in K$. Hence the initial segements are all small.

For (5): Item (3) (Basic Closure) is a prerequisite to (Basic Closure) on N in (5). (Weak Intersection Property) entails (Intersection Property) by induction. We liked to consider $\mathrm{dom}(f^p) = \{j < i \mid K_j \in A^p\}$ over $\mathrm{dom}(f^p) = T^p$ in our formulation. We demand $\mathrm{dom}(f^p) = \{j < i \mid K_j \in A^p\} = \{j < i \mid K_j \in T^p\}$, the whole indices of transitive type in A^p. We also demand $\Vdash_{P_j}$ "$f^p(j)$ is $(\dot{Q}_j, N[G_j])$-generic" with the Boolean value one. These two for simplification. The witness $w(p,j)$ defined if and only if $K_j \in T^p$.

For (6): This equivalence is to be used to establish $q \leq p$.

For (7): $P_i \in H_{|\mathrm{TC}(P_i)|^+} \in K_{i+1} \prec K$ and $\Vdash_{P_i}$ "$\dot{Q}_i \in (H_{|\mathrm{TC}(\dot{Q}_i)|^+})^{V[G_i]} \in K_{i+1}[G_i] \prec K[G_i]$". The minimal spaces to talk about generic conditions are available as points in K_{i+1} and $K_{i+1}[G_i]$, respectively.

□

Lemma 1.2. Let $I = \langle (K_i, P_i, \dot{Q}_i, P_i * \dot{Q}_i) \mid i < \kappa \rangle$ be as above. Then for each $i < \kappa$

(1) If $j < i$, then P_j is a suborder of P_i.

(2) If $\mathrm{cf}(i) \geq \omega_1$, then $P_i = \bigcup\{P_j \mid j < i\}$.

(3) (Local Projection) Let $p \in P_i$ and $K_j \in A^p$. Then

$$P_j * \dot{Q}_j \lceil (w(p,j), f^p(j)) \longleftarrow P_i \lceil p$$

defined by $x \mapsto (w(x,j), f^x(j))$ is a projection. Namely,

- (Order) If $x \leq y \leq p$ in P_i, then $(w(x,j), f^x(j)) \leq (w(y,j), f^y(j)) \leq (w(p,j), f^p(j))$ in $P_j * \dot{Q}_j$.
- ("Reduction") If $y \leq p$ in P_i and $(h,\pi) \leq (w(y,j), f^y(j))$ in $P_j * \dot{Q}_j$, then there exists $x \leq y$ in P_i such that $((w(x,j), f^x(j)) \leq (h,\pi)$.

Proof. For (1): Directly by the recursive definition. Let $j < i < \kappa$ and $p \in P_j$.

Case (I): $p = (\emptyset, A^p)$. Since $A^p \subset S \cap K_j \subset S \cap K_i$, we are done.

Case (II): $p \in P_j$ is constructed from some $r \in P_l$, $l < j$, τ and A. Since $l < j < i$ and so $l < i$, we conclude $p \in P_i$.

Hence $P_j \subset P_i$. Let $p, q \in P_j$. It is straightforward to observe that $q \leq p$ in P_j iff $q \leq p$ in P_i.

For (2): Let $p \in P_i$ and $\mathrm{cf}(i) \geq \omega_1$. Assume we are in case (II). Then p is constructed from some $q \in P_j$, $j < i$, τ and A. Since A is finite and $S \cap K_i = S \cap \bigcup\{K_j \mid j < i\}$, we conclude $p \in P_m$ for some $m < i$.

For (3): (Well-def) Let $x \leq p$ in P_i. Then $K_j \in A^p \subseteq A^x$. Hence $w(x,j) \in P_j$, $(w(x,j), f^x(j)) \in P_j * \dot{Q}_j$ and $\Vdash_{P_j}$ "$f^x(j) \in \dot{Q}_j$". We also have $(w(x,j), f^x(j)) \leq (w(p,j), f^p(j))$ in $P_j * \dot{Q}_j$.

(Order) Let $x \leq y$ in $P_i \lceil p$. Then by definition, $w(x,j) \leq w(y,j)$ in P_j and $w(x,j) \Vdash_{P_j}$ "$f^x(j) \leq f^y(j)$ in $\dot{Q}_j$". This means $(w(x,j), f^x(j)) \leq (w(y,j), f^y(j))$ in $P_j * \dot{Q}_j$.

("Reduction") Let $(h,\pi) \leq (w(y,j), f^y(j))$ in $P_j * \dot{Q}_j$. Then let π^* be a P_j-name such that $\Vdash_{P_j}$ "if $h \in G_j$, then $\pi^* = \pi$, else $\pi^* = f^y(j)$" and so $\Vdash_{P_j}$ "$\pi^* \in \dot{Q}_j$". Then for all $M \in (K_j, K_i) \cap S \cap A^y$, we have $\Vdash_{P_j}$ "π^* is $(\dot{Q}_j, M[G_j])$-generic". Let $x = (f^x, A^x)$, where

- $A^x = A^h \cup A^y$.

- $f^x = f^h \cup \{(j, \pi^*)\} \cup f^y \lceil (j,i)$.

Then we may confirm $x \in P_i$ and $x \leq y$ in P_i by possible repeated uses of (II). We also have $(w(x,j), f^x(j)) = (h, \pi^*) \leq (h, \pi)$.

□

Definition 1.3. Let I be as above. Let us write $K_\kappa = K$. Let $P_\kappa = \bigcup\{P_i \mid i < \kappa\}$ and for $p, q \in P_\kappa$, let $q \leq p$ in P_κ, if $q \leq p$ in some (all) P_i with $p, q \in P_i$.

We may view P_κ constructed from $I = I\lceil\kappa$ as in the recursive construction of I with $i = \kappa$. But $\dot{Q}_\kappa$ is not yet. We study the preorder P_κ in the next section.

§ 2. Amalgamations

In this section, whenever we write $M \prec H_\chi$, we mean M is countable and χ is a regular uncountable cardinal. Given $p \in P_i$, we may add new $N \in S$ to S^p.

Lemma 2.1. Let $i \leq \kappa$ and let $I\lceil i, K_i, P_i \in M \prec H_\chi$ so that $M \cap K_i \in S \cap K_i$ satisfies (Basic Closure). Let $p \in M \cap P_i$. Then there exists $q \in P_i$ such that $q \leq p$ in P_i and $M \cap K_i \in A^q$.

Proof. We define $q = (f^q, A^q)$ and $A^q = T^q \cup S^q$, where

(1) $T^q = T^p$.

(2) $S^q = S^p \cup \{M \cap K_l \mid K_l \in T^p \cup \{K_i\}\}$.

Let $\langle l \mapsto \tau_l \mid K_l \in T^p \rangle$ satisfy

(3) $\Vdash_{P_l}$ "$\tau_l \leq f^p(l)$ in $\dot{Q}_l$ and τ_l is $(\dot{Q}_l, M[G_l])$-generic"

Let

(4) $f^q = \{(l, \tau_l) \mid K_l \in T^p\}$.

Then it is routine to check that $q = (f^q, A^q) \in P_i$, $q \leq p$ in P_i and $M \cap K_i \in A^q$.

□

Given $q \in P_i$, we consider an appropriate copy of q in $M \cap P_i$ and possibly an extension r of it in $M \cap P_i$, where $P_i \in M \prec H_\chi$ and $M \cap K_i \in S^q$. Then we may cook a common extension of q and r in P_i, which we call an *amalgamation.* There are four cases depending on how $M \cap K_i$ is listed in A^q.

Lemma (Amalgamation 00). Let $i \leq \kappa$ and let $I\lceil i, K_i, P_i \in M \prec H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

(00) $A^q \cap T = \emptyset$.

(2) $S_0 = N \cap A^q$ and $S_1 = [N, K_i) \cap A^q$.

(3) $r \in M \cap P_i$ and $S_0 \subseteq A^r$.

Then r and q are compatible in P_i.

Proof. We define $e = (f^e, A^e)$ with $A^e = T^e \cup S^e$. Let

- $T^e = T^r$.
- $S^e = S^r \cup \{\overline{N} \cap K_l \mid \overline{N} \in S_1, K_l \in T^r\} \cup S_1$.

Fix $\langle l \mapsto \tau_l \mid K_l \in T^r \rangle$ such that

- $\Vdash_{P_l}$ "$\tau_l \leq f^r(l)$ in $\dot{Q}_l$ and for all $\overline{N} \in S_1$, τ_l is $(\dot{Q}_l, \overline{N}[G_l])$-generic".
- $f^e = \{(l, \tau_l) \mid K_l \in T^r\}$.

Then it is routine to check that $e = (f^e, A^e) \in P_i$ and that $e \leq r, q$ in P_i.

□

Lemma (Amalgamation 01). Let $i \leq \kappa$ and let $I\lceil i, K_i, P_i \in M \prec H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

(01) $K_{j^*} \in A^q$, $K_{j^*} \cap A^q \cap T = \emptyset$ and $N \in K_{j^*} \cap A^q$.

(2) $S_0 = N \cap A^q$ and $S_1 = [N, K_{j^*}) \cap A^q$.

(3) $r \in M \cap P_i$ and $S_0 \subseteq A^r$.

Then r and q are compatible in P_i.

Proof. We may use (Amalgamation 00). Let us consider $w(q, j^*)$ and r. Apply (Amalgamation 00) to get $e \in P_i$ with $e \leq w(q, j^*), r$ in P_i. Since $e \leq w(q, j^*)$ in P_{j^*} holds, we may take $e' \in P_i$ with $e' \leq q$ in P_i and $w(e', j^*) = e$. Then $e' \leq w(e', j^*)$ in P_i and so $e' \leq q, r$ in P_i holds.

□

Lemma (Amalgamation 10). Let $i \leq \kappa$ and let $I\lceil i, K_i, P_i \in M \prec H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

(10) $K_j \in A^q$, $(K_j, K_i) \cap A^q \cap T = \emptyset$ and $N \in (K_j, K_i)$.

(2) $S_0 = (K_j, N) \cap A^q$ and $S_1 = [N, K_i) \cap A^q$.

(3) $r \in M \cap P_i$ and $\{K_j\} \cup S_0 \subseteq A^r$.

(4) (Head) $\big(w(q, j), f^q(j)\big)$ and $\big(w(r, j), f^r(j)\big)$ are compatible in $P_j * \dot{Q}_j$.

Then r and q are compatible in P_i.

Proof. We define $e = (f^e, A^e)$ with $A^e = T^e \cup S^e$. Let

- $(h, \pi) \leq \big(w(q, j), f^q(j)\big)$, $\big(w(r, j), f^r(j)\big)$ in $P_j * \dot{Q}_j$.

Let

- $T^e = T^h \cup (T^r \setminus K_j)$.
- $S^e = S^h \cup S^r \cup \{\overline{N} \cap K_l \mid \overline{N} \in S_1, K_l \in T^r \setminus (\{K_j\} \cup K_j)\} \cup S_1$.

Let

- $\Vdash_{P_j}$ "$\pi^* \in \dot{Q}_j$ and $\pi^* \leq f^q(j)$".
- $h \Vdash_{P_j}$ "$\pi^* = \pi$".
- And so for all $\overline{N} \in S_1$, $\Vdash_{P_j}$ "π^* is $(\dot{Q}_j, \overline{N}[G_j])$-generic".

Fix $\langle l \mapsto \tau_l \mid K_l \in T^r \setminus (\{K_j\} \cup K_j) \rangle$ such that

- $\Vdash_{P_l}$ "$\tau_l \leq f^r(l)$ in $\dot{Q}_l$ and for all $\overline{N} \in S_1$, τ_l is $(\dot{Q}_l, \overline{N}[G_l])$-generic".
- $f^e = f^h \cup \{(j, \pi^*)\} \cup \{(l, \tau_l) \mid K_l \in T^r \setminus (\{K_j\} \cup K_j)\}$.

Then it is routine to check that $e = (f^e, A^e) \in P_i$ and that $e \leq r, q$ in P_i.

□

Lemma (Amalgamation 11). Let $i \leq \kappa$ and let $I\lceil i, K_i, P_i \in M \prec H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

(11) $K_j, K_{j^*} \in A^q$, $(K_j, K_{j^*}) \cap A^q \cap T = \emptyset$ and $N \in (K_j, K_{j^*}) \cap A^q$.

(2) $S_0 = (K_j, N) \cap A^q$ and $S_1 = [N, K_{j^*}) \cap A^q$.

(3) $r \in M \cap P_i$ and $\{K_j\} \cup S_0 \subseteq A^r$.

(4) (Head) $(w(q,j), f^q(j))$ and $(w(r,j), f^r(j))$ are compatible in $P_j * \dot{Q}_j$.

Then r and q are compatible in P_i.

Proof. We may use (Amalgamation 10). Let us consider $w(q,j^*)$ and r. Apply (Amalgamation 10) to get $e \in P_i$ with $e \leq w(q,j^*), r$ in P_i. Since $e \leq w(q,j^*)$ in P_{j^*} holds, we may take $e' \in P_i$ with $e' \leq q$ in P_i and $w(e',j^*) = e$. Then $e' \leq w(e',j^*)$ in P_i and so $e' \leq q, r$ in P_i holds.

□

§ 3. Preservation of Properness

Let $p \in P_l$. Then the proper initial segments $w(p,i)$ defined in P_i are (P_i, M)-generic for right $M \in A^p$.

Lemma 3.1. Let $l \leq \kappa$, $p \in P_l$, $\{K_i, M\} \subseteq A^p$ and $K_i \in M$. Then $w(p,i)$ is (P_i, M)-generic and so $(w(p,i), f^p(i))$ is $(P_i * \dot{Q}_i, M)$-generic.

Proof. By induction on l. Let $D \subseteq P_i$ be dense with $D \in M$. We want to show that $D \cap M$ is predense below $w(p,i)$. Let $q \leq w(p,i)$ in P_i. We may assume that $q \in D$. We have four cases depending on how $M \cap K_i$ is listed in A^q. Let us assume that there are $K_j, K_{j^*} \in A^q$ as in (Amalgamation 11). Other cases are similar. Let $G_j * H_j$ be a $P_j * \dot{Q}_j$-generic filter over V with $(w(q,j), f^q(j)) \in G_j * H_j$. We have $M[G_j * H_j] \prec H_\kappa[G_j * H_j] = (H_\kappa)^{V[G_j * H_j]}$ in $V[G_j * H_j]$. Since $i < l$, $q \in P_i$, $\{K_j, M \cap K_i\} \subseteq A^q$ and $K_j \in M \cap K_i$, we may apply induction hypothesis to conclude $w(q,j)$ is (P_j, M)-generic. Hence $(w(q,j), f^q(j))$ is $(P_j * \dot{Q}_j, M)$-generic. Hence in $V[G_j * H_j]$, $M[G_j * H_j] \cap V = M$ holds. Therefore there exists $r \in D \cap M$ such that $\{K_j\} \cup S_0 \subseteq A^r$ and $(w(r,j), f^r(j)) \in G_j * H_j$ due to q. Since $(w(q,j), f^q(j))$, $(w(r,j), f^r(j)) \in G_j * H_j$, they are compatible in $P_j * \dot{Q}_j$. Hence by (Amalgamation 11), we know that q and r are compatible in P_i. Hence we are done.

□

We show $\langle P_i \mid i \leq \kappa \rangle$ is a sequence of proper preorders. Recall we set $I = \langle (K_i, P_i, \dot{Q}_i, P_i * \dot{Q}_i) \mid i < \kappa \rangle$. For $m \leq \kappa$, we write $I \lceil m$ to denote the initial segment of I by m. Namely, $I \lceil m = \langle (K_i, P_i, \dot{Q}_i, P_i * \dot{Q}_i) \mid i < m \rangle$.

Lemma 3.2. Let $i \leq \kappa$ and $I \lceil i, K_i, P_i \in M \prec H_\chi$. Then for all $p \in P_i \cap M$, there exists $q \in P_i$ such that $q \leq p$ in P_i and $M \cap K_i \in A^q$. This q is (P_i, M)-generic and P_i is proper.

Proof. Given p, we construct q by Lemma 2.1. We know that the proper initial segments $w(q,j)$ and $(w(q,j), f^q(j))$ are (P_j, N)-generic and $(P_j * \dot{Q}_j, N)$-generic, respectively, for the right $N \in S^q$ by Lemma 3.1. We have to show q itself is (P_i, M)-generic. To this end we repeat the same argument. Let $D \subseteq P_i$ be dense and $D \in M$. We want to show $D \cap M$ is predense below q in P_i. Let $d \leq q$ in P_i. We assume that $d \in D$. Since $M \cap K_i \in S^d$, we have four cases depending on how $M \cap K_i$ is listed in A^d. We assume that there are K_j and K_{j^*} as in (Amalgamation 11). Other cases are similar. Since $P_j * \dot{Q}_j \in K_{j+1} \subseteq K_i$ and $(w(d,j), f^d(j))$ is $(P_j * \dot{Q}_j, M \cap K_i)$-generic, it is also $(P_j * \dot{Q}_j, M)$-generic. Let $G_j * H_j$ be $P_j * \dot{Q}_j$-generic with $(w(d,j), f^d(j)) \in G_j * H_j$. We have

$$M[G_j * H_j] \cap V = M.$$

We consider a suitable copy r of d in $M \cap D$. By (Amagamation 11), we know that r and d are compatible in P_i.

□

The cardinals strictly between ω_1 and κ are collapsed.

Lemma 3.3. (1) Let $i < \kappa$ and let $p \in P_{i+1}$ with $K_i \in T^p$. Then $p \Vdash_{P_{i+1}}$ "$| K_{i+1} | = \omega_1$".

(2) For all $i < \kappa$, we have $\Vdash_{P_\kappa}$ "$| K_i | = \omega_1$".

Proof. For any $q \leq p$ in P_{i+1} and any $a \in K_{i+1}$, there exists $r \in P_{i+1}$ such that $r \leq q$ in P_{i+1} and that there exists $N \in S^r$ with $a \in N$. Hence $K_{i+1} = \bigcup\{N \mid N \in (K_i, K_{i+1}) \cap S^r, r \in G_{i+1}\}$ holds, where G_{i+1} is any P_{i+1}-generic filter with $p \in G_{i+1}$ over the ground model V. But $\langle N \mapsto N \cap \omega_1 \mid N \in (K_i, K_{i+1}) \cap S^r, r \in G_{i+1}\rangle$ is one-to-one. Hence $|K_{i+1}| = \omega_1$ holds.

□

P_κ does not have the κ-chain condition. To see this, let $\langle N_\eta \mid \eta < \kappa\rangle$ be a sequence of different elements in S such that $N_\eta \cap \omega_1$ are constant. Then this gives rise to an antichain of size κ in P_κ.

Lemma 3.4. P_κ preserves κ to be a cardinal. Hence $\Vdash_{P_\kappa}$ "$\kappa = \omega_2^{V[G_\kappa]}$".

Proof. Let $\Vdash_{P_\kappa}$ "$\dot{f} : \omega_1 \longrightarrow \kappa$" and $p \in P_\kappa$. Since for all $j < \kappa$, $|P_j| < \kappa$, we may choose $i < \kappa$ with $\mathrm{cf}(i) \geq \omega_1$ such that $p \in P_i = \bigcup\{P_j \mid j < i\}$ and that for all $\xi < \omega_1$, $D_\xi = \{y \in P_i \mid \exists v < \kappa\ y \Vdash_{P_\kappa}$ "$\dot{f}(\xi) = v$"$\}$ are dense in P_i. Let $q \leq p$ in P_κ with $K_i \in T^q$. Since there are projections

$$P_i\lceil w(q,i) \longleftarrow P_i * \dot{Q}_i\lceil (w(q,i), f^q(i)) \longleftarrow P_\kappa\lceil q$$

we have $q \Vdash_{P_\kappa}$ "$\forall\ \xi < \omega_1\ D_\xi \cap G_i \neq \emptyset$", where G_i is the P_i-generic filter constructed by upward-closing $\{w(x,i) \mid x \in G_\kappa, K_i \in A^x\}$ in P_i. Hence $q \Vdash_{P_\kappa}$ "$\dot{f} \in V[G_i]$" and so $q \Vdash_{P_\kappa}$ "$\dot{f}$ is bounded below κ".

□

§ 4. Reals and Souslin trees

We first observe that P_κ may add κ-many Cohen reals based on [UY]. Let $i < \kappa$, we define a suborder of P_{i+1} in $V[G_i * H_i]$,

$$P_{i+1}/G_i * H_i = \{y \in P_{i+1} \mid K_i \in T^y,\ (w(y,i), f^y(i)) \in G_i * H_i\},$$

where $G_i * H_i$ is any $P_i * \dot{Q}_i$-generic filter over V.

Lemma 4.1. Let $i < \kappa$. Let $I\lceil(i+1), K_{i+1}, P_{i+1} \in M \prec H_\chi$. Let $p \in P_{i+1}$ with $\{K_i, M \cap K_{i+1}\} \subseteq A^p$. Then $p \Vdash_{P_{i+1}}$ "$\forall D \in V[G_i * H_i]$, $D \subseteq M \cap (P_{i+1}/G_i * H_i)$ dense, $D \cap G_{i+1} \neq \emptyset$". And so if $(w(p,i), f^p(i)) \Vdash_{P_i * \dot{Q}_i}$ "$M \cap (P_{i+1}/G_i * H_i)$ is atomless", then $p \Vdash_{P_{i+1}}$ "$\exists$ a Cohen real over $V[G_i * H_i]$".

Proof. Let $p' \leq p$ in P_{i+1} and let $\dot{D}$ be a $P_i * \dot{Q}_i$-name with $p' \Vdash_{P_{i+1}}$ "$\dot{D}_{G_i * H_i} \subseteq M \cap (P_{i+1}/G_i * H_i)$ is dense in $M \cap (P_{i+1}/G_i * H_i)$". We show $E = \{y \in P_{i+1} \mid \exists\ r\ y \leq r$ in P_{i+1} and $y \Vdash_{P_{i+1}}$ "$r \in \dot{D}_{G_i * H_i}$"$\}$ is dense below p' in P_{i+1}. Hence $p' \Vdash_{P_{i+1}}$ "$\dot{D}_{G_i * H_i} \cap G_{i+1} \neq \emptyset$".

Let $q \leq p'$ in P_{i+1}. Let $G_i * H_i$ be $P_i * \dot{Q}_i$-generic over V with $(w(q,i), f^q(i)) \in G_i * H_i$. Since $(w(q,i), f^q(i))$ is $(P_i * \dot{Q}_i, M)$-generic, we have

$$M[G_i * H_i] \cap V = M.$$

In $V[G_i * H_i]$, there exists $q' \in P_{i+1} \cap M$ such that $\{K_i\} \cup (K_i, M \cap K_{i+1}) \cap S^q \subseteq A^{q'}$ and $(w(q', i), f^{q'}(i)) \in G_i * H_i$. This holds due to q. Since $q' \in M \cap (P_{i+1}/G_i * H_i)$ and $D = \dot{D}_{G_i * H_i}$ is dense, we have $r \in D \subseteq M \cap (P_{i+1}/G_i * H_i)$ with $r \leq q'$ in $P_{i+1}/G_i * H_i$ (in P_{i+1}). Take $(x, \pi) \leq (w(q,i), f^q(i))$ in $P_i * \dot{Q}_i$ and $(x, \pi) \in G_i * H_i$ and $(x, \pi) \Vdash_{P_i * \dot{Q}_i}$ "$r \in \dot{D}$". We may apply (Amalgamation 10) to conclude r and a condition formed from q whose head is strengthened by (x, π) are compatible in P_{i+1}. Hence we may take $y \leq r, q$ in P_{i+1} such that $y \Vdash_{P_{i+1}}$ "$r \in \dot{D}_{G_i * H_i}$".

□

To have that $P_{i+1}/G_i * H_i$ is atomless, we assume that κ is a measurable cardinal.

Lemma 4.2. Let κ be measurable. Then we may assume that $\Vdash_{P_i * \dot{Q}_i}$ "$P_{i+1}/G_i * H_i$ is atomless".

Proof. Given K_i, we construct K_{i+1} closed under an appropriate function so that the pair of M_1 and M_2 as in Lemma D are found in K_{i+1}. These M_1 and M_2 provide incompatible conditions p_1 and p_2 in P_{i+1} with the common head $\left(w(p_1, i), f^{p_1}(i)\right) = \left(w(p_2, i), f^{p_2}(i)\right)$ in $G_i * H_i$.

Here is some details: Let $(q, \tau) \in P_i * \dot{Q}_i$ and $A \subset (K_i, K) \cap S$ be such that

- $\Vdash_{P_i}$ "$\tau \in \dot{Q}_i$" and A is a finite $\in$-chain.
- (Explicit Basic Closure) If $N \in A$, $m \leq i$ and $K_m \in N$, then $I\lceil(m+1) \in N$.
- For all $N \in A$, $K_i \in N$ and $\Vdash_{P_i}$ "τ is $(\dot{Q}_i, N[G_i])$-generic".

Then by Lemma D, there exist $(h, \pi) \in P_i * \dot{Q}_i$, M_1 and M_2 such that

- $h \leq q$ in P_i.
- $I\lceil(i+1), (q, \tau), A \in M_1 \cap M_2$ and $M_1, M_2 \in S$.

And so

- (Explicit Basic Closure) If $m \leq i$ and $K_m \in M_l$, then $I\lceil(m+1) \in M_l$ $(l = 1, 2)$.
- $M_1 \cap K_i = M_2 \cap K_i \in A^h$.
- $\Vdash_{P_i}$ "$\pi \leq \tau$ in $\dot{Q}_i$ and π is $(\dot{Q}_i, M_l[G_i])$-generic" $(l = 1, 2)$.
- $M_1 \cap \kappa \neq M_2 \cap \kappa$.

Since κ is measurable, there is a regular uncountable cardinal $\theta < \kappa$ such that

- $I\lceil(i+1) \in H_\theta$.
- $H_\theta \in T$.
- For any $(q, \tau) \in P_i * \dot{Q}_i$ and any $A \subset (K_i, H_\theta) \cap S$ as above, there exist (h, π), M_1 and M_2 as above with $M_1, M_2 \in H_\theta$.

Let $K_{i+1} = H_\theta$. Then for any $p = \left(f^q \cup \{(i, \tau)\}, A^q \cup \{K_i\} \cup A\right) \in P_{i+1}$, there exist

$$p_1 = \left(f^h \cup \{(i, \pi)\}, A^h \cup \{K_i\} \cup A \cup \{M_1\}\right),$$

$$p_2 = \left(f^h \cup \{(i, \pi)\}, A^h \cup \{K_i\} \cup A \cup \{M_2\}\right)$$

in P_{i+1} such that $p_1, p_2 \leq p$ in P_{i+1} and that p_1 and p_2 are incompatible in P_{i+1}. Hence $\Vdash_{P_i * \dot{Q}_i}$ "$P_{i+1}/G_i * H_i$ is atomless".

□

We show that Souslin trees may be preserved by P_κ.

Theorem 4.3. Let T be a Souslin tree. If for all $i < \kappa$, $\Vdash_{P_i}$ "$\dot{Q}_i$ preserves T, if T were Souslin". Then $\Vdash_{P_\kappa}$ "T remains Souslin".

Proof. Let T be a Souslin tree in V. By induction on $i \leq \kappa$, we show that $\Vdash_{P_i}$ "T remains Souslin". Let $p \Vdash_{P_i}$ "$\dot{X} \subset T$ be a maximal antichain". We want $q \leq p$ such that $q \Vdash_{P_i}$ "$\dot{X}$ is countable".

Let $p, T, \dot{X}, I\lceil i, K_i, P_i \in M \prec H_\chi$. Let $q \leq p$ be (P_i, M)-generic. Let $\langle s_n \mid n < \omega \rangle$ enumerate the elements of $T_{M \cap \omega_1}$, the $(M \cap \omega_1)$-th level of T. We show that for all $n < \omega$, $q \Vdash_{P_i}$ "$\exists\, t < s_n\ t \in \dot{X}$" and so $q \Vdash_{P_i}$ "$\dot{X}$ is countable". To this end let $r \leq q$ in P_i. We find $t < s_n$ and $y \leq r$ with $y \Vdash_{P_i}$ "$t \in \dot{X}$". We have four cases depending on how $M \cap K_i$ is listed in A^r. Assume we are as in (Amalgamation 11). Let $G_j * H_j$ be $P_j * \dot{Q}_j$-generic over V with $\left(w(r, j), f^r(j)\right) \in G_j * H_j$. In $V[G_j * H_j]$, T remains Souslin by induction. Let us take a copy

$$r' \in M[G_j * H_j] \cap (P_i/G_j * H_j) = M \cap (P_i/G_j * H_j)$$

of r. We may assume $r' \leq p$ in P_i and $\{K_j\} \cup S_0 \subseteq A^{r'}$. Let $E = \{s \in T \mid \exists\, t < s \text{ in } T\ \exists\, x \leq r' \text{ in } P_i/G_j * H_j\ x \Vdash_{P_i} \text{“} t \in \dot{X} \text{”}\}$. Then this E is a dense subset of T and $E \in M[G_j * H_j]$. Since s_n is $(T, M[G_j * H_j])$-generic, we have $s, t, x \in M$ such that $t < s < s_n$ in T, $x \leq r'$ in $P_i/G_j * H_j$ and $x \Vdash_{P_i}$ "$t \in \dot{X}$". By (Amalgamation 11), x and r are compatible in P_i. Hence there exists $y \in P_i$ such that $y \leq r$ and $y \Vdash_{P_i}$ "$t \in \dot{X}$".

□

There are two operations to form proper preorders. We want a new theory of iterated forcing with local projections that puts forward these two operations together with the direct limit.

Note 4.4. (1) Let P be a proper preorder and let χ be a sufficiently large regular uncountable cardinal. We define an associated preorder Q such that $(p, A) \in Q$, if A is a finite $\in$-chain of countable elementary substructures N of H_χ with $P \in N$ and $p \in P$ is (P, N)-generic for all $N \in A$. For $(p_2, A_2), (p_1, A_1) \in Q$, let $(p_2, A_2) \leq (p_1, A_1)$ in Q, if $A_2 \supseteq A_1$ and $p_2 \leq p_1$ in P. Then we may show that Q is proper and there exists a natural projection $P \longleftarrow Q$.

(2) Let $\langle P_n \mid n < \omega \rangle$ be an iterated forcing such that all P_n are proper. Then we may associate a proper preorder Q that is in a similar situation as (1). However, projections from Q to P_n are formed locally.

References

[AM] D. Aspero, M. Mota, A GENERALIZATION OF MARTIN'S AXIOM, preprint, 2012.

[N] I. Neeman, FORCING WITH SEQUENCES OF MODELS OF TWO TYPES, preprint.

[UY] T. Usuba, T. Yorioka, communication, 2012.

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