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非線形汎関数のショケ積分表示可能性条件

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ABSTRACT. We give an alternative and direct approach to the Choquet integral representability of a comonotonically additive, bounded, monotone functional I defined on the space of all continuous, real-valued functions on a locally compact space X with compact support and on the space of all continuous, real-valued functions on X vanishing at infinity. To this end, we introduce the notion of the asymptotic translatability of the functional I and reveal that this simple notion is equivalent to the Choquet integral representability of I with respect to a monotone measure on X with appropriate regularity.

1. INTRODUCTION

This is an announcement of the forthcoming paper [10]. Most of functionals, appeared in popular mathematical models for uncertainty and partial ignorance, are monotone, real-valued functionals defined on a vector sublattice of the space $B(X)$ of all bounded, real-valued functions on a non-empty set X with additional properties such as the superadditivity, the n-monotonicity, the comonotonic additivity, the translation invariance (or the constant additivity), and others. See, for instance, coherent lower previsions in Walley's behavioral approach to decision making and probability [22], exact cooperative games and expected utility without additivity by Schmeidler [18, 20] and Gilboa [6], coherent risk measures by Artzner *et al.* [1], and exact functionals by Maaß [11] and n-exact functionals by G. de Cooman *et al.* [4]. In those studies, it is important to clarify under what conditions a given functional I defined on a given vector sublattice $\mathcal{F}$ of $B(X)$ can be represented as

$$I(f) = (\mathrm{C})\int_X f d\mu, \quad f \in \mathcal{F},$$

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using the Choquet integral with respect to a monotone measure on X with appropriate regularity. This type of problem is often called the *Choquet integral representability* of a functional.

In this paper, we discuss the Choquet integral representability of comonotonically additive, monotone functionals. Schmeidler [19] already obtained such a representation in the case that I is defined on the space $B(X, \Sigma)$ of all bounded, real-valued functions on a non-empty set X that is measurable with respect to a field Σ of subsets of X. Murofushi *et al.* [12] and Rébillé [17] extended it to the case that I is not necessarily monotone. see also [2]. When X is a Hausdorff space and I is a functional on the space $C_b(X)$ of all bounded, continuous, real-valued functions on X, some Choquet integral representations of I can be deduced from Zhou [24] and Cerreia-Vioglio *et al.* [2].

However, these results do not cover the case of functionals defined on the space $C_{00}(X)$ of all continuous, real-valued functions on a locally compact space X with compact support and the space $C_0(X)$ of all continuous, real-valued functions on X vanishing at infinity, since these spaces do not contain the constant functions on X unless X is compact. In fact, there is a comonotonically additive, bounded, monotone functional on $C_{00}(\mathbb{R})$, any of whose extension to a larger space cannot be represented as the Choquet integral; see Remark 4 further on.

The preceding detailed study of the Choquet integral representability of a functional I on the space $K := C_{00}(X)$ was published in a series of papers by Narukawa *et al.* [13, 14, 15]. In particular, in [15] they introduced the notion of the ε-symmetry and the M-uniform continuity to show that every comonotonically additive, bounded, monotone functional I having these properties can be represented by the Choquet integral with respect to a finite monotone measure on X. This has been accomplished by extending the domain space K to the larger vector lattice K^* and by extending the functional I to the functional I^* on K^* in well-defined ways.

In this paper, we will adopt an alternative and direct approach to this issue. Firstly, we give an improvement of [21, Theorem 3.7] and its extension to the space $C_0(X)$ using the Greco theorem [7], which is the most general Daniell-Stone type integral representation theorem for functionals on function spaces. Next, we will introduce the notion of the asymptotic translatability of a functional I on $C_{00}(X)$ and on $C_0(X)$ and show that this simple notion is equivalent to the Choquet integral representability of I with respect to a monotone measure on X with appropriate regularity.

2. Notation and preliminaries

Let X be a non-empty set and 2^X denote the family of all subsets of X. For each $A \subset X$, χ_A denotes the characteristic function of A. $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all nonnegative real numbers, respectively. Also, $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ denote the set of all extended real numbers and the set of all nonnegative extended real numbers, respectively. $\mathbb{N}$ denotes the set of all natural numbers. For any functions $f, g : X \to \overline{\mathbb{R}}$, let $f \vee g := \max(f, g)$, $f \wedge g := \min(f, g)$, $f^+ := f \vee 0$, $f^- := (-f) \vee 0$, $|f| := f \vee (-f)$, and $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

We say that a set function $\mu : 2^X \to \overline{\mathbb{R}}^+$ is a *monotone measure* on X if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subset B$. When μ is finite, that is, $\mu(X) < \infty$, the *conjugate* $\bar{\mu}$ of μ is defined by $\bar{\mu}(A) := \mu(X) - \mu(A^c)$ for each $A \subset X$, where A^c denotes the complement of the set A.

For any function $f : X \to \overline{\mathbb{R}}$, the decreasing distribution function $t \in \mathbb{R} \mapsto \mu(\{f > t\})$ is Lebesgue measurable. Thus, the following formalization is well-defined; see [3] and [19].

Definition 1. Let μ be a monotone measure on X. The *Choquet integral* of a nonnegative function $f : X \to \overline{\mathbb{R}}^+$ with respect to μ is defined by

$$(\mathrm{C})\int_X f d\mu := \int_0^\infty \mu(\{f > t\})dt,$$

where the integral on the right-hand side is the usual Lebesgue integral.

When $\mu(X) < \infty$, the *Choquet integral* of a function $f : X \to \overline{\mathbb{R}}$ with respect to μ is defined by

$$(\mathrm{C})\int_X f d\mu := (\mathrm{C})\int_X f^+ d\mu - (\mathrm{C})\int_X f^- d\bar{\mu}$$

whenever the Choquet integrals on the right-hand side are not both ∞.

Remark 1. For any monotone measure μ on X and any function $f : X \to \overline{\mathbb{R}}$, the decreasing distribution function $t \in \mathbb{R} \mapsto \mu(\{f \geq t\})$ is also Lebesgue measurable, and the function $\mu(\{f > t\})$ in the above definition may be replaced with the function $\mu(\{f \geq t\})$, since $\mu(\{f \geq t\}) \geq \mu(\{f > t\}) \geq \mu(f \geq t + \varepsilon\})$ for every $\varepsilon > 0$ and $0 \leq t < \infty$. This fact will be used implicitly in this paper.

See [5], [16], and [23] for more information on monotone measures and Choquet integrals.

For readers' convenience, we introduce the Greco theorem [7, Proposition 2.2], which is the most general Choquet integral representation theorem for comonotonically additive, monotone, extended real-valued functionals; see also [5, Theorem 13.2]. Recall that two functions $f, g : X \to \overline{\mathbb{R}}$ are *comonotonic* and they are written by $f \sim g$ if, for every $x, x' \in X$, $f(x) < f(x')$ implies $g(x) \leq g(x')$.

Definition 2. Let $\mathcal{F}$ be a non-empty family of functions $f : X \to \overline{\mathbb{R}}$ with pointwise ordering. Let $I : \mathcal{F} \to \overline{\mathbb{R}}$ be a functional.

(1) I is said to be *monotone* if $I(f) \leq I(g)$ whenever $f, g \in \mathcal{F}$ and $f \leq g$.
(2) I is said to be *comonotonically additive* if $I(f+g) = I(f) + I(g)$ whenever $f, g, f+g \in \mathcal{F}$ and $f \sim g$.
(3) I is said to be *bounded* if there is a constant $M > 0$ such that $|I(f)| \leq M\|f\|_\infty$ for all $f \in \mathcal{F}$.

Theorem 1 (The Greco theorem). *Let $\mathcal{F}^+$ be a non-empty family of nonnegative functions $f : X \to \overline{\mathbb{R}}^+$. Assume that $\mathcal{F}^+$ satisfies*

(i) *if $f \in \mathcal{F}^+$ and $c \in \mathbb{R}^+$, then cf, $f \wedge c$, $f - f \wedge c = (f-c)^+ \in \mathcal{F}^+$. In particular, $0 \in \mathcal{F}^+$.*

Assume that $I : \mathcal{F}^+ \to \overline{\mathbb{R}}$ is a comonotonically additive, monotone functional satisfying

(ii) $I(0) = 0$,
(iii) $\sup_{a>0} I(f - f \wedge a) = I(f)$ *for every* $f \in \mathcal{F}^+$, *and*
(iv) $\sup_{b>0} I(f \wedge b) = I(f)$ *for every* $f \in \mathcal{F}^+$.

For each $A \subset X$, define the set functions $\alpha, \beta : 2^X \to \overline{\mathbb{R}}^+$ by

$$\alpha(A) := \sup\{I(f) : f \in \mathcal{F}^+,\ f \leq \chi_A\},$$
$$\beta(A) := \inf\{I(f) : f \in \mathcal{F}^+,\ \chi_A \leq f\},$$

where let $\inf \emptyset := \infty$.

(1) *The set functions α and β are monotone measures on X with $\alpha \leq \beta$.*
(2) *For any monotone measure λ on X, the following two conditions are equivalent:*
   (a) $\alpha \leq \lambda \leq \beta$.
   (b) $I(f) = (\mathrm{C})\int_X f d\lambda$ *for every* $f \in \mathcal{F}^+$.

Remark 2. Every comonotonically additive, monotone functional $I : \mathcal{F}^+ \to \overline{\mathbb{R}}$ satisfying assumptions (i) and (ii) of Theorem 1 is nonnegative, that is, $I(f) \geq 0$ for every $f \in \mathcal{F}^+$, and it is positively homogeneous, that is, $I(cf) = cI(f)$ for every $f \in \mathcal{F}^+$ and $c \in \mathbb{R}^+$. See, for instance, [5, page 159] and [13, Proposition 4.2].

3. The Choquet integral representability on $C_{00}^+(X)$

In this section, firstly we give an alternative proof of [21, Theorem 3.7] and its improvement using the Greco theorem.

From this point forwards, let X be a locally compact Hausdorff space. $C_{00}(X)$ denotes the space of all continuous, real-valued functions on X with compact support and $C_0(X)$ denotes the space of all continuous, real-valued functions on X vanishing at infinity. $C_{00}^+(X)$ and $C_0^+(X)$ denote the positive cone of $C_{00}(X)$ and the positive cone of $C_0(X)$, respectively. For any function f on X, $S(f)$ denotes the support of f, which is defined by the closure of $\{f \neq 0\}$.

A bounded set in X is a set that is contained in a compact subset of X. A subset A of X is said to be G_δ if there is a sequence $\{G_n\}_{n\in\mathbb{N}}$ of open sets such that $A = \bigcap_{n=1}^\infty G_n$. The class of all G_δ sets is closed under the formulation of finite unions and countable intersections. If $f \in C_0(X)$ and $c > 0$, then the set $\{|f| \geq c\}$ is compact G_δ. If X is metrizable, then every closed subset of X is G_δ. A subset A of X is said to be K_σ if there is a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets such that $A = \bigcup_{n=1}^\infty K_n$. The class of all K_σ sets is closed under the formulation of countable unions and finite intersections. If $f \in C_0(X)$ and $c \geq 0$, then the set $\{|f| > c\}$ is open K_σ and it is bounded if $c > 0$. X is said to be σ-compact if it is K_σ. If X is σ-compact and metrizable, then every open subset of X is K_σ. The complement of every G_δ subset of a σ-compact space is K_σ.

For any compact K and any open G with $K \subset G$, there is $f \in C_{00}(X)$ such that $\chi_K \leq f \leq \chi_G$. Thus, for every open subset G of X, there is an increasing net $\{f_\tau\}_{\tau\in\Gamma}$ of functions in $C_{00}(X)$ such that $0 \leq f_\tau \leq 1$ for all $\tau \in \Gamma$ and $f_\tau \uparrow \chi_G$. By contrast, for every compact subset K of X, there is a decreasing net $\{f_\tau\}_{\tau\in\Gamma}$ of functions in $C_{00}(X)$ such that $0 \leq f_\tau \leq 1$ for all $\tau \in \Gamma$ and $f_\tau \downarrow \chi_K$. When G is open K_σ and K is compact G_δ, in the above statement, the net $\{f_\tau\}_{\tau\in\Gamma}$ may be replaced with a sequence $\{f_n\}_{n\in\mathbb{N}}$.

The following regularity properties give a tool to approximate general sets by more tractable sets such as open and compact sets. They are still important in monotone measure theory.

Definition 3. Let μ be a monotone measure on X.

(1) μ is said to be *outer regular* (respectively, *outer K_σ regular*) if, for every subset A of X, $\mu(A) = \inf\{\mu(G) : A \subset G, G \text{ is open}\}$ (respectively, $\mu(A) = \inf\{\mu(H) : A \subset H, H \text{ is open } K_\sigma\}$).

(2) μ is said to be *quasi outer regular* (respectively, *quasi outer K_σ regular*) if, for every compact subset K of X, $\mu(K) = \inf\{\mu(G) : K \subset G, G \text{ is open}\}$ (respectively, for every compact G_δ subset L of X, $\mu(L) = \inf\{\mu(H) : L \subset H, H \text{ is open } K_\sigma\}$).

(3) μ is said to be *inner regular* (respectively, *inner G_δ regular*) if, for every subset A of X, $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}$ (respectively, $\mu(A) = \sup\{\mu(L) : L \subset A, L \text{ is compact } G_\delta\}$).

(4) μ is said to be *quasi inner regular* (respectively, *quasi inner G_δ regular*) if, for every open subset G of X, $\mu(G) = \sup\{\mu(K) : K \subset G, K \text{ is compact}\}$ (respectively, for every open K_σ subset H of X, $\mu(H) = \sup\{\mu(L) : L \subset H, L \text{ is compact } G_\delta\}$).

Every outer K_σ regular (respectively, inner G_δ regular) monotone measure on X is outer regular (respectively, inner regular). By contrast, every quasi outer regular (respectively, quasi inner regular) monotone measure on X is quasi outer K_σ regular (respectively, quasi inner G_δ regular).

For later use we collect some basic properties of the regularity of monotone measures on locally compact spaces.

Proposition 1. *Let μ be a monotone measure on X.*

(1) *μ is quasi outer regular if and only if $\mu(K) = \inf_{\tau\in\Gamma} \mu(K_\tau)$ whenever $\{K_\tau\}_{\tau\in\Gamma}$ is a decreasing net of compact sets and $K = \bigcap_{\tau\in\Gamma} K_\tau$.*

(2) *μ is quasi inner regular if and only if $\mu(G) = \sup_{\tau\in\Gamma} \mu(G_\tau)$ whenever $\{G_\tau\}_{\tau\in\Gamma}$ is an increasing net of open sets and $G = \bigcup_{\tau\in\Gamma} G_\tau$.*

(3) *μ is quasi outer K_σ regular if and only if $\mu(L) = \inf_{n\in\mathbb{N}} \mu(L_n)$ whenever $\{L_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of compact G_δ sets and $L = \bigcap_{n\in\mathbb{N}} L_n$.*

(4) *μ is quasi inner G_δ regular if and only if $\mu(H) = \sup_{n\in\mathbb{N}} \mu(H_n)$ whenever $\{H_n\}_{n\in\mathbb{N}}$ is an increasing sequence of open K_σ sets and $H = \bigcup_{n\in\mathbb{N}} H_n$.*

A nonnegative, real-valued function f on X is said to be *lower semicontinuous* if the set $\{f > r\}$ is open for every $r \geq 0$ and it is said to be *upper semicontinuous* if the set $\{f \geq r\}$ is closed for every $r \geq 0$. Thus, f is upper semicontinuous and vanishing at infinity if and only if the set $\{f \geq r\}$ is compact for every $r > 0$.

By Proposition 1, the first assertion of the following proposition can be proved in the same way as [8, Theorem 7]. Other assertions can also be proved in a similar fashion.

Proposition 2. *Let μ be a finite monotone measure on X.*

(1) *μ is quasi inner regular if and only if it holds that $\lim_{\tau\in\Gamma} (\mathrm{C})\int_X f_\tau d\mu = \sup_{\tau\in\Gamma} (\mathrm{C})\int_X f_\tau d\mu = (\mathrm{C})\int_X f d\mu$ whenever a uniformly bounded, increasing net $\{f_\tau\}_{\tau\in\Gamma}$ of lower semicontinuous, nonnegative, real-valued functions on X converges pointwise to such a function f on X.*

(2) *μ is quasi outer regular if and only if it holds that $\lim_{\tau\in\Gamma} (\mathrm{C})\int_X f_\tau d\mu = \inf_{\tau\in\Gamma} (\mathrm{C})\int_X f_\tau d\mu = (\mathrm{C})\int_X f d\mu$ whenever a uniformly bounded, decreasing net $\{f_\tau\}_{\tau\in\Gamma}$ of upper semicontinuous, nonnegative, real-valued functions on X vanishing at infinity converges pointwise to such a function f on X.*

(3) μ *is quasi inner* G_δ *regular if and only if it holds that* $\lim_{n\to\infty}(\mathrm{C})\int_X f_n d\mu = \sup_{n\in\mathbb{N}}(\mathrm{C})\int_X f_n d\mu = (\mathrm{C})\int_X f d\mu$ *whenever a uniformly bounded, increasing sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of lower semicontinuous, nonnegative, real-valued functions on* X *converges pointwise to such a function* f *on* X.

(4) μ *is quasi outer* K_σ *regular if and only if it holds that* $\lim_{n\to\infty}(\mathrm{C})\int_X f_n d\mu = \inf_{n\in\mathbb{N}}(\mathrm{C})\int_X f_n d\mu = (\mathrm{C})\int_X f d\mu$ *whenever a uniformly bounded, decreasing sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of upper semicontinuous, nonnegative, real-valued functions on* X *vanishing at infinity converges pointwise to such a function* f *on* X.

The following theorem is an improvement of [21, Theorem 3.7] and [9, Theorem 2]. It has essentially been derived from the Greco theorem.

Theorem 2. *Let* $I : C_{00}^+(X) \to \mathbb{R}$ *be a comonotonically additive, monotone functional. For each* $A \subset X$, *define the set functions* $\alpha, \beta, \gamma : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha(A) := \sup\{I(f) : f \in C_{00}^+(X), f \leq \chi_A\},$$
$$\beta(A) := \inf\{I(f) : f \in C_{00}^+(X), \chi_A \leq f\},$$
$$\gamma(A) := \sup\{I(f) : f \in C_{00}^+(X), 0 \leq f \leq 1, S(f) \subset A\},$$

where let $\inf\emptyset := \infty$, *and define their regularizations* $\alpha^*, \beta^*, \gamma^*, \alpha^{**}, \beta^{**}, \gamma^{**} : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha^*(A) := \inf\{\alpha(G) : A \subset G, G \text{ is open}\},$$
$$\beta^*(A) := \sup\{\beta(K) : K \subset A, K \text{ is compact}\},$$
$$\gamma^*(A) := \inf\{\gamma(G) : A \subset G, G \text{ is open}\},$$
$$\alpha^{**}(A) := \inf\{\alpha(H) : A \subset H, H \text{ is open } K_\sigma\},$$
$$\beta^{**}(A) := \sup\{\beta(L) : L \subset A, L \text{ is compact } G_\delta\},$$
$$\gamma^{**}(A) := \inf\{\gamma(H) : A \subset H, H \text{ is open } K_\sigma\}.$$

(1) *The set functions* α, β, γ, α^*, β^*, γ^*, α^{**}, β^{**}, *and* γ^{**} *are monotone measures on* X.

(2) *For any monotone measure* λ *on* X, *the following two conditions are equivalent*:
   (a) $\alpha \leq \lambda \leq \beta$.
   (b) $I(f) = (\mathrm{C})\int_X f d\lambda$ *for every* $f \in C_{00}^+(X)$.

(3) $\gamma^*(K) = \gamma^{**}(K) = \beta(K) < \infty$ *for every compact subset* K *of* X.

(4) *The defined monotone measures are comparable, that is,* $\alpha = \gamma \leq \beta^{**} \leq \beta^* \leq \alpha^* = \gamma^* \leq \gamma^{**} = \alpha^{**} \leq \beta$, *so that any of them is a representing measure of* I.

(5) $\beta^*(G) = \beta^{**}(G) = \alpha(G)$ *for every open subset* G *of* X.

(6) α^* *is quasi inner regular and outer regular.*

(7) β^* *is inner regular and quasi outer regular.*

(8) α^{**} *is quasi inner* G_δ *regular and outer* K_σ *regular.*

(9) β^{**} *is inner* G_δ *regular and quasi outer* K_σ *regular.*

(10) $\beta(X) < \infty$ *if and only if* X *is compact.*

(11) $\alpha(X) < \infty$ *if and only if* I *is bounded.*

(12) $\alpha(X) = \gamma(X) = \beta^{**}(X) = \beta^*(X) = \alpha^*(X) = \gamma^*(X)$.

(13) *Assume that* X *is* σ*-compact. Then* $\alpha(X) = \alpha^{**}(X) = \gamma^{**}(X)$.

(14) *Let* λ *be a monotone measure on* X. *Let* $I(f) := (\mathrm{C})\int_X f d\lambda$ *for every* $f \in C_{00}^+(X)$. *Then* I *is comonotonically additive and monotone. Moreover, the following conditions are equivalent:*
  (a) I *is real-valued.*
  (b) $\lambda(\{f > 0\}) < \infty$ *for every* $f \in C_{00}^+(X)$.

4. The Choquet integral representability on $C_{00}(X)$

In this section, we formalize a Choquet integral representation theorem for comonotonically additive functionals defined on the entire space $C_{00}(X)$.

Lemma 1. *For any* $f \in C_0(X)$ *and any constant* $c > 0$ *with* $|f| \le c$, *there is an increasing net* $\{g_\tau\}_{\tau\in\Gamma}$ *of functions in* $C_0(X)$ *such that* $0 \le g_\tau \le c$ *and* $g_\tau \pm f \ge 0$ *for all* $\tau \in \Gamma$ *and that* $g_\tau \uparrow c$. *If* $f \in C_{00}(X)$, *then the net* $\{g_\tau\}_{\tau\in\Gamma}$ *can be chosen from* $C_{00}(X)$. *When* X *is* σ*-compact, the net* $\{g_\tau\}_{\tau\in\Gamma}$ *may be replaced with a sequence* $\{g_n\}_{n\in\mathbb{N}}$.

The property given in the next proposition is called the *asymptotic translatability* of the Choquet integral. It is important for formalizing Choquet integral representation theorems for functionals defined on the entire space $C_{00}(X)$ and $C_0(X)$.

Proposition 3. *Let* μ *be a quasi inner regular, finite monotone measure on* X. *For any* $f \in C_0(X)$, *any increasing net* $\{g_\tau\}_{\tau\in\Gamma}$ *of functions in* $C_0(X)$, *and any constant* $c > 0$, *if* $0 \le g_\tau \le c$ *and* $f + g_\tau \ge 0$ *for all* $\tau \in \Gamma$ *and if* $g_\tau \uparrow c$, *then* $\lim_{\tau\in\Gamma} (\mathrm{C})\int_X (f + g_\tau) d\mu = (\mathrm{C})\int_X f d\mu + \lim_{\tau\in\Gamma} (\mathrm{C})\int_X g_\tau d\mu$.

Theorem 3. *Let* $I : C_{00}(X) \to \mathbb{R}$ *be a comonotonically additive, bounded, monotone functional. Assume that* I *has the asymptotic translatability, that is, for any* $f \in C_{00}(X)$, *any increasing net* $\{g_\tau\}_{\tau\in\Gamma}$ *of functions in* $C_{00}(X)$, *and any constant* $c > 0$, *if* $0 \le g_\tau \le c$ *and* $f + g_\tau \ge 0$ *for all* $\tau \in \Gamma$ *and if* $g_\tau \uparrow c$, *then* $\lim_{\tau\in\Gamma} I(f + g_\tau) = I(f) + \lim_{\tau\in\Gamma} I(g_\tau)$. *Then, there is a finite monotone measure* μ *on* X *satisfying the following conditions:*

(a) $I(f) = (\mathrm{C})\int_X f d\mu$ *for all* $f \in C_{00}(X)$.

(b) μ *is quasi inner regular.*

(c) μ *is outer regular.*

Moreover, the finite monotone measure μ *on* X *satisfying* (a)–(c) *is uniquely determined.*

Conversely, let λ *be a finite monotone measure on* X *satisfying* (b) *and let* I *be defined by* (a). *Then,* I *is a comonotonically additive, bounded, monotone, real-valued functional on* $C_{00}(X)$ *and it has the asymptotic translatability.*

Remark 3. When X is σ-compact, the asymptotic translatability condition in the above theorem may be replaced with its sequential version: for any $f \in C_{00}(X)$, any increasing sequence $\{g_n\}_{n\in\mathbb{N}}$ of functions in $C_{00}(X)$, and any constant $c > 0$, if $0 \le g_n \le c$ and $f + g_n \ge 0$ for all $n \in \mathbb{N}$ and if $g_n \uparrow c$, then $\lim_{n\to\infty} I(f + g_n) = I(f) + \lim_{n\to\infty} I(g_n)$. In this case, if we let $\mu := \alpha^{**}$ given in Theorem 2, then μ is a unique quasi inner G_δ regular, outer K_σ regular, finite monotone measure on X that represents I.

The following proposition shows that the asymptotic translatability does not follow from the comonotonic additivity, the boundedness, and the monotonicity of a functional.

Proposition 4. *Let* $D := [0, 1]$. *Define the set function* $\lambda : 2^{\mathbb{R}} \to \{0, 1\}$ *by*

$$\lambda(A) := \begin{cases} 1 & \text{if } D \subset A \\ 0 & \text{if } D \cap A^c \ne \emptyset \end{cases}$$

for each $A \subset \mathbb{R}$.

(1) λ *is a monotone measure on* $\mathbb{R}$ *and its conjugate* $\bar{\lambda}$ *is given by*

$$\bar{\lambda}(A) := \begin{cases} 1 & \text{if } D \cap A \ne \emptyset \\ 0 & \text{if } A \subset D^c \end{cases}$$

for each $A \subset \mathbb{R}$.

(2) λ *is outer regular and quasi inner regular.*

(3) $\bar{\lambda}$ *is quasi outer regular and inner regular.*

Define the functional $I : C_0(\mathbb{R}) \to \mathbb{R}$ *by*

$$I(f) := (\mathrm{C})\int_{\mathbb{R}} f^+ d\lambda, \quad f \in C_0(\mathbb{R})$$

and let I_0 *be the restriction of* I *onto* $C_{00}(X)$.

(4) I *and* I_0 *are comonotonically additive, bounded, and monotone, but they do not have the asymptotic translatability.*

Remark 4. Proposition 4 also shows that any extension of I_0 to a larger space of bounded functions on $\mathbb{R}$, which contains $C_{00}(\mathbb{R})$, cannot be represented by a quasi inner regular, finite monotone measure on $\mathbb{R}$.

5. The Choquet integral representability on $C_0(X)$

From Theorem 2 we can derive a Choquet integral representation theorem for comonotonically additive, monotone functionals on $C_0^+(X)$ having a continuity condition given by Greco.

Theorem 4. *Let $I : C_0^+(X) \to \mathbb{R}$ be a comonotonically additive, monotone functional satisfying $\inf_{a>0} I(f \wedge a) = 0$.*

For each $A \subset X$, define the set functions $\alpha, \beta, \gamma : 2^X \to \overline{\mathbb{R}}^+$ by

$$\alpha(A) := \sup\{I(f) : f \in C_0^+(X), f \leq \chi_A\},$$
$$\beta(A) := \inf\{I(f) : f \in C_0^+(X), \chi_A \leq f\},$$
$$\gamma(A) := \sup\{I(f) : f \in C_0^+(X), 0 \leq f \leq 1, S(f) \subset A\},$$

where let $\inf \emptyset := \infty$, and define their regularizations α^*, β^*, γ^*, α^{**}, β^{**}, and γ^{**} in the same way as Theorem 2.

(1) *The set functions α, β, γ, α^*, β^*, γ^*, α^{**}, β^{**}, and γ^{**} are monotone measures on X and they satisfy properties (3)–(13) of Theorem 2.*

(2) *For any monotone measure λ on X, the following two conditions are equivalent:*
   (a) $\alpha \leq \lambda \leq \beta$.
   (b) $I(f) = (\mathrm{C})\int_X f d\lambda$ *for every* $f \in C_0^+(X)$.

(3) *Let λ be a monotone measure on X such that $\lambda(\{f > 0\}) < \infty$ for every $f \in C_0^+(X)$. Let $I(f) := (\mathrm{C})\int_X f d\lambda$ for every $f \in C_0^+(X)$. Then, I is a comonotonically additive, monotone, real-valued functional on $C_0^+(X)$ satisfying $\inf_{a>0} I(f \wedge a) = 0$.*

Remark 5. There is a locally compact space X and a monotone measure λ on X such that $\lambda(\{f > 0\}) < \infty$ for all $f \in C_0^+(X)$ but $\lambda(X) = \infty$. An example is as follows: Let X be an uncountable set with the discrete topology. Then, X is locally compact, but it is not σ-compact. Therefore, $X \neq \{f > 0\}$ for any $f \in C_0^+(X)$. Now define the monotone measure $\lambda : 2^X \to \overline{\mathbb{R}}_+$ by $\lambda(A) = \infty$ if $A = X$ and $\lambda(A) = 0$ if $A \neq X$.

From Theorem 4 the following theorem can be proved in the same way as Theorem 3.

Theorem 5. *Let $I : C_0(X) \to \mathbb{R}$ be a comonotonically additive, bounded, monotone functional. Assume that I has the asymptotic translatability, that is, for any $f \in C_0(X)$, any increasing net $\{g_\tau\}_{\tau\in\Gamma}$ of functions in $C_0(X)$, and any constant $c > 0$, if $0 \leq g_\tau \leq c$ and $f + g_\tau \geq 0$ for all $\tau \in \Gamma$ and if $g_\tau \uparrow c$, then $\lim_{\tau\in\Gamma} I(f + g_\tau) = I(f) + \lim_{\tau\in\Gamma} I(g_\tau)$. Then, there is a finite monotone measure μ on X satisfying the following conditions:*

(a) $I(f) = (\mathrm{C})\int_X f d\mu$ *for every* $f \in C_0(X)$.

(b) μ *is quasi inner regular.*

(c) μ *is outer regular.*

Moreover, the finite monotone measure μ on X satisfying (a)–(c) *is uniquely determined.*

Conversely, let λ be a finite monotone measure on X satisfying (b) *and let I be defined by* (a). *Then, I is a comonotonically additive, bounded, monotone, real-valued functional and it has the asymptotic translatability.*

6. Conclusion

In this paper, we discussed the Choquet integral representability of a comonotonically additive, bounded, monotone functional I on $C_{00}(X)$ and on $C_0(X)$ with locally compact X. We need to impose some additional conditions on the functional I, since there is a comonotonically additive, bounded, monotone functional on $C_{00}(\mathbb{R})$, any of whose extension to a larger space cannot be represented as the Choquet integral with respect to a finite monotone measure. This seems to come from the lack of constant functions in $C_{00}(X)$ and $C_0(X)$ and due to the asymmetry of the Choquet integral. For this reason, we introduced the notion of the asymptotic translatability and revealed that this simple notion is equivalent to the Choquet integral representability of I with respect to a finite monotone measure on X with appropriate regularity.

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