数理解析研究所講究録
第 1861 巻 2013 年 46-61

Analytic continuation of eigenvalues of Daubechies operator and Fourier ultra-hyperfunctions

By

Kunio YOSHINO*

Abstract

We will analyze the analytic properties of eigenvalues of Daubechies operator by using the theory of Fourier ultra - hyperfunctions. The reconstruction formula for symbol function from eigenvalues will be given by using Borel summability method. We establish the relationship between symbol functions of Daubechies operators and Fourier ultra - hyperfunctions.

§ 1. Introduction

Daubechies (localization) operator was introduced by Ingrid Daubechies in *A Time Frequency Localization Operator: A Geometric Phase Space Approach,* IEEE. Trans. Inform. Theory **34** (1988), pp. 605–612. We will analyze the analytic properties of eigenvalues of Daubechies operator by using the theory of Fourier ultra-hyperfunctions. Several reconstruction formulas for symbol function from eigenvalues will be given. We establish the relationship between symbol function of Daubechies operator and Fourier ultra-hyperfunction.

§ 2. Bargmann-Fock space and Bargmann Transform

In this section we recall the definition of Bargmann-Fock space and Bargmann transform. Bargmann kernel $A_n(z,x)$ is defined as follows:

$$A_n(z,x) = \pi^{-n/4} \exp\left\{-\frac{1}{2}(z^2+x^2) + \sqrt{2}z\cdot x\right\}, \quad (z \in \mathbb{C}^n, x \in \mathbb{R}^n).$$

We define Bargmann transform $B(\psi)$ by following manner.

$$B(\psi)(z) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \psi(x)A_n(z,x)dx, \qquad (\psi \in L^2(\mathbb{R}^n)).$$

2010 Mathematics Subject Classification(s): Primary 46F; Secondary 46E.
Key Words: Daubechies operator, Fourier ultra-hyperfunctions, Borel transform
*Tokyo City University, Tokyo 158-8557, Japan.

§ 2.1. Bargmann-Fock space BF

We put

$$BF = \{g \in H(\mathbb{C}^n) : \int_{\mathbb{C}^n} |g(z)|^2 e^{-|z|^2} dz \wedge d\bar{z} < \infty\}$$

where $H(\mathbb{C}^n)$ is the space of entire functions.
BF is called Bargmann-Fock space.

Theorem 2.1 ([1]).
1. *Bargmann-Fock space BF is Hilbert space.*
2. *Bargmann transform is a unitary mapping from $L^2(\mathbb{R}^n)$ to Bargmann-Fock space BF.*

Example 2.2. For $\phi_{p,q}(x) = \pi^{-1/4} e^{ipx} e^{-(x-q)^2/2}$, we have
$B(\phi_{p,q})(z) = e^{zw-|w|^2/2+ipq/2}, \quad (w = \frac{p+iq}{\sqrt{2}})$.

§ 2.2. Hermite functions

Definition 2.3 ([6], [20]). There are several ways to define Hermite functions $h_m(x)$. Following definitions are all equivalent.
1. $\pi^{-1/4} \exp\left\{-\frac{1}{2}(z^2 + x^2) + \sqrt{2}z \cdot x\right\} = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} h_m(x)$,
2. $h_m(x) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \exp(x^2/2) \frac{d^m}{dx^m} \exp(-x^2)$,
3. $h_m(x) = \frac{1}{\sqrt{2^m m!}} \left(\frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\right)^m h_0(x), \quad h_0(x) = \pi^{-1/4} \exp(-x^2/2)$.

Example 2.4 ([4]).
1. $h_0(x) = \pi^{-1/4} \exp(-x^2/2)$ is called coherent state.
2. $h_2(x) = \pi^{-1/4} \frac{2x^2 - 1}{\sqrt{2}} \exp(-x^2/2)$ is called Mexican hat wavelet.

Hermite functions of several variables are defined by
$h_{[m]}(x_1, x_2, ...x_n) = \prod_{i=1}^n h_{m_i}(x_i), \quad [m] = (m_1, \dots, m_n) \in N^m$

Proposition 2.5 ([1],[6]).
1. $\{h_{[m]}(x)\}_{m=0}^{\infty}$ *is complete orthonormal basis in* $L^2(\mathbb{R}^n)$.
2. $\left\{\frac{z^m}{\sqrt{m!}}\right\}_{m=0}^{\infty}$ *is complete orthonormal basis in* $BF(\mathbb{C}^n)$.

Proposition 2.6 ([1],[6]).

1. $(-\frac{\partial^2}{\partial x^2} + x^2 - 1)h_m(x) = mh_m(x)$,
2. $B(h_m)(z) = \frac{z^m}{\sqrt{m!}}$,
3. $\mathfrak{F}(h_m)(x) = (-i)^m h_m(x)$,

where $\mathfrak{F}$ is Fourier transformation.

Example 2.7. Suppose that $f(x) \in L^2(\mathbb{R}^n)$. Then we have following expansion.

$$f(x) = \sum_{m=0}^{\infty} f_{[m]} h_{[m]}(x), \quad \{f_{[m]}\}_{m=0}^{\infty} \in l^2.$$

$$B(f)(z) = \sum_{m=0}^{\infty} f_{[m]} \frac{z^m}{\sqrt{m!}}.$$

Proposition 2.8. *We have the following commutative diagram.*

$$\begin{tikzcd} L^2(\mathbb{R}^n) \arrow[r, "B"] \arrow[d, "T"'] & BF \arrow[d, "B \circ T \circ B^{-1}"] \\ L^2(\mathbb{R}^n) \arrow[r, "B"] & BF \end{tikzcd}$$

Example 2.9 ([1],[6]). For $g(z) \in BF$, we have

1. $(B \circ L \circ B^{-1})g(z) = z\frac{\partial}{\partial z}g(z), \quad (L = -\frac{\partial^2}{\partial x^2} + x^2 - 1)$
2. $(B \circ \mathfrak{F} \circ B^{-1})g(z) = g(-iz)$.

§ 3. Windowed Fourier transform and Gabor transform

§ 3.1. Windowed Fourier transform

Definition 3.1. We define windowed Fourier transform of $f(x)$ as follows:

$$W_\phi(f)(p,q) = \int_{\mathbb{R}^n} e^{-ipx}\overline{\phi(x-q)}f(x)dx, \quad f(x) \in L^2(\mathbb{R}^n), \quad (p,q \in R^n)$$

$\phi(x)$ is called window function.

If we put $\phi_{p,q}(x) = e^{ipx}\phi(x-q)$, then we have $W_\phi(f)(p,q) = \langle \phi_{p,q}, f \rangle$. Following inversion formula is known.

Proposition 3.2 ([7]). *If* $\langle g, h\rangle = 1$, *then*

$$f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} h_{p,q}(x) W_g(f)(p,q) dpdq$$

valids.

Proof.

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} h_{p,q}(x)\, W_g(f)\,(p,q)\,dp\,dq &= \iiint_{\mathbb{R}^{3n}} h_{p,q}(x)\,\overline{g_{p,q}(y)}\,f(y)\,dy\,dp\,dq\\ &= \iiint_{\mathbb{R}^{3n}} e^{ip(x-y)} h(x-q)\,\overline{g(y-q)}\,f(y)\,dy\,dp\,dq\\ &= \int_{\mathbb{R}^{2n}} (2\pi)^n \delta(x-y)\,h(x-q)\,\overline{g(y-q)}\,f(y)\,dy\,dq\\ &= (2\pi)^n f(x) \int_{\mathbb{R}^n} \overline{g(x-q)}\,h(x-q)\,dq\\ &= (2\pi)^n f(x) \langle g, h\rangle = (2\pi)^n f(x).\end{aligned}$$

Here we used the plane wave expansion of delta function:
$\delta(x) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} e^{ipx} dp.$ □

§ 3.2. Gabor transform

Windowed Fourier transformation with Gaussian window function $\pi^{-n/4} e^{-x^2/2}$ is called Gabor transformation. The theory of Gabor transform is already applied to Iris identification and signal analysis of human voice (consonant, vowel, etc). Gabor transform is closely related to FBI transform, Bargmann transform and Wigner distributions ([6], [11]).

Definition 3.3 (Gabor transform).

$$W_\phi(f)(p,q) = \pi^{-n/4} \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x) dx, \quad f(x) \in L^2(\mathbb{R}^n)$$

$\pi^{-n/4} e^{ipx} e^{-(x-q)^2/2}$ is called Gabor function,

§ 3.3. The relationship between FBI transform, Bargmann transform and Gabor transform

Definition 3.4 (FBI (Fourier-Bros-Iagolnitzer) transform ([6])).
FBI transform $P^t(f)(p,q)$ is defined by

$$P^t(f)(p,q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-t(x-q)^2} f(x) dx.$$

Proposition 3.5. *we have*

1. $P^{1/2}(f)(p,q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x)dx$
2. $B(f)(z) = \pi^{-n/4} e^{1/4(p^2+q^2+2ipq)} \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x)dx,$

$(z = \dfrac{q+ip}{\sqrt{2}}, p, q \in \mathbb{R}^n).$

§ 3.4. Inversion formula of Gabor transform

As a special case of Proposition 3.2, we have following inversion formula.

Proposition 3.6 ([3], [4]). *Assume that $f(x) \in L^2(\mathbb{R}^n)$. Then we have*

$$f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_\phi(f)(p,q) dpdq$$

where $\phi_{p,q}(x) = \pi^{-n/4} e^{ipx} e^{-(x-q)^2/2}$.

This identity is so called **resolution of the identity** ([4]).

§ 3.5. Unitarity of Gabor transform

Proposition 3.7. *Gabor transform satisfies the following unitary relation.*

1. $\langle W_\phi(f), W_\phi(g)\rangle = (2\pi)^{-n}\langle f, g\rangle$
2. $\|W_\phi(f)\|_{L^2} = (2\pi)^{-n/2}\|f\|_{L^2}$.

§ 4. Windowed Fourier transform and the Heisenberg group

We have following exact sequence.

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \longrightarrow 0 \quad \text{(exact)}$$

where $\mathbb{C}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$ is phase space and $\mathbb{R} \times \mathbb{C}^n = H_n$ is the Heisenberg group (polarized). $\mathbb{R}$ is center of the Heisenberg group. So the Heisenberg group is central extension of phase space. Modulation operator M_p and translation operator T_q are defined as follows:

$$M_p f(x) = e^{ipx} f(x), \quad T_q f(x) = f(x-q), \quad f(x) \in L^2(\mathbb{R}^2).$$

M_p and T_q satisfy $M_p T_q = e^{-ipq} T_q M_p$.
$\pi(p,q) = M_p T_q$ is projective representation of phase space. Namely it satisfies following relation $\pi(p_1, q_1)\pi(p_2, q_2) = e^{-ip_2 q_1}\pi(p_1 + p_2, q_1 + q_2)$.
Projective representation $\pi(p,q)$ becomes unitary represenration $\pi(t,p,q)$ of the Heisenberg group as follows:

Put $\pi(t,p,q)g(x) = e^{it}e^{ipx}g(x-q)$, $g \in L^2(\mathbb{R}^n)$ and $(t,p,q) \in H_n$. Then $\pi(t,p,q)$ is unitary representation of the Heisenberg group and $\pi(0,p,q) = \pi(p,q)$
Since $W_\phi(f)(p,q) = \langle \phi_{p,q}, f\rangle$ with $\phi_{p,q}(x) = \pi(p,q)\phi(x)$, windowed Fourier transform W_ϕ is related to the Heisenberg group. For the details of Heisenberg group, we refer the reader to [6], [8], [9], [17].

§ 5. Daubechies localization operator

In this section we will recall the definition of Daubechies operator and its properties.

Definition 5.1 ([3], [4]). For $f(x) \in L^2(\mathbb{R}^n)$, we put

$$P_F(f)(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} F(p,q)\phi_{p,q}(x)W_\phi(f)(p,q)dpdq,$$

where $\phi_{p,q}(x) = \pi^{-n/4}e^{ipx}e^{-(x-q)^2/2}$ and $F(p,q)$ is symbol function.
P_F is called Daubechies operator.

§ 5.1. Remark on P_F

If $F(p,q) = 1$, then we have $f(x) = P_F(f)(x)$ (resolution of the identity). So in this case, P_F is identity operator.

Proposition 5.2 ([3],[4]). *Suppose that $F(p,q) \in L^1(\mathbb{R}^{2n})$ and $f \in L^2(\mathbb{R}^n)$.*
1. *If $F(p,q) \geqq 0$, then P_F is positive operator. i.e. $\langle P_F(f), f\rangle \geqq 0$.*
2. *P_F is bounded linear operator. i.e. $\|P_F(f)\|_{L^2} \leqq (2\pi)^{-n/2}\|f\|_{L^2}\|F\|_{L^1}$.*
3. *P_F is trace class operator. i.e. Trace of $P_F = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} F(p,q)dpdq$.*

Proposition 5.3 ([3],[4]).
If $F(p,q) \in L^1(\mathbb{R}^{2n})$ and $F(p,q)$ is polyradial function.
i.e. $F(p_1,q_1,\ldots,p_n,q_n) = \tilde{F}({r_1}^2,\ldots,{r_n}^2)$, ${r_i}^2 = {p_i}^2 + {q_i}^2$ $(1 \le i \le n)$.
then
1. *Hermite functions $h_m(x)$ are eigenfunctions of Daubechies operator.*

$$P_F(h_{[m]})(x) = \lambda_{[m]}h_{[m]}(x), \quad ([m] \in \mathbb{N}^n),$$

2. $\lambda_{[m]} = \dfrac{1}{m!}\displaystyle\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n e^{-s_i}{s_i}^{m_i}\tilde{F}(2s_1,\ldots,2s_n)ds_1\cdots ds_n.$

Proof. Original proof was given in [4]. But it is a little bit complicated. If we employ the Bargmann-Fock space, then we can simplify the proof of this proposition ([21]). □

§ 5.2. Commutativity of Daubechies operator P_F,

Proposition 5.4 ([3], [4]).
1. *If symbol function $F(p,q)$ is polyradial function, then P_F commutes with harmonic oscillator Hamiltonian* $-\dfrac{\partial^2}{\partial x^2}+x^2-1$ *and Fourier transform.*
2. *If F_1, F_2 are polyradial functions, then $P_{F_1}P_{F_2}=P_{F_2}P_{F_1}$.*

Proof. Daubechies operator P_F, harmonic oscillator Hamiltonian and Fourier transform have Hermite functions $\{h_{[m]}(x)\}_{m=0}^{\infty}$ as eigenfunction. $\{h_{[m]}(x)\}_{m=0}^{\infty}$ is complete orthonormal basis in $L^2(\mathbb{R}^n)$. Hence they commute each other. □

§ 6. Analytic continuation of eigenvalues of Daubechies operator

In this section we assume that $n=1$.

$$\lambda_m=\frac{1}{m!}\int_0^{\infty}e^{-s}s^m\tilde{F}(2s)ds$$

are eigenvalues of Daubechies operator, $\lambda(z)=\dfrac{1}{\Gamma(z+1)}\displaystyle\int_0^{\infty}e^{-s}s^z\tilde{F}(2s)ds$ ($\mathrm{Re}(z)>-1$), where $\Gamma(z)$ is Euler Gamma function.

Proposition 6.1 ([21], [22], [23]).
1. $|\lambda(z)|\leq\dfrac{C}{\sqrt{|z|}}e^{\frac{\pi}{2}|\mathrm{Im}(z)|}$, ($C$ *is constant*, $\mathrm{Re}(z)>0$)
2. $\lambda(z)$ *is holomorphic in the right half plane* $\mathrm{Re}(z)>0$.
3. $\lambda(m)=\lambda_m$, $(m\in\mathbb{N})$.
4, $\lambda(z)$ *is unique analytic continuation of* λ_m.

§ 7. Generating function of eigenvalues of Daubechies operator

Let $\{\lambda_{[m]}\}$ be eigenvalues of Daubechies operator. We put

$$\Lambda(w)=\sum_{[m]=0}^{\infty}\lambda_{[m]}w^{[m]}.$$

$\Lambda(w)$ is called generating function of eigenvalues of Daubechies operator.
In digital signal processing $\Lambda(w)$ is called causal Z-transform ($w=z^{-1}$) instead of generating function.

Proposition 7.1 ([21]). *Let $\lambda_{[m]}$ be eigenvalues of P_F. Then we have*

1. *There exists a positive constant C such that*
$|\lambda_{[m]}| \leq \dfrac{C}{\sqrt{|m|}}, \quad ([m] \in \mathbb{N}^n).$

2. $\Lambda(w) = \displaystyle\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n e^{-s_i(1-w_i)} \tilde{F}(2s_1, \ldots, 2s_n) ds_1 \cdots ds_n.$

3. *$\Lambda(w)$ is holomorphic in $\prod_{i=1}^n \{w \in \mathbb{C}^n : \mathrm{Re}(w_i) < 1\}$ and bounded in its closure. Moreover, $\Lambda(iv) \in C_0(\mathbb{R}^n), (v \in \mathbb{R}^n)$. i.e. $\Lambda(iv) \in C(\mathbb{R}^n)$ and $\lim_{|v| \to \infty} \Lambda(iv) = 0$.*

Proof. Without loss of genelarity, we can assume that $n = 1$.

1. By Proposition 5.3,

$$\lambda_m = \frac{1}{m!} \int_0^\infty e^{-s} \tilde{F}(2s) s^m ds.$$

Since $e^{-s} s^m \leq e^{-m} m^m$, we have

$$|\lambda_m| \leq \frac{1}{m!} e^{-m} m^m \int_0^\infty |\tilde{F}(2s)| ds.$$

By Stirling's formula $m! \sim \sqrt{2\pi m} e^{-m} m^m$, we have $|\lambda_m| \leq \dfrac{C}{\sqrt{m}}$.

2. $$\Lambda(w) = \sum_{m=0}^\infty \lambda_m w^m = \sum_{m=0}^\infty \frac{w^m}{m!} \int_0^\infty e^{-s} s^m \tilde{F}(2s) ds$$
$$= \int_0^\infty e^{-s} \tilde{F}(2s) \sum_{m=0}^\infty \frac{(ws)^m}{m!} ds = \int_0^\infty e^{-s(1-w)} \tilde{F}(2s) ds.$$

3. For $\mathrm{Re}(w) \leq 1$, we have

$$|\Lambda(w)| \leq \int_0^\infty |e^{-s(1-w)}| |\tilde{F}(2s)| ds \leq \|\tilde{F}\|_{L^1}.$$

Since $\Lambda(iv)$ is Fourier transform of L^1 function $e^{-s}\tilde{F}(2s)$, we can conclude that it is in $C_0(\mathbb{R}^n)$ by Riemann-Lebesgue theorem. □

§ 8. Fourier ultra-hyperfunctions

In this section we will recall the definition of Fourier ultra-hyperfunctions and their properties. Put $L = [a, \infty) + i[-b, b], \quad L_\varepsilon = [a - \varepsilon, \infty) + i[-b - \varepsilon, b + \varepsilon], \quad (b \geq 0)$. We consider following test function space:

$$Q(L_\varepsilon : \varepsilon') = \{f(t) \in H(L_\varepsilon) \cap C(L_\varepsilon) : \sup_{t \in L_\varepsilon} |f(t) e^{\varepsilon' |t|}| < \infty\}$$

$H(L_\varepsilon)$ is the space of holomorphic functions defined in the interior of L_ε and $C(L_\varepsilon)$ is the space of continuous functions on L_ε respectively.
$Q(L:\{0\}) = \lim\operatorname{ind}_{\varepsilon>0,\varepsilon'>0}Q(L_\varepsilon:\varepsilon')$ and $Q'(L:\{0\})$ is the dual space of $Q(L:\{0\})$.
The element of $Q'(L:\{0\})$ is called Fourier ultra-hyperfunction carried by L.
$\widetilde{T}(z) = \langle T, e^{-zt}\rangle$ denotes the Fourier-Laplace transform of $T \in Q'(L:\{0\})$.

Theorem 8.1 ([14]). *Suppose that T is Fourier ultra-hyperfunction carried by L.*
1. *$\widetilde{T}(z)$ is holomorphic in the right half plane $Re(z) > 0$.*
2. *$\forall\varepsilon > 0, \varepsilon' > 0, \exists C_{\varepsilon,\varepsilon'} > 0$ s.t. $|\widetilde{T}(z)| \leq C_{\varepsilon,\varepsilon'}e^{-(a-\varepsilon)x+(b+\varepsilon)|y|}$,*
   $(x \geq \varepsilon', z = x+iy \in \mathbb{C})$

Conversely, if $g(z)$ is holomorphic in the right half plane and satisfies above estimate then there exists a unique Fourier ultra-hyperfunction T such that $g(z) = \widetilde{T}(z)$.

Example 8.2. For $a = 0, b = \pi$, we put

$$\langle T, h\rangle = \frac{1}{2\pi i}\int_{L_\varepsilon} e^{t\sinh w}h(w)dw, \quad h(w) \in Q(L:\{0\}).$$

Then $\widetilde{T}(z) = J_z(t)$, $\quad$ ($J_z(t)$ is Bessel function([5])).

For the details of the theory of Fourier ultra-hyperfunction, please refer to [14], [18] and [19].

§ 9. Characterization of Fourier ultra-hyperfunctions by heat kernel method

In this section we will recall the heat kernel method for genelarized functions introduced by T. Matsuzawa ([12], [13]). Fundamental solution $E(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{x^2}{4t}}$ of heat equation is called heat kernel. For the characterization of Fourier-hyperfunctions and Fourier ultra-hyperfunctions by heat kernel method, we have following results.

Theorem 9.1 ([2]). *For Fourier hyperfunction T, we put $U(x,t) = (E*T)(x,t)$. Then $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,1))$ satisfies following conditions 1 and 2.*
1. $\frac{\partial}{\partial t}U(x,t) = \Delta U(x,t)$
2. $\forall\varepsilon > 0, \exists C_\varepsilon > 0$ *s.t.*
   $|U(x,t)| \leqq C_\varepsilon \exp(\varepsilon(|x| + \frac{1}{t}))$, $\quad (x \in \mathbb{R}^n, 0 < t < T)$
3. $\lim_{t\to 0} U(x,t) = T$

Conversely if $U(x,t)$ satisfies conditions 1 and 2, then there exists a Fourier hyperfunction T such that $U(x,t) = (E*T)(x,t)$.

Theorem 9.2 ([2]). *For Fourier ultra-hyperfunction T, put $U(x,t) = (E * T)(x,t)$. Then $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,1))$ satisfies following conditions* 1 *and* 2.

1. $\frac{\partial}{\partial t} U(x,t) = \Delta U(x,t)$
2. $\exists A > 0, \exists B > 0$ *s.t.*

$|U(x,t)| \leqq A \exp(B(|x| + \frac{1}{t})), \quad (x \in \mathbb{R}^n, 0 < t < T)$

3. $\lim_{t \to 0} U(x,t) = T$

Conversely if $U(x,t)$ satisfies conditions 1 *and* 2, *then there exists a Fourier ultra hyperfunction T such that $U(x,t) = (E * T)(x,t)$.*

Example 9.3. We give two examples.

1. (Dirac delta function $\delta(x)$)

$$U(x,t) = (\delta * E)(x,t) = E(x,t). \quad \lim_{t \to 0} U(x,t) = \lim_{t \to 0} E(x,t) = \delta(x).$$

2. (Heaviside function $H(x)$)

$$U(x,t) = (H * E)(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds.$$

$$\lim_{t \to 0} U(x,t) = \lim_{t \to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds = H(x).$$

§ 10. Representation of $\lambda(z)$ and $\Lambda(w)$ by Fourier ultra-hyperfunction

Proposition 10.1. *$\lambda(z)$ and $\Lambda(w)$ can be expressed by Fourier ultra-hyperfunction $T_t \in Q'([0,\infty) + i[-\frac{\pi}{2}, \frac{\pi}{2}], \{0\})$ as follows:*

1. $\lambda(z) = \langle T_t, e^{-zt} \rangle$,
2. $\Lambda(w) = \langle T_t, \frac{we^{-t}}{1 - we^{-t}} \rangle + \lambda_0$.

Proof. 1. Since $\lambda(z)$ satisfies the conditions 1 and 2 in Proposition 6.1, there exists Fourier ultra-hyperfunction $T_t \in Q'([0,\infty) + i[-\frac{\pi}{2}, \frac{\pi}{2}], \{0\})$ such that $\lambda(z) = \langle T_t, e^{-zt} \rangle$ (Theorem 8.1).

2. $$\begin{aligned} \Lambda(w) - \lambda_0 &= \sum_{m=1}^{\infty} \lambda_m w^m = \sum_{m=1}^{\infty} \lambda(m) w^m = \sum_{m=1}^{\infty} \langle T_t, e^{-tm} \rangle w^m \\ &= \langle T_t, \sum_{m=1}^{\infty} (e^{-t} w)^m \rangle = \langle T_t, \frac{we^{-t}}{1 - we^{-t}} \rangle. \end{aligned}$$

□

§ 10.1. Remark

We can express Fourier ultra-hyperfunction T by $\Lambda(w)$ as follows ([15]):

$$\langle T, h\rangle = \frac{1}{2\pi i}\int_{\partial L_\varepsilon} \Lambda(e^t)h(t)dt, \quad h(t) \in Q(L : \{0\}).$$

§ 11. Reconstruction formulas for symbol function

In this section we will show two reconstruction formulas.

Proposition 11.1 (The first reconstruction formula ([21])**).**

$$\tilde{F}(2s) = \frac{e^s}{s}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda(z)\Gamma(z+1)s^{-z}dz, \quad (c > 0).$$

Proof. Since $\lambda(z) = \dfrac{1}{\Gamma(z+1)}\displaystyle\int_0^\infty e^{-s}s\tilde{F}(2s)s^{z-1}ds$, by inverse Mellin transform, we obtain above formula.

□

Proposition 11.2 (The second reconstruction formula([21]**)).**

$$\tilde{F}(2s) = \frac{1}{2\pi}e^s\int_{-\infty}^{+\infty} e^{isv}\Lambda(-iv)dv$$

valids.

Proof. Since $\Lambda(-iv) = \displaystyle\int_{-\infty}^{+\infty} e^{-s}\tilde{F}(2s)e^{-isv}ds$, by inversion formula of Fourier transform, we obtain our desired result. □

§ 12. The Relationship between $\lambda(z)$ and $\Lambda(w)$

Theorem 12.1. *The relationship between $\lambda(z)$ and $\Lambda(w)$ are given by following formulas:*

1. $\lambda(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial L_\varepsilon} \Lambda(e^\zeta)e^{-z\zeta}d\zeta$

where $L_\varepsilon = [-\varepsilon, \infty) + i\left[-\dfrac{\pi}{2} - \varepsilon, \dfrac{\pi}{2} + \varepsilon\right]$.

2. $\lambda(z) = \dfrac{1}{2\pi i}\displaystyle\int_{exp(L_\varepsilon)} \Lambda(w)w^{-z-1}dw$
3. $\Lambda(e^{-t}) = \dfrac{1}{2}\lambda(0) + \displaystyle\int_0^\infty \lambda(x)e^{-xt}dx + i\int_0^\infty \frac{e^{-itx}\lambda(ix) - e^{itx}\lambda(-ix)}{e^{2\pi x} - 1}dx$

Proof. For the proof of 1 and 2, we refer the reader to [15], [21], [22], [23].
3. If we apply Plana's summation formula([5], [10]) to $f(z) = \lambda(z)e^{-tz}$, then we obtain the above formula. □

§ 12.1. Examples

Example 12.2. $F(p,q) = e^{\frac{a-1}{2a}(p^2+q^2)} = e^{\frac{a-1}{2a}r^2}$, $(\mathrm{Re}(\frac{1}{a}) > 1)$.

$$\lambda_m = a^{m+1}, \quad \lambda(z) = a^{z+1}, \quad \Lambda(w) = \frac{a}{1-aw}, \quad T_t = a\delta(t+\log a).$$

Example 12.3. Assume that $\widetilde{F}(2s) = e^s \sum_{n=2}^{\infty} e^{-n^2 s}$ is theta function.

$$\lambda_m = \zeta(2m+2) - 1, \quad \lambda(z) = \zeta(2z+2) - 1,$$

where $\zeta(z)$ is Riemann zeta function.

$$\Lambda(w) = \frac{-1}{2w}\left(\frac{\pi\sqrt{w}}{\sin\pi\sqrt{w}}e^{-\pi i\sqrt{w}} + \pi i\sqrt{w} - 1\right) + \frac{1}{w-1},$$

$$T_t = \sum_{n=2}^{\infty} \frac{1}{n^2}\delta(t - 2\log n).$$

To calculate $\Lambda(w)$, we used $\zeta(2n) = 2^{2n-1}\pi^{2n}\frac{B_n}{(2n)!}$ and

$$\frac{x}{e^x-1} + \frac{x}{2} - 1 = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n}, \quad (B_n \text{ is Bernoulli number}).$$

Example 12.4. $\widetilde{F}(2s) = e\delta(s-1)$, $(\delta(s)$ is Dirac's delta function$)$.

$$\lambda_m = \frac{1}{m!}, \quad \lambda(z) = \frac{1}{\Gamma(z+1)}, \quad \Lambda(w) = e^w,$$

$$\langle T_t, h(t)\rangle = \int_{\partial L_\varepsilon} e^{e^t}h(t)dt, \quad h(t) \in Q(L:\{0\}),$$

where $L = [a,\infty) + i[-\frac{\pi}{2}, \frac{\pi}{2}]$.
Since a is arbitrary positive number, T is carried by $\infty + i[\frac{\pi}{2}, -\frac{\pi}{2}]$ ([16]).

§ 12.2. Integral representation of Riemann zeta function

$$\zeta(2z+2) = \frac{1}{2\pi i}\int_{\partial\exp(-L_\varepsilon)} \frac{1}{2w}\left(\frac{\pi\sqrt{w}}{\sin\pi\sqrt{w}}e^{-\pi i\sqrt{w}} + \pi i\sqrt{w} - 1\right)w^{-z-1}dw.$$

Proof. By 2 in Theorem 12.1 and Example 12.3, we obtain above formula. □

§ 13. Relationship between symbol function and Fourier ultra-hyperfunctions

Theorem 13.1. *Suppose that T is Fourier ultra-hyperfunction such that $\lambda(z) = \widetilde{T}(z)$ and $\lambda(z) = \frac{1}{\Gamma(z+1)}\int_0^\infty e^{-s}s^z\widetilde{F}(2s)ds$. Then we have*

1. $\langle T_t, h(t)\rangle = \frac{1}{2\pi i}\int_{\partial L_\varepsilon}\left\{\int_0^\infty e^{se^t}e^{-s}\widetilde{F}(2s)ds\right\}h(t)dt, \quad h(t)\in Q(L:\{0\})$

where $L = [0,\infty) + i[-\frac{\pi}{2}, \frac{\pi}{2}]$.

2. $\widetilde{F}(2s) = \langle T_t, e^{s+t-se^t}\rangle, \quad (s>0)$.

Proof. 1.

$$\begin{aligned}
&\int_{\partial L_\varepsilon}\left\{\int_0^\infty e^{se^t}e^{-s}\widetilde{F}(2s)ds\right\}h(t)dt \\
&\quad = \int_{\partial L_\varepsilon}\int_0^\infty e^{se^t}\left\{\frac{1}{2\pi}\int_{-\infty}^\infty e^{-isv}\Lambda(-iv)dv\right\}dsh(t)dt \\
&\quad = \frac{1}{2\pi}\int_{\partial L_\varepsilon}\int_0^\infty\int_{-\infty}^\infty e^{se^t}e^{-isv}\Lambda(-iv)dvdsh(t)dt \\
&\quad = \frac{1}{2\pi}\int_{\partial L_\varepsilon}\int_0^\infty e^{se^t}e^{-isv}ds\int_{-\infty}^\infty \Lambda(-iv)dvh(t)dt \\
&\quad = \frac{1}{2\pi i}\int_{\partial L_\varepsilon}\int_{-\infty}^\infty \frac{\Lambda(-iv)}{v-ie^t}dvh(t)dt = \int_{\partial L_\varepsilon}\Lambda(e^t)h(t)dt \\
&\quad = \int_{\partial L_\varepsilon}\langle T_u, \frac{e^{t-u}}{1-e^{t-u}}\rangle h(t)dt = \langle T_u, \int_{\partial L_\varepsilon}\frac{e^{t-u}}{1-e^{t-u}}h(t)dt\rangle = 2\pi i\langle T_t, h(t)\rangle.
\end{aligned}$$

2. $$\begin{aligned}
&\int_0^\infty e^{-s}s^z\langle T_t, e^{s+t-se^t}\rangle ds = \langle T_t, e^t\int_0^\infty s^z e^{-se^t}ds\rangle \\
&\quad = \langle T_t, e^{-tz}\int_0^\infty u^z e^{-u}du\rangle = \Gamma(1+z)\langle T_t, e^{-tz}\rangle = \Gamma(1+z)\lambda(z).
\end{aligned}$$

On the other hand we have

$$\int_0^\infty e^{-s}s^z\widetilde{F}(2s)ds = \Gamma(1+z)\lambda(z).$$

Therefore, by the uniqueness of Mellin transform , we obtain our desired result. □

§ 14. The Reconstruction formula by Borel summability method

Put

$$G(t)=\int_0^{\infty}\frac{\tilde{F}(2s)e^{-s}}{t-s}ds,\quad (t\in\mathbb{C}\setminus[0,\infty]).$$

Proposition 14.1 ([22]). *$G(t)$ has following properties:*

1. *$G(t)$ is holomorphic in $\mathbb{C}\setminus[0,\infty]$.*
2. $G(t)\sim\sum_{m=0}^{\infty}m!\lambda_m t^{-m-1}$,

i.e. Formal power series $\sum_{m=0}^{\infty}m!\lambda_m t^{-m-1}$ is an asymptotic expansion of $G(t)$.

Proposition 14.2 ([22]).

1. *$\Lambda(w)$ is the Borel transform of formal power series $\sum_{m=0}^{\infty}m!\lambda_m t^{-m-1}$.*
2. *Laplace transform of $\Lambda(w)$ is $G(t)$.*

Proof. 1. This is the definition of Borel transform.
2. $\Lambda(w)$ is bounded in left half plane. So we can consider Laplace transform of $\Lambda(w)$ along negative real axis.

$$\begin{aligned}\int_0^{-\infty}\Lambda(w)e^{-tw}dw&=\int_0^{-\infty}\left\{\int_0^{\infty}\widetilde{F}(2s)e^{-s(1-w)}ds\right\}e^{-tw}dw\\&=\int_0^{\infty}\widetilde{F}(2s)e^{-s}\left\{\int_0^{-\infty}e^{w(s-t)}dw\right\}ds\\&=\int_0^{\infty}\frac{\widetilde{F}(2s)e^{-s}}{t-s}ds=G(t),\quad (\mathrm{Re}(t)<0).\end{aligned}$$

□

Proposition 14.3. *We have following diagram:*

$$\begin{tikzcd}\widetilde{F}(2s)e^{-s} \arrow[r, mapsto, "H"] \arrow[d, mapsto] & G(t) \\ \sum_{m=0}^{\infty}m!\lambda_m t^{-m-1} \arrow[r, mapsto, "B"] & \Lambda(w) \arrow[u, mapsto, "L"]\end{tikzcd}$$

B is Borel transformation, L is Laplace transformation and H is Hilbert transformation.

Theorem 14.4 (Third Reconstruction Formula).

$$\tilde{F}(2s) = e^s \frac{-1}{2\pi i} \left(G(s+i0) - G(s-i0) \right).$$

Proof. Since $G(t)$ is Hilbert transform of $\tilde{F}(2s)e^{-s}$, $\tilde{F}(2s)e^{-s}$ is boundary value of $G(t)$. □

References

[1] Bargmann, V., On a Hilbert space of analytic functions and an associated integral transform. Part I, *Comm. Pure. Appl. Math* (1961), 187–214.

[2] Chung, S. Y., Kim, D. and Kim, S. K., Structure of extended Fourier hyperfunctions, *Japanese J. Math.* **19** (1994), 217–226.

[3] Daubechies, I., A time frequency localization operator; A geometric phase space approach, *IEEE. Trans. Inform. Theory* **34** (1988), 605–612.

[4] ———, *Ten Lectures on Wavelets*, SIAM., Philadelphia, Pennsylvania, 1992.

[5] A. Erdelyi, A., Magnus, W., Oberhettinger F. and Tricomi, F. G., *Higher Transcendental Functions* I, Bateman Manuscript Project, New York, Tront, London, 1953.

[6] Folland, G. B., *Harmonic Analysis in Phase Space*, Princeton University Press, 1989.

[7] Gröchenig K., *Foundations of Time-Frequency Analysis*, Birkhäuser, 2000.

[8] Kashiwara, M. and M. Vergne, M., Functions on the Shilov boundary of the generalized half plane, *Non-Commutative Harmonic Analysis*, Lecture Notes in Math. **728**, Springer, 1979, pp. 136-176.

[9] Kawazoe, T., *Harmonic Analysis on Group*, Asakura Shoten, 2000, in Japanese.

[10] Kurokawa, S. and Wakayama, M., *Absolute Casimir Element*, Iwanami, 2002, in Japanese.

[11] Martinez, A., *An Introduction to Semiclassical and Microlocal Analysis*, Springer-Verlag, New York, 2002.

[12] Matsuzawa, T., A calculus approach to hyperfunctions I, *Nagoya Math. J.* **108** (1987), 53–66.

[13] ———, A calculus approach to hyperfunctions II, *Trans. Amer. Math.* 313 (1989), 619–654.

[14] Morimoto, M., Analytic functionals with non-compact carrier, *Tokyo J. Math.* **1** (1978), 77–103.

[15] Morimoto, M. and Yoshino, K., A uniqueness theorem for holomorphic functions of exponential type, *Hokkaido Math. J.* **7** (1978), 259–270.

[16] ———, Some examples of analytic functionals with carrier at the infinity, *Proc. Japan Acad.* **56** (1980), 357–361.

[17] Mumford, D., Nori, D. and Norman, P., *Tata Lectures on Theta* III, Birkhäuser, 1991.

[18] Park, Y. S. and Morimoto, M., Fourier ultrahyperfunctions in the Eucledean n-spaces, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.* **20** (1973), 121–127.

[19] Sargos, P. and Morimoto, M., Transformation des fonctionnelles analytiques a porteur non compacts, *Tokyo J. Math.* **4** (1981), 457–492.

[20] Wong, M. W., *Weyl Transforms*, Springer-Verlag, New York, 1998.

[21] Yoshino, K., Daubechies localization operator in Bargmann-Fock Space, *Proceedings of SAMPTA* 2009, Marseille, France, May, 2009.
[22] ———, Analytic continuations and applications of eigenvalues of Daubechies' localization operator, *CUBO A Mathematical J.* **12** (2010), 196–204.
[23] ———, Complex analytic study of Daubechies localization operators, *IEEE, ISCIT* 2010 (2010), 695–700.