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Perturbative Expansion of the Chern-Simons Integral

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1 Introduction

The necessity of the asymptotic expansion in infinite dimensions arises from the pioneering work of the Chern-Simons theory by Witten [10] in 1989.

However, the exponential 3rd term of the Chern-Simons theory is less likely to be handled by techniques known at present, so that we will challenge a new method called the Fujiwara-Kumano-go method [5, 8] as a possibility.

Let M be a compact oriented smooth 3-manifold, G a simply connected, connected compact simple Lie group, and $P \to M$ a principal G-bundle over M. Let denote by $\Omega^r(M, \mathfrak{g})$, the space of $\mathfrak{g}$-valued smooth r-forms on M.

Let $\mathcal{A}$ denote the space of connections on P and $\mathcal{G}$ the group of gauge transformations on P. Note that, by fixing a reference connection on P as the origin, we may identify $\mathcal{A}$ with the (infinite-dimensional) vector space $\Omega^1(M, \mathfrak{g})$, and $\mathcal{G}$ with the space $C^\infty(M, G)$ of smooth maps from M to G, respectively. Then the *Chern-Simons integral* of an integrand $F(A)$ is given by

$$\int_{\mathcal{A}/\mathcal{G}} F(A)\, e^{L(A)}\, \mathcal{D}(A), \tag{1.1}$$

where the Chern-Simons Lagrangian L is defined by

$$L(A) = -\frac{ik}{4\pi}\int_M \mathrm{Tr}\left\{A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right\}, \tag{1.2}$$

$\mathcal{D}(A)$ is the *Feynman measure* and the parameter k is a positive integer called the *level of charges.*

Among various integrands, the most typical example of gauge invariant observables is the *Wilson line* defined by

$$F(A) = \prod_{j=1}^{s} \mathrm{Tr}_{R_j}\, \mathcal{P} \exp \int_{\gamma_j} A, \tag{1.3}$$

where $\mathcal{P}$ denotes the product integral (see [4]), γ_j, $j = 1, 2, \ldots, s$, are closed oriented loops, and the trace Tr is taken with respect to some irreducible representation R_j of G assigned to each γ_j.

In Section 2, we give several relevant basic notions, after which we state our results precisely. In Section 3, we prove theorems stated in Section 2 by using the Fujiwara-Kumano-go method.

Throughout this paper, $\sqrt{z}$ is understood to denote the branch of the root of $z \in \mathbf{C}$ where $-\frac{\pi}{2} < arg\sqrt{z} < \frac{\pi}{2}$.

2 Definitions and Results

From the method of superfields of the perturbative formulation of the Chern-Simons integral [2, 3], we have the Lorentz gauge fixed form of the Chern-Simons integral written as

$$\begin{aligned}(2.1)\quad &\int_{\mathcal{A}}\int_{\Phi}\int_{\widehat{C}'}\int_{C'} \mathcal{D}(A)\mathcal{D}(\phi)\mathcal{D}(\widehat{c}')\mathcal{D}(c')F(A_0+A)\\ &\times \exp\left[ik((A,\phi),Q_{A_0}(A,\phi))_+ - \frac{ik}{4\pi}\int_M \mathrm{Tr}\,\frac{2}{3}A\bigwedge A\bigwedge A + \int_M \mathrm{Tr}\,\widehat{c}'d_{A_0} * D_A c'\right].\end{aligned}$$

Here A_0 is a background connection, Q_{A_0} is a twisted Dirac operator and $(\ ,\ )_+$ is the inner product of the Hilbert space $L^2(\Omega_+) = L^2\left(\Omega^1(M,\mathfrak{g}) \oplus \Omega^3(M,\mathfrak{g})\right)$ given by

$$((A,\phi),\,(B,\varphi))_+ = (A,\,B) + (\phi,\,\varphi),$$

where the inner product and the norm on $\Omega^r(M,\mathfrak{g})$ are defined by

$$(2.2)\qquad (\omega,\eta) = -\int_M \mathrm{Tr}\,\omega\wedge *\eta, \quad |\cdot| = \sqrt{(\cdot\,,\,\cdot)}.$$

By heuristically considering

$$\begin{aligned}&\int_{\widehat{C}'}\int_{C'} \mathcal{D}(\widehat{c}')\mathcal{D}(c')\exp\left[\int_M \mathrm{Tr}\,\widehat{c}'d_{A_0} * D_A c'\right]\\ &= \det *d_{A_0} * D_A = \det\Delta_0 \det{}_R *d_{A_0} * D_A,\end{aligned}$$

and balancing out $\det\Delta_0$ by a normalizaion of (2.1), we arrive at the perturbative heuristic formulation of the normalized Chern-Simons integral such that

$$\begin{aligned}(2.3)\quad &\frac{1}{Z}\int_{\mathcal{A}}\int_{\Phi} \mathcal{D}(A)\mathcal{D}(\phi)F(A_0+A)\\ &\times \det{}_R *d_{A_0} * D_A \exp\left[ik((A,\phi),Q_{A_0}(A,\phi))_+ - \frac{ik}{4\pi}\int_M \mathrm{Tr}\,\frac{2}{3}A\bigwedge A\bigwedge A\right],\end{aligned}$$

where

$$(2.4)\qquad Z = \int_{\mathcal{A}}\int_{\Phi} \mathcal{D}(A)\mathcal{D}(\phi)\exp\left[ik((A,\phi),Q_{A_0}(A,\phi))_+\right].$$

To provide mathematical meaning to this, we have first of all to regularize the formal determinant $\det_R *d_{A_0} * D_A$. Briefly we recall Albeverio-Mitoma[1].

Since Q_{A_0} is self-adjoint and elliptic, Q_{A_0} has pure point spectrum [8]. Let $\lambda_j, e_j = (e_j^A, e_j^\phi), j = 1, 2, \cdots$ be the eigenvalues and vectors of Q_{A_0} in $L^2(\Omega_+)$.

Let $\{\nu_j, \xi_j, j = 1, 2, \cdots\}$ be the eigensystem of Δ_0 in $L^2(\Omega^0)$,

$$\vec{c}' = \sum_{j=1}^{\infty} \xi_j \vec{c}'_j \quad \text{and} \quad c' = \sum_{j=1}^{\infty} \xi_j c'_j.$$

Define

$$a_{R,ij}^{\ell} = -\frac{1}{\nu_i} \int_M \mathrm{Tr}\, r_i d_{A_0} \xi_i \bigwedge *[e_\ell^A, r_j \xi_j].$$

Since M is compact, we can choose appropriate real numbers $r_j > 0, j = 1, 2, \ldots$ such that

$$\sum_{i,j} \mid a_{R,ij}^{\ell} \mid < 1. \tag{2.5}$$

For an integer p, we define the Hilbert subspace $H_p(\Omega_+)$ of $L^2(\Omega_+)$ with new inner product $(\ ,\ )_p$ defined by

$$\left((A, \phi), (B, \varphi)\right)_p = \left(A, \left(I + Q_{A_0}^2\right)^p B\right) + \left(\phi, \left(I + Q_{A_0}^2\right)^p \varphi\right),$$

where I is the identity operator on $L^2(\Omega_+)$, and the p-norm on $H_p(\Omega_+)$ is defined as usual by $\|\cdot\|_p = \sqrt{(\cdot\ ,\ \cdot)_p}$.Henceforth we denote $H_p(\Omega_+)$ briefly by H_p whenever no confusion may occur. Let $h_i = \left(1 + \lambda_i^2\right)^{-p/2} e_i$ be the CONS of H_p. Choose a sufficiently large p satisfying the condition

$$\sum_{i=1}^{\infty} \left(1 + \lambda_i^2\right)^{-p} |\lambda_i| < \infty,$$

and guaranteeing the regularizations in what follows.

From now on, we use the brief notations such that

$$\beta_j = \left(1 + \lambda_i^2\right)^{-p/2}, \quad \text{and} \quad a_j = \beta_j^2 \lambda_j.$$

Then for $(A, \phi) \epsilon H_p$, we consider instead of $\det_R *d_{A_0} * D_A$, the regularized determinant defined by

$$\det_{Reg}(A) = \lim_{m \to \infty} \det_{Reg}^m(A),$$

where

$$\det_{Reg}^m(A) = \begin{vmatrix} a_{11}^R(A) & a_{12}^R(A) & \cdot & a_{1m}^R(A) \\ a_{21}^R(A) & a_{22}^R(A) & \cdot & a_{2m}^R(A) \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}^R(A) & a_{m2}^R(A) & \cdot & a_{mm}^R(A) \end{vmatrix},$$

$$a_{ii}^R(A) = 1 + \sum_{\ell=1}^{\infty} \beta_\ell((A,\phi), h_\ell)_p a_{R,ii}^\ell$$

and

$$a_{ij}^R(A) = \sum_{\ell=1}^{\infty} \beta_\ell((A,\phi), h_\ell)_p a_{R,ij}^\ell.$$

The regularized determinant is well defined for sufficiently large p, which is guaranteed by the increasig rates of eigenvalues of Q_{A_0} ((c) of Lemma 1.6.3 in [6]).

Next we proceed to a regularizing the holonomy. From Mitoma-Nishikawa [9], for a given closed smooth curve $\gamma : [0,1] \to M$ in M, for each $t \in [0,1]$ and sufficiently small $\epsilon > 0$ there exists a Poincare dual $C_\gamma^\epsilon(t)$ such that

$$\lim_{\epsilon \to 0} \sup_{0 \leq t \leq 1} \left| (A, C_\gamma^\epsilon(t)) - \sum_{i=1}^{3} \int_0^t A_i{}^\alpha(\gamma(\tau))\dot{\gamma}^i(\tau) d\tau \right| = 0.$$

Since

$$(A,\, C_\gamma^\epsilon(t)) = \Big((A,\phi),\, \big(I + Q_{A_0}^2\big)^{-p} (C_\gamma^\epsilon(t), 0)\Big)_p,$$

by setting

$$\tilde{C}_\gamma^\epsilon(t) = \big(I + Q_{A_0}^2\big)^{-p} (C_\gamma^\epsilon(t), 0), \tag{2.6}$$

we define

$$A_\gamma^\epsilon(t) = \sum_{\alpha=1}^{d} \Big((A,\phi),\, \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha\Big)_p E_\alpha, \tag{2.7}$$

where $\tilde{C}_\gamma^\epsilon(t) = \sum \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha$, $\{E_\alpha\}$ is a basis of $\mathfrak{g}$, and define

$$\bar{A}(t) = \int_{\gamma[0,t]} A.$$

By Chen's iterated integral [4], we define the ϵ-regularization of the holonomy by

$$W_\gamma^\epsilon(A) = I + \sum_{r=1}^{\infty} W_\gamma^{\epsilon,r}(A), \tag{2.8}$$

where

$$W_\gamma^{\epsilon,r}(A) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} d(\bar{A}_0 + A_\gamma^\epsilon)(t_1) d(\bar{A}_0 + A_\gamma^\epsilon)(t_2) \cdots d(\bar{A}_0 + A_\gamma^\epsilon)(t_r),$$

and the ϵ-regularized Wilson line by

$$F^{\epsilon}_{A_0}(A) = \prod_{j=1}^{s} \mathrm{Tr}_{R_j} W^{\epsilon}_{\gamma_j}(A), \tag{2.9}$$

where the trace Tr is taken in the representation R_j of G assigned to each loop γ_j.

Thus adding regularizations of the determinant and holonomy to (2.3) , we have

$$\begin{aligned} &\frac{1}{Z}\int_{\mathcal{A}}\int_{\Phi}\mathcal{D}(A)\mathcal{D}(\phi)F^{\epsilon}_{A_0}(A) \\ &\times \det_{Reg}(A)\exp\Big[ik((A,\phi),Q_{A_0}(A,\phi))_+ - \frac{ik}{4\pi}\int_M \mathrm{Tr}\,\frac{2}{3}A\bigwedge A\bigwedge A\Big], \end{aligned} \tag{2.10}$$

which is a heuristic version of the normalized Chern-Simons integral.

Let us exploit compensations in the numerator and denominator.(see [1]). Then setting

$$\widetilde{F}^{\epsilon}_{A_0}(A) = \det_{Reg}(A)F^{\epsilon}_{A_0}(A),$$

we have a heuristic form of (2.10) such that

$$\begin{aligned} &\frac{1}{\widetilde{Z}}\int_{\mathcal{A}}\int_{\Phi}\mathcal{D}(A)\mathcal{D}(\phi)\widetilde{F}^{\epsilon}_{A_0}(\frac{1}{\sqrt[3]{k}}A) \\ &\times \exp\Big[i\sqrt[3]{k}((A,\phi),Q_{A_0}(A,\phi))_+ - \frac{i}{4\pi}\int_M \mathrm{Tr}\,\frac{2}{3}A\bigwedge A\bigwedge A\Big], \end{aligned} \tag{2.11}$$

where

$$\widetilde{Z} = \int_{\mathcal{A}}\int_{\Phi}\mathcal{D}(A)\mathcal{D}(\phi)\exp\Big[i\sqrt[3]{k}((A,\phi),Q_{A_0}(A,\phi))_+\Big]. \tag{2.12}$$

Based on the heuristic idea such that the asymptotic expansion up to the order $2N$ of (2.11) may be equal to the asymptotic expansion up to the order $2N$ of

$$\begin{aligned} &\frac{1}{\widetilde{Z}}\int_{\mathcal{A}}\int_{\Phi}\mathcal{D}(A)\mathcal{D}(\phi)\widetilde{F}^{\epsilon}_{A_0}(\frac{1}{\sqrt[3]{k}}A) \\ &\times \exp\Big[i\sqrt[3]{k}((A,\phi),Q_{A_0}(A,\phi))_+ - \frac{i}{4\pi}\int_M \mathrm{Tr}\,\frac{2}{3}A\bigwedge A\bigwedge A\Big] \\ &\times \exp\Big[-(\sum_{j=1}^{\infty}|((A,\phi),e_j)_+|)^{2N+2}\Big], \end{aligned} \tag{2.13}$$

from now on, we discuss the asymptotic expansion of (2.13).

For $x = (A,\phi) \in H_p$,

$$x = \sum_{j=1}^{\infty} (x, h_j)_p h_j,$$

$$(x, Q_{A_0} x)_+ = \sum_{j=1}^{\infty} a_j (x, h_j)_p^2,$$

and

$$\frac{-1}{4\pi} \int_M \mathrm{Tr} \frac{2}{3} A \bigwedge A \bigwedge A = \sum_{a,b,c=1}^{\infty} (x, h_a)_p (x, h_b)_p (x, h_c)_p \beta_a \beta_b \beta_c T_{abc},$$

where $T_{abc} = -\int_M \mathrm{Tr} \frac{1}{6\pi} e_a^A \bigwedge e_b^A \bigwedge e_c^A$.

By the idea of justifying the Feynman integral due to Itô [7] such that the convergent factors

$$\exp\left[-\frac{(x,x)_m}{2n}\right] \quad \text{with} \quad n > 0$$

implemented into the m-dimensional approximation of (2.13) and using the formula of changing variables , by setting $x = \sqrt{n} y$, we have

$$\begin{aligned} (2.14) \qquad & \frac{1}{\hat{Z}_{m,n}} \int_{R^m} \widetilde{F}_{A_0}^{\epsilon}\left(\frac{\sqrt{n}}{\sqrt[3]{k}} y^m\right) \exp\left[in\sqrt[3]{k}(y^m, Q_{A_0} y^m)_+\right] \\ & \times \exp\left[i \sum_{abc=1}^{\infty} \sqrt{n} y_a \sqrt{n} y_b \sqrt{n} y_c \beta_a \beta_b \beta_c T_{abc}\right] \\ & \exp\left[-\frac{(y,y)_m}{2} - \left(\sum_{j=1}^{m} \beta_j \sqrt{n} |y_j|\right)^{2N+2}\right] \frac{\nu_m(dy)}{(\sqrt{2\pi})^m}, \end{aligned}$$

where $\nu_m(dy)$ is the m-dimensional Lebesgue measure, $y^m = \sum_{j=1}^m y_j h_j$, and

$$\hat{Z}_{m,n} = \int_{R^m} \exp\left[in\sqrt[3]{k}(y^m, Q_{A_0} y^m)_+\right] \exp\left[-\frac{(y,y)_m}{2}\right] \frac{\nu_m(dy)}{(\sqrt{2\pi})^m}.$$

Setting

$$\begin{aligned} f_n^L(x_L, x_{L-1}, \cdots, x_1) &= \widetilde{F}_{A_0}^{\epsilon}\left(\frac{\sqrt{n}}{\sqrt[3]{k}} x^L\right) \\ & \times \exp\left[i \sum_{abc=1}^{L} \sqrt{n} x_a \sqrt{n} x_b \sqrt{n} x_c \beta_a \beta_b \beta_c T_{abc}\right] \\ & \exp\left[-\left(\sum_{j=1}^{L} \beta_j \sqrt{n} |x_j|\right)^{2N+2}\right], \end{aligned}$$

we give the definition of a perturbative Chern-Simons integral (2.13) by setting it as equal to

$$\lim_{L\to\infty}\lim_{n\to\infty}\frac{1}{\hat{Z}_{L,n}}\int_{R^L} f_n^L(x_L,x_{L-1},\cdots,x_1)\exp\left[in\sqrt[3]{k}(x^L,Q_{A_0}x^L)_+\right] \tag{2.15}$$
$$\exp\left[-\frac{(x,x)_L}{2}\right]\frac{\nu_L(dx)}{(\sqrt{2\pi})^L}.$$

Replace the eigenvalues λ_j of Q_{A_0} by λ_j^ζ for large ζ satisfying

Assumption 1. RENORMALIZATION.

$$\sum_{j=1}^{\infty} j^{2(8N^2+12N+4)}\frac{1}{\sqrt{|\lambda_j|}}\le c<+\infty,$$

which is guaranteed by Lemma 1.6.3 in [6].
Then we have

Theorem 1. *Under the renormalization Assumption 1 and the assumption $T_{abc}\le T<+\infty$, we have* (2.15) *is equal to*

$$\widetilde{F}^\epsilon_{A_0}(0)$$
$$+\lim_{L\to\infty}\left\{\sum_{s=1}^{N}\left(\sum_{r=1}^{s}\left(\sum_{1\le j_1<j_2<j_3\cdots<j_r\le L}\left(\sum_{m_1,m_2,\cdots,m_r\ge1,m_1+m_2+\cdots+m_r=s}\right.\right.\right.\right.$$
$$\lim_{n\to\infty}\left(\left(\prod_{q=1}^{r}\frac{1}{2^{m_q}m_q!(1-2in\sqrt[3]{k}\beta_{j_q}^2\lambda_{j_q})^{m_q}}\partial_{x_{j_q}}^{2m_q}\right)\widetilde{F}^\epsilon_{A_0}\left(\frac{\sqrt{n}}{\sqrt[3]{k}}x^L\right)\right. \tag{2.16}$$
$$\left.\left.\left.\left.\left.\times\exp\left[i\sum_{abc=1}^{L}\sqrt{n}x_a\sqrt{n}x_b\sqrt{n}x_c\beta_a\beta_b\beta_cT_{abc}\right]\right)(0)\right)\right)\right)\right\}$$
$$+O\left(\left(\frac{1}{\sqrt[3]{k}}\right)^{N+1}\right),$$

for sufficiently large k, where $\frac{\partial}{\partial x}=\partial_x$.

3 Proof of Theorem

The non-normalized form of (2.15) **is equal to**

$$\lim_{L\to\infty}\lim_{n\to\infty}\left(\frac{1}{\sqrt{2\pi}}\right)^L\int_{R^L} e^{-\sum_{j=1}^L\frac{1}{2}(1-2i\sqrt[3]{k}na_j)x_j^2}f^L(x_L,x_{L-1},\cdots,x_1)\prod_{j=1}^{L}dx_j, \tag{3.1}$$

where $f^L(x_L,x_{L-1},\cdots,x_1)$ is a version of $f_n^L(x_L,x_{L-1},\cdots,x_1)$ such that

$$\exp\left[-(\sum_{j=1}^{L}\sqrt{n}\beta_j|x_j|)^{2N+2}\right]$$

in $f_n^L(x)$ is replaced by

$$\begin{cases} \exp\left[-(\sum_{j=1}^L \sqrt{n}\beta_j x_j)^{2N+2}\right], & \text{if} x_j \geq 0, \\ \exp\left[-(-\sum_{j=1}^L \sqrt{n}\beta_j x_j)^{2N+2}\right], & \text{if} x_j \leq 0. \end{cases}$$

When we consider the function $f^L(x_N, x_{L-1}, \cdots, x_j, \cdots, x_1)$ as a function of j-th coordinate x_j, denote it by $\hat{f}^L(x_j)$.

Since

$$\hat{f}^L(x_j) = \hat{f}^L(0) + x_j \partial_{x_j} \hat{f}^L(0) + \cdots + \frac{x_j^{2N}}{(2N)!} \partial_{x_j}^{2N} \hat{f}^L(0)$$
$$+ x_j^{2N+1} \int_0^1 \frac{(1-\theta_j)^{2N}}{(2N)!} \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) d\theta_j,$$

so that set

$$Q_j^0 f^L = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2} \left\{ \hat{f}^L(0) + x_j \partial_{x_j} \hat{f}^L(0) + \cdots + \frac{x_j^{2N}}{(2N)!} \partial_{x_j}^{2N} \hat{f}^L(0) \right\} dx_j$$

and

$$Q_j^1 f^L = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2} x_j^{2N+1} \left(\int_0^1 \frac{(1-\theta_j)^{2N}}{(2N)!} \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) d\theta_j \right) dx_j.$$

Setting for any well differentiable function $g^L(x_L, x_{L-1}, \cdots, x_2, x_1)$,

$$D_j^0 g^L = \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \hat{g}^L(0),$$
$$D_j^m g^L = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2} x_j^m \partial_{x_j}^m \hat{g}^L(0) dx_j, \quad 1 \leq m \leq 2N,$$

we get

$$Q_L^0 \cdots Q_2^0 Q_1^0 f^L$$
$$= (\sum_{m=0}^{2N} D_L^m)(\sum_{m=0}^{2N} D_{L-1}^m) \cdots (\sum_{m=0}^{2N} D_1^m f^L). \tag{3.2}$$

Before proceeding to the estimate of the leading terms, we remark

Lemma 1. *For any integer* $0 \leq m \leq 2N$,

$$\left| D_j^m f^L \right| \leq \frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|} \left| \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \right|^m \left| \partial_{x_j}^m f^L(x_L, \cdots, x_{j+1}, 0, x_{j-1}, \cdots, x_1) \right|.$$

Now we estimate the leading terms . The absolute value of the above (3.2) is dominated by

(3.3)

$$\Big|\Big(\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big)f^L(0,0,\cdots,0)\Big| + \sum_{1\le j_1\le L}\Big(\sum_{m_1=1}^{2N}|D_L^0D_{L-1}^0\cdots D_{j_1}^{m_1}\cdots D_2^0D_1^0f^L|\Big)$$

$$+\sum_{1\le j_1<j_2\le L}\Big(\sum_{m_1,m_2=1}^{2N}|D_L^0D_{L-1}^0\cdots D_{j_2}^{m_2}\cdots D_{j_1}^{m_1}\cdots D_2^0D_1^0f^L|\Big)$$

$$+\cdots\cdots+\cdots+\cdots$$

$$+\sum_{1\le j_1<j_2<j_3\cdots<j_r\le L}\Big(\sum_{m_1,m_2,\cdots,m_r=1}^{2N}|D_L^0D_{L-1}^0\cdots D_{j_r}^{m_r}\cdots D_{j_2}^{m_2}\cdots D_{j_1}^{m_1}\cdots D_2^0D_1^0f^L|\Big)$$

$$+\cdots\cdots+\cdots+\cdots.$$

By Lemma 1,

(3.4)

$$\Big|D_L^0D_{L-1}^0\cdots D_{j_r}^{m_r}\cdots D_{j_2}^{m_2}\cdots D_{j_1}^{m_1}\cdots D_2^0D_1^0f^L\Big|$$

$$\le\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_L}|}\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_{L-1}}|}\cdots\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_{j_r}}|}\left|\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_{j_r}}}\right|^{m_r}$$

$$\times\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_{j_r-1}}|}\cdots\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_{j_{r-1}}}|}\left|\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_{j_{r-1}}}}\right|^{m_{r-1}}$$

$$\times\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_{j_{r-1}-1}}|}\cdots\cdots\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_{j_1}}|}\left|\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_{j_1}}}\right|^{m_1}\cdots\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_1}|}$$

$$\times\Big|(\partial_{x_{j_r}}^{m_r}\cdots\partial_{x_{j_1}}^{m_1}f^L)(0,0,\cdots,0,0,\cdots,0,0)\Big|.$$

Here we state the key lemma will be omitted the proof because of the restriction of the pages.

Lemma 2. *For any non-negative integer N and L, there exists some constants $A_{2N}\ge 0$, $B_{2N}\ge 1$ and $\alpha(=2)\ge 1$ such that for any non-negative integers $1\le j_1<j_2<\cdots<j_r\le L$ and $m_j\le 2N$,*

$$\Big|\Big(\prod_{s=1}^{r}\big(\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_{j_s}}}\big)^{\alpha_{j_s}}\Big)\Big(\prod_{j=1,\cdots,L,\ne j_1,j_2,\cdots,j_r}\big(\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\big)^{m_j}\Big)\Big|$$

$$\sup_{2N+1\le\alpha_{j_s}\le 4N+4,x_{j_s},1\le s\le r}\Big|e^{-\sum_{s=1}^{r}\frac{1}{2}x_{j_s}^2}\partial_{x_L}^{m_L}\cdots\partial_{x_{j_r}}^{\alpha_{j_r}}$$

$$\cdots \partial_{x_{j_2}}^{\alpha_{j_2}} \cdots \partial_{x_{j_1}}^{\alpha_{j_1}} \cdots \partial_{x_1}^{m_1} f^L(0,\cdots,x_{j_r},0,\cdots,x_{j_1},0,\cdots,0)\Big|$$
$$\leq A_{2N}\Big(\{(2N+1)\sum_{j=1,\cdots,L,\neq j_1,j_2,\cdots,j_r} m_j + (2N+1)\sum_{s=1}^{r}\alpha_{j_s}\}!\Big)^{\alpha}$$
$$\Big(\prod_{s=1}^{r}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{\alpha_{j_s}}\Big)$$
$$\Big(\prod_{j=1,\cdots,L,\neq j_1,j_2,\cdots,j_r}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{m_j}\Big).$$

Then Lemma 2 implies

$$\Big|\Big(\prod_{s=1}^{r}(\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_{j_s}}})^{m_s}\Big)(\partial_{x_{j_r}}^{m_r}\cdots\partial_{x_{j_1}}^{m_1}f^L)(0,0,\cdots,0,0,\cdots,0,0)\Big|$$
$$\leq A_{2N}(\{(2N+1)(\sum_{s=1}^{r}m_s)\}!)^{\alpha}\Big(\prod_{s=1}^{r}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{m_s}\Big),$$

which, together with (3.4), yields

$$\Big|D_L^0 D_{L-1}^0\cdots D_{j_r}^{m_r}\cdots D_{j_2}^{m_2}\cdots D_{j_1}^{m_1}\cdots D_2^0 D_1^0 f^L\Big|$$
$$\leq\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\Big)A_{2N}(\{(2N+1)(\sum_{s=1}^{r}m_s)\}!)^{\alpha}\Big(\prod_{s=1}^{r}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{m_s}\Big). \tag{3.5}$$

Combining (3.3) and (3.5), we get that the absolute value of (3.2) is dominated by

(3.6)
$$\Big|\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big|\Big\{A_{2N}+A_{2N}\sum_{1\leq j_1\leq L}\Big(\sum_{m_1=1}^{2N}B_{2N}^{m_1}(\{(2N+1)m_1\}!)^{\alpha}\Big|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\Big|^{m_1}\Big)$$
$$+A_{2N}\sum_{1\leq j_1<j_2\leq L}\Big(\sum_{m_1,m_2=1}^{2N}B_{2N}^{m_1+m_2}(\{(2N+1)(m_1+m_2)\}!)^{\alpha}\Big|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\Big|^{m_1}\Big|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\Big|^{m_2}\Big)$$
$$+\cdots\cdots+\cdots\cdots$$
$$+A_{2N}\sum_{1\leq j_1<j_2<j_3\cdots<j_r\leq L}\Big(\sum_{m_1,m_2,\cdots,m_r=1}^{2N}B_{2N}^{m_1+m_2+\cdots+m_r}$$
$$(\{(2N+1)(m_1+m_2+\cdots+m_r)\}!)^{\alpha}\Big|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\Big|^{m_1}\Big|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\Big|^{m_2}\cdots\Big|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}}\Big|^{m_r}\Big)$$
$$+\cdots\cdots+\cdots\cdots\Big\}.$$

Since

$$
\begin{aligned}
&(\{(2N+1)(m_1+m_2+\cdots+m_r)\}!)^{\alpha}\\
&\le (\{r(2N+1)2N\}!)^{\alpha} \le \{((p_N)^{p_N})^r r!^{p_N}\}^{\alpha}\\
&= ((p_N)^{p_N\alpha})^r r!^{p_N\alpha},
\end{aligned}
$$

where

$$p_N = (2N+1)2N,$$

we have the above equation is dominated by

$$
\begin{aligned}
&\left|\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\right|A_{2N}\Bigg\{1+B_{2N}^{2N}\sum_{1\le j_1\le L}\left(\sum_{m_1=1}^{2N}((p_N)^{p_N\alpha})\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}\right)\\
&+(B_{2N}^{2N})^2\sum_{1\le j_1<j_2\le L}\Bigg(\sum_{m_1,m_2=1}^{2N}((p_N)^{p_N\alpha})^2 2!^{p_N\alpha}\frac{1}{1^{p_N\alpha}2^{p_N\alpha}}\\
&j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1} j_2^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\right|^{m_2}\Bigg)\\
&+\cdots\cdots+\cdots\cdots\\
&+(B_{2N}^{2N})^r\sum_{1\le j_1<j_2<j_3\cdots<j_r\le L}\Bigg(\sum_{m_1,m_2,\cdots,m_r=1}^{2N}((p_N)^{p_N\alpha})^r r!^{p_N\alpha}\frac{1}{1^{p_N\alpha}2^{p_N\alpha}\ldots r^{p_N\alpha}}\\
&j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1} j_2^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\right|^{m_2}\cdots j_r^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}}\right|^{m_r}\Bigg)\\
&+\cdots\cdots+\cdots\cdots\Bigg\}\\
&\le\left|\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\right|A_{2N}\Bigg\{1+B_{2N}^{2N}\sum_{1\le j_1\le L}\left(\sum_{m_1=1}^{2N}((p_N)^{p_N\alpha})\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}\right)\\
&+(B_{2N}^{2N})^2\sum_{1\le j_1<j_2\le L}\Bigg(\sum_{m_1,m_2=1}^{2N}((p_N)^{p_N\alpha})^2\\
&j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1} j_2^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\right|^{m_2}\Bigg)\\
&+\cdots\cdots+\cdots\cdots\\
&+(B_{2N}^{2N})^r\sum_{1\le j_1<j_2<j_3\cdots<j_r\le L}\Bigg(\sum_{m_1,m_2,\cdots,m_r=1}^{2N}((p_N)^{p_N\alpha})^r j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}
\end{aligned}
$$

$$j_2^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\right|^{m_2}\cdots j_r^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}}\right|^{m_r}\Bigg)$$

$$+\cdots\cdots+\cdots\cdots\Bigg\}$$

$$\leq\left|\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\right|A_{2N}\Bigg\{1+[B_{2N}^{2N}(p_N)^{p_N\alpha}]\sum_{1\leq j_1\leq L}\left(\sum_{m_1=1}^{2N}j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}\right)$$

$$+\frac{1}{2!}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^2\left(\sum_{j_1=1}^{L}\left(\sum_{m_1=1}^{2N}j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}\right)\right)^2$$

$$+\cdots\cdots+\cdots\cdots$$

$$+\frac{1}{r!}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^r\left(\sum_{j_1=1}^{L}\left(\sum_{m_1=1}^{2N}j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}\right)\right)^r$$

$$+\cdots\cdots+\cdots\cdots\Bigg\}.$$

By the Assumption 1 such that

$$\sum_{j=1}^{\infty}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\leq c<+\infty,$$

we have

$$\sum_{j=1}^{L}\left(\sum_{m=1}^{2N}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|^{m}\right)$$
$$\leq\left(\sum_{j=1}^{L}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\right)\left(1+\sum_{j=1}^{L}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\right)^{2N-1}$$
$$\leq c(1+c)^{2N-1}.$$

Hence we get that the absolute value of (3.2) is dominated by

(3.7)
$$\Big|\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big|A_{2N}\Big\{1+[B_{2N}^{2N}(p_N)^{p_N\alpha}](c(1+c)^{2N-1})$$
$$+\frac{1}{2!}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^2(c(1+c)^{2N-1})^2+\cdots\cdots+\cdots\cdots$$
$$+\frac{1}{r!}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^r(c(1+c)^{2N-1})^r+\cdots\cdots+\cdots\cdots\Big\}$$
$$\leq\Big|\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big|A_{2N}\exp\Big[[B_{2N}^{2N}(p_N)^{p_N\alpha}]c(1+c)^{2N-1}\Big]$$

and we also have

(3.8)
$$\sum_{r=2N+2}^{\infty}A_{2N}\frac{1}{r!}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^r\left(\sum_{j=1}^{L}\left(\sum_{m=1}^{2N}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|^m\right)\right)^r$$
$$\leq\Big(\sum_{j=1}^{L}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\Big)^{2N+2}A_{2N}(1+c)^{(2N-1)(2N+1)}\exp\Big[[B_{2N}^{2N}(p_N)^{p_N\alpha}](1+c)^{2N}\Big].$$

Further for any natural number $2\leq r\leq 2N+1$, we get

(3.9)
$$\sum_{1\leq j_1<j_2<j_3\cdots<j_r\leq L}\Big(\sum_{1\leq m_1,m_2,\cdots,m_r\leq 2N+1,\sum_{s=1}^{r}m_s\geq 2N+2}A_{2N}B_{2N}^{m_1+m_2+\cdots+m_r}$$
$$(\{(2N+1)(m_1+m_2+\cdots+m_r)\}!)^{\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\right|^{m_2}\cdots\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}}\right|^{m_r}\Big)$$
$$\leq\sum_{1\leq j_1<j_2<j_3\cdots<j_r\leq L}\Big(\sum_{1\leq m_1,m_2,\cdots,m_r\leq 2N+1,\sum_{s=1}^{r}m_s\geq 2N+2}$$
$$A_{2N}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^r j_1^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|^{m_1}j_2^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}}\right|^{m_2}\cdots j_r^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}}\right|^{m_r}\Big)$$
$$\leq A_{2N}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^{2N+1}\sum_{1\leq j_1<j_2<j_3\cdots<j_r\leq L}\Big(\sum_{1\leq m_1,m_2,\cdots,m_r\leq 2N+1,\sum_{s=1}^{r}m_s\geq 2N+2}$$
$$\Big(\sum_{j=1}^{L}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\Big)^{\sum_{s=1}^{r}m_s}$$
$$\leq\Big(\sum_{j=1}^{\infty}j^{p_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\Big)^{2N+2}A_{2N}[B_{2N}^{2N}(p_N)^{p_N\alpha}]^{2N+1}(2N+1)^{2N+1}(1+c)^{(2N+1)^2}.$$

Since

$$\partial^{m_r}_{x_{j_r}}\partial^{m_{r-1}}_{x_{j_{r-1}}}\cdots\partial^{m_1}_{x_{j_1}}\exp\left[-(\sum_{j=1}^{L}\sqrt{n}\beta_j|x_j|)^{2p+2}\right](0)=0$$

if $m_1+\cdots+m_{r-1}+m_r\le 2N+1$, **we have, together with** (3.6), (3.8) **and** (3.9),

$$Q^0_L\cdots Q^0_2Q^0_1f^L$$

is rewritten as

(3.10)

$$\Big(\prod_{j=1}^{L}\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big)\Big\{f^L(0,0,\cdots,0)$$

$$+\sum_{s=1}^{N}\Bigg(\sum_{r=1}^{s}\Bigg(\sum_{1\le j_1<j_2<j_3\cdots<j_r\le L}\Bigg(\sum_{m_1,m_2,\cdots,m_r\ge1,m_1+m_2+\cdots+m_r=s}\frac{1}{2^{m_1}m_1!(1-2in\sqrt[3]{k}a_{j_1})^{m_1}}\partial^{2m_1}_{x_{j_1}}$$

$$\frac{1}{2^{m_2}m_2!(1-2in\sqrt[3]{k}a_{j_2})^{m_2}}\partial^{2m_2}_{x_{j_2}}\cdots\frac{1}{2^{m_r}m_r!(1-2in\sqrt[3]{k}a_{j_r})^{m_r}}\partial^{2m_r}_{x_{j_r}}f^L(0,0,\cdots,0)\Bigg)\Bigg)\Bigg)$$

$$+O\Bigg(\Big(\sum_{j=1}^{\infty}j^{pN\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\Big)^{2N+2}\Bigg)\Big\}.$$

Hence letting $L\to\infty$ **in** (3.10), **we have** (2.16).

Now we estimate the remaining terms again by the method similar to that in the leading terms. In this turn, we also begin with the proof of

Lemma 3. *There exists a positive constant* C_N *such that*

(3.11)

$$|Q^1_jf^L|\le\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}C_N\Big(\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\Big)^{\alpha_j}$$
$$\sup_{2N+1\le|\alpha_j|\le4N+4,x_j}e^{-\frac{x_j^2}{2}}\left|\partial^{\alpha_j}_{x_j}\hat{f}^L(x_j)\right|.$$

Proof. Apply the Fubini theorem and use the integration by parts formula, we have

$$Q^1_jf^L=\int_0^1d\theta_j\frac{(1-\theta_j)^{2N}}{(2N)!}\frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2}x_j^{2N+1}$$
$$\partial^{2N+1}_{x_j}\hat{f}^L(\theta_jx_j)dx_j$$
$$=\int_0^1d\theta_j\frac{(1-\theta_j)^{2N}}{(2N)!}\frac{1}{\sqrt{2\pi}}\int\frac{-\partial_{x_j}}{1-2in\sqrt[3]{k}a_j}(e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2})x_j^{2N}$$

$$\begin{aligned}
&\partial_{x_j}^{2N+1}\hat{f}^L(\theta_j x_j)dx_j \\
&= \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!}\frac{1}{(1-2in\sqrt[3]{k}a_j)\sqrt{2\pi}}\int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2}2Nx_j^{2N-1} \\
&\partial_{x_j}^{2N+1}\hat{f}^L(\theta_j x_j)dx_j \\
&+ \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!}\frac{1}{(1-2in\sqrt[3]{k}a_j)\sqrt{2\pi}}\int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2}x_j^{2N} \\
&\theta_j\partial_{x_j}^{2N+2}\hat{f}^L(\theta_j x_j)dx_j.
\end{aligned}$$

Repeating this process until x_j vanishes, we have

$$\begin{aligned}
&Q_j^1 f^L = \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!}\frac{1}{(1-2in\sqrt[3]{k}a_j)^{N+1}\sqrt{2\pi}}2N(2N-2)(2N-4)\cdots 2 \\
&\left\{-\partial_{x_j}^{2N+1}\hat{f}^L(0) + \int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2}\right. \\
&\left.\times\, \theta_j\partial_{x_j}^{2N+2}\hat{f}^L(\theta_j x_j)dx_j\right\} \\
&+\cdots+\cdots\cdots \\
&+ \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!}\frac{1}{(1-2in\sqrt[3]{k}a_j)^{2N+1}\sqrt{2\pi}}\int e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2} \\
&\theta_j^{2N+1}\partial_{x_j}^{4N+2}\hat{f}^L(\theta_j x_j)dx_j.
\end{aligned} \tag{3.12}$$

Setting

$$M_j = \frac{1 - x_j\partial_{x_j}}{1+(1-2in\sqrt[3]{k}a_j)x_j^2},$$

we have

$$M_j e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2} = e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2}.$$

Noticing M_j^* denotes the adjoint operator of M_j in $L^2(dx_j)$, we get for $G(x)$ of being well differentiable function and of satisfying

$$\lim_{|x_j|\to\infty} e^{-\frac{1}{2}x_j^2}|G(x)| = 0,$$

$$M_j^* G(x) = \frac{2G(x)}{(1+(1-2in\sqrt[3]{k}a_j)x_j^2)^2} + \frac{x_j\partial_{x_j}G(x)}{1+(1-2in\sqrt[3]{k}a_j)x_j^2},$$

so that we have

(3.13)

$$|(M_j^*)^2G(x)| \le \Big\{\Big|\frac{4G(x)}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big| + 8\Big|G(x)\Big|\Big|\frac{(1-2in\sqrt[3]{k}a_j)x_j^2}{(1+(1-2in\sqrt[3]{k}a_j)x_j^2)^4}\Big|$$

$$+4\Big|\frac{x_j}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|\Big|\frac{\partial_{x_j}G(x)}{(1+(1-2in\sqrt[3]{k}a_j)x_j^2)^2}\Big| + \Big|\frac{x_j}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|\Big|\frac{\partial_{x_j}G(x)}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|$$

$$+2\Big|\partial_{x_j}G(x)\Big|\Big|\frac{x_j}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|\Big|\frac{(1-2in\sqrt[3]{k}a_j)x_j^2}{(1+(1-2in\sqrt[3]{k}a_j)x_j^2)^2}\Big|$$

$$+\Big|\frac{x_j^2}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|\Big|\frac{\partial_{x_j}^2G(x)}{1+(1-2in\sqrt{3}]ka_j)x_j^2}\Big|\Big\}.$$

Therefore for any positive integer q, we have

$$|e^{-\frac{1}{2}(1-2in\sqrt[3]{k}a_j)x_j^2}(M_j^*)^2(\partial_{x_j}^q\hat{f}^L)(\theta_jx_j)|$$

$$\le 20\Big(\sup_{|\alpha_j|\le 2,\,x_j,\,|\theta_j|\le 1}\Big(\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\Big)^{\alpha_j}\Big|e^{-\frac{1}{2}x_j^2}\partial_{x_j}^{\alpha_j+q}\hat{f}^L(\theta_jx_j)\Big|\Big)\times\Big|\frac{1}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|.$$

Notcing that

$$\int\Big|\frac{1}{1+(1-2in\sqrt[3]{k}a_j)x_j^2}\Big|dx_j \le \frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\int\frac{1}{\sqrt{1+y^4}}dy$$

and taking

$$\sup_{2N+2\le|\alpha_j|\le 4N+4,\,x_j,\,|\theta_j|\le 1}\Big(\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\Big)^{\alpha_j}\Big|e^{-\frac{1}{2}x_j^2}\partial_{x_j}^{\alpha_j}\hat{f}^L(\theta_jx_j)\Big|$$

$$\le \sup_{2N+2\le|\alpha_j|\le 4N+4,\,x_j,\,|\theta_j|\le 1}\Big(\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\Big)^{\alpha_j}\Big|e^{-\frac{1}{2}\theta_j^2x_j^2}\partial_{x_j}^{\alpha_j}\hat{f}^L(\theta_jx_j)\Big|$$

$$\le \sup_{2N+2\le|\alpha_j|\le 4N+4,\,x_j}\Big(\frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}\Big)^{\alpha_j}\Big|e^{-\frac{1}{2}x_j^2}\partial_{x_j}^{\alpha_j}\hat{f}^L(x_j)\Big|$$

and for example,

$$\Big|\frac{1}{(1-2in\sqrt[3]{k}a_j)^{N+1}\sqrt{2\pi}}\Big\{-\partial_{x_j}^{2N+1}\hat{f}^L(0)\Big\}\Big|$$

$$\le \Big|\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big|\Big|\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}}\Big|^{2N+1}\sup_{x_j}\Big|e^{-\frac{1}{2}x_j^2}\partial_{x_j}^{2N+1}\hat{f}^L(x_j)\Big|$$

into account, we have the desired inequality from (3.12). □

Now we return to the estimate the remaining terms such that

$$
(3.14)\quad
\begin{aligned}
&\left|\sum_{k_L,\cdots,k_2,k_1=0,1,(k_L,\cdots,k_2,k_1)\neq(0,0,\cdots,0)} Q_L^{k_L}\cdots Q_2^{k_2}Q_1^{k_1}f^L\right|\\
&\le\sum_{r=1}^{L}\sum_{1\le j_1<j_2<\cdots<j_r\le L}\left|Q_L^0\cdots Q_{j_r}^1\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots Q_1^0 f^L\right|.
\end{aligned}
$$

First we discuss the case of $r\ge 2$.

$$
(3.15)\quad
\begin{aligned}
&\left|Q_L^0\cdots Q_{j_r}^1\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots Q_1^0 f^L\right|\\
&=\left|(\sum_{m=0}^{2N} D_L^m)\cdots\cdots Q_{j_r}^1\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots(\sum_{m=0}^{2N} D_1^m)f^L\right|\\
&\le\left|D_L^0\cdots\cdots Q_{j_r}^1\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots D_1^0 f^L\right|\\
&+\cdots+\cdots+\cdots\\
&+\Big(\sum_{1\le l_1<l_2<\cdots<l_r\le L,l_1,l_2,\cdots,l_r\neq j_1,j_2,\cdots,j_r}\Big)\\
&\Big(\sum_{m_{l_1},m_{l_2},\cdots,m_{l_r}=1,m_i=0,i\neq\{l_1,l_2,\cdots,l_r\}}^{2N}\Big)\\
&\left|D_L^{m_L}\cdots Q_{j_r}^1\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots D_1^{m_1}f^L\right|+\cdots+\cdots.
\end{aligned}
$$

Next we estimate

$$
(3.16)\quad \left|D_L^{m_L}\cdots D_{j_r+1}^{m_{j_r+1}}Q_{j_r}^1 D_{j_r-1}^{m_{j_r-1}}\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots D_1^{m_1}f^L\right|.
$$

By the manner similar to that in (3.4), **we arrive at** (3.16) **is dominated by**

$$
(3.17)\quad
\begin{aligned}
&\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)\Big(\prod_{j\neq j_1,j_2,\cdots,j_r}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}^{m_j}\Big)\\
&\times C_N^r\sup_{2N+1\le|\alpha_{j_s}|\le 4N+4,x_{j_s},1\le s\le r}\Big(\prod_{s=1}^{r}\Big(\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)^{\alpha_{j_s}}\Big)\\
&\times\Big|e^{-\sum_{s=1}^{r}\frac12 x_{j_s}^2}\Big(\partial_{x_1}^{m_1}\cdots\partial_{x_{j_1}}^{\alpha_{j_1}}\cdots\partial_{x_{j_2}}^{\alpha_{j_2}}\cdots\partial_{x_{j_r-1}}^{m_{j_r-1}}\partial_{x_{j_r}}^{\alpha_{j_r}}\partial_{x_{j_r+1}}^{m_{j_r+1}}\cdots\partial_{x_L}^{m_L}\\
&f^L((0,\cdots,0,x_{j_r},\cdots,x_{j_2},\cdots,x_{j_1},\cdots,0)\Big|.
\end{aligned}
$$

Suppose that

$$1 \le l_1 < l_2 < \cdots < l_t \le L, l_1, l_2, \cdots, l_t \ne j_1, j_2, \cdots, j_r$$

and

$$m_{l_j} \ge 1, j = 1, 2, \cdots s, m_i = 0, i \ne \{l_1, l_2, \cdots, l_t\}.$$

Setting

$$\hat{p}_N = (2N+1)(4N+4),$$

we have , by Lemma 2,

$$\begin{aligned}
&\Big(\prod_{j\ne j_1, j_2, \cdots, j_r} \frac{1}{|\sqrt{1-2in\sqrt[3]{k}a_j}|}^{m_j}\Big) \\
&\sup_{2N+1\le|\alpha_{j_s}|\le 4N+4, x_{j_s}, 1\le s\le r} \Big(\prod_{s=1}^{r}\big(\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\big)^{\alpha_{j_s}}\Big) \\
&\times \Big|e^{-\sum_{s=1}^{r}\frac{1}{2}x_{j_s}^2}(\partial_{x_L}^{m_L}\cdots\partial_{x_{j_r}}^{\alpha_{j_r}} \\
&\cdots\partial_{x_{j_2}}^{\alpha_{j_2}}\cdots\partial_{x_{j_1}}^{\alpha_{j_1}}\cdots\partial_{x_1}^{m_1} f^L((0,\cdots,0,x_{j_r},\cdots,x_{j_2},\cdots,x_{j_1},\cdots,0)\Big| \\
&\le A_{2N} \sup_{2N+1\le|\alpha_{j_s}|\le 4N+4} (\{(2N+1)\sum_{s=1}^{r}\alpha_{j_s} + (2N+1)\sum_{j=1}^{t} m_{l_j}\}!)^{\alpha} \\
&\Big(\prod_{s=1}^{r}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{\alpha_{j_s}}\Big)\Big(\prod_{j=1}^{t}(B_{2N}\frac{1}{|\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{m_{l_j}}\Big) \\
&\le A_{2N}\hat{p}_N^{\hat{p}_N\alpha(t+r)}\{(t+r)!\}^{\hat{p}_N\alpha} \\
&\frac{1}{1^{\hat{p}_N\alpha}2^{\hat{p}_N\alpha}\cdots(t+r)^{\hat{p}_N\alpha}} \\
&\times\Big(\prod_{s=1}^{r} j_s^{\hat{p}_N\alpha}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{\alpha_{j_s}}\Big) \\
&\Big(\prod_{j=1}^{t} l_j^{\hat{p}_N\alpha}(B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{m_{l_j}}\Big).
\end{aligned} \tag{3.18}$$

Summing up, (3.16),(3.17) **and** (3.18),**and setting** $c_N = \max\{1, C_N\hat{p}_N^{\hat{p}_N\alpha}B_{2N}^{4N+4}(1+c)^{2N+3}\}$, **we have for sufficiently large** k,

$$
\begin{aligned}
&\left|D_L^{m_L}D_{L-1}^{m_{L-1}}\cdots D_{j_r+1}^{m_{j_r+1}}Q_{j_r}^1D_{j_r-1}^{m_{j_r-1}}\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots D_1^{m_1}f^L\right|\\
&\le\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)A_{2N}\hat{p}_N^{\hat{p}_N\alpha t}\Big(\prod_{j=1}^{t}l_j^{\hat{p}_N\alpha}\left|B_{2N}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|^{m_{l_j}}\Big)\\
&\quad\times C_N^r\hat{p}_N^{\hat{p}_N\alpha r}\Big(\prod_{s=1}^{r}j_s^{\hat{p}_N\alpha}B_{2N}^{4N+4}(1+c)^{2N+3}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}}\right|^{2N+1}\Big)\\
&\le\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)A_{2N}\hat{p}_N^{\hat{p}_N\alpha t}\Big(\prod_{j=1}^{t}l_j^{\hat{p}_N\alpha}\left|c_N\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|^{m_{l_j}}\Big)\\
&\quad\times c_N^r\Big(\prod_{s=1}^{r}j_s^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}}\right|^{2N+1}\Big).
\end{aligned}\tag{3.19}
$$

Therefore by (3.19) **and** (3.15), **we have**

$$
\begin{aligned}
&\left|Q_L^0\cdots Q_{j_r}^1\cdots Q_{j_2}^1\cdots Q_{j_1}^1\cdots Q_1^0f^L\right|\\
&\le\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)c_N^r\Big(\prod_{s=1}^{r}j_s^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}}\right|^{2N+1}\Big)A_{2N}\Big\{1+\\
&+\hat{p}_N^{\hat{p}_N\alpha}\Big(\sum_{1\le l_1\le L,l_1\neq j_1,j_2,\cdots,j_r}\Big)\Big(\sum_{m_{l_1}=1,m_i=0,i\neq l_1}^{2N}\Big)l_1^{\hat{p}_N\alpha}\left|c_N\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{l_1}|}}\right|^{m_{l_1}}\\
&+\cdots+\cdots\\
&+\hat{p}_N^{\hat{p}_N\alpha t}\Big(\sum_{1\le l_1<l_2<\cdots<l_t\le L,l_1,l_2,\cdots,l_t\neq j_1,j_2,\cdots,j_r}\Big)\Big(\sum_{m_{l_1},m_{l_2},\cdots,m_{l_t}=1,m_i=0,i\neq\{l_1,l_2,\cdots,l_t\}}^{2N}\Big)\\
&\Big(\prod_{j=1}^{t}l_j^{\hat{p}_N\alpha}\left|c_N\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{l_j}|}}\right|^{m_{l_j}}\Big)+\cdots+\cdots\Big\}.
\end{aligned}\tag{3.20}
$$

Since $m_{l_i}\le 2N$, **setting**

$$
M=A_{2N}\sum_{t=0}^{\infty}\frac{1}{t!}\left(c_N^{2N}\hat{p}_N^{\hat{p}_N\alpha}\right)^t\left(\sum_{j=1}^{\infty}\left(\sum_{m=1}^{2N}j^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|^m\right)\right)^t,
$$

by the manner similar to that in estimating the leading terms, (3.20) **is dominated by**

$$(3.21)\qquad \Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)Mc_N^r\Big(\prod_{s=1}^{r} j_s^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}}\right|^{2N+1}\Big).$$

Therefore by (3.14) and (3.21), we arrive at for sufficiently large k,

$$\left|\sum_{k_L,\cdots,k_2,k_1=0,1,(k_L,\cdots,k_2,k_1)\neq(0,0,\cdots,0),\sum_{j=1}^{L}k_j\geq 2} Q_L^{k_L}\cdots Q_2^{k_2}Q_1^{k_1}f^L\right|$$

$$\leq\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)\Big\{\sum_{r=2}^{L}\frac{1}{r!}Mc_N^r\Big(\sum_{j=1}^{L}j^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}\right|\Big)^{(2N+1)r}\Big\}$$

$$\leq\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)M\Big(\sum_{j=1}^{L}j_s^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}}\right|\Big)^{(2N+2)}c_N^2c^{2N}\exp[c_Nc^{2N+1}]$$

$$\leq\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)\Big\{O\Big(\Big(\frac{1}{\sqrt[3]{k}}\Big)^{N+1}\Big)\Big\}.$$

Next we discuss the case of $r=1$. Since

$$\partial_{x_j}^{2N+1}f^L(0)=0,$$

so that the discussion after (3.12) implies there exists a constant C such that for sufficiently large k,

$$|D_L^0\cdots\cdots Q_{j_1}^1\cdots D_1^0f^L|$$
$$\leq C\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)\Big(j_1^{\hat{p}_N\alpha}\left|\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}\right|\Big)^{(2N+2)}.$$

Further, since

$$\left|Q_L^0\cdots Q_{j_1}^1\cdots Q_1^0f^N-D_L^0\cdots Q_{j_1}^1\cdots D_1^0f^L\right|$$
$$\leq\Big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\Big)\Big[A_{2N}\sum_{r=1}^{\infty}\frac{1}{r!}\big(c_N^{2N}\hat{p}_N^{\hat{p}_N\alpha}\big)^r\Big(\sum_{j=1}^{\infty}\Big(\sum_{m=1}^{2N}j^{\hat{p}_N\alpha}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}^{m}\Big)\Big)^r\Big]$$
$$\times c_Nj_1^{\hat{p}_N\alpha}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}^{2N+1},$$

and

$$\left(\sum_{j=1}^{\infty}\left(\sum_{m=1}^{2N} j^{\hat{p}_N\alpha}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}}^{m}\right)\right)^{r})$$

$$\leq \left((\sum_{j=1}^{\infty} j^{\hat{p}_N\alpha}\frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})(1+c)^{2N-1}\right)^{r},$$

we have for sufficiently large k,

$$\sum_{j_1=1}^{L}\left|Q_L^0\cdots Q_{j_1}^1\cdots Q_1^0 f^L\right| \leq \big(\prod_{j=1}^{L}\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}\big)\left\{O\left(\left(\frac{1}{\sqrt[3]{k}}\right)^{N+1}\right)\right\},$$

which completes the proof.

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