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Existence of the solutions of Lewy equation as the tempered ultrahyperfunctions

By

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Abstract

The aim of this article is to show that there exist the solutions of the Lewy equation in the space of the tempered ultrahyperfunctions and give the example.

§ 1. Introduction and Main result

In the middle of 1950's, B. Malgrange and L. Ehrenpreis independently obtained the result that every linear differential operator with constant coefficients has a fundamental solution (see [2] and [9]). This implies that if L is a linear differential operator with constant coefficients on $\mathbb{R}^d$ and $f\in C_0^\infty(\mathbb{R}^d)$, there exists $u\in C^\infty(\mathbb{R}^d)$ such that

$$Lu=f$$

(see [3] and so on). Therefore everyone believed that a linear differential equation with variable coefficients

$$P(x,\partial)u=\sum_{|\alpha|\leq m}a_\alpha(x)\partial^\alpha u=f$$

can be also solved for an arbitrary right-hand side f, especially $f\in C_0^\infty(\mathbb{R}^d)$. But in 1957, H. Lewy destroyed all hopes in the world by the following result:

Theorem 1.1 ([4], [16]). *There exist the functions $f\in C_0^\infty(\mathbb{R}^3_{x,y,t})$ so that the following linear partial differential equation has no solution in the space C^1 in any neighborhood of the point $(x,y,t)=(0,0,t_0)$:*

$$(1.1)\quad -\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u(x,y,t)+2i(x+iy)\frac{\partial}{\partial t}u(x,y,t)=f(x,y,t),\ f\in C_0^\infty(\mathbb{R}^3_{x,y,t}).$$

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Moreover in [7] and [8], L. Hörmander showed that the Lewy equation (1.1) has no solution in the space of Schwartz's distributions in any open non-void subset of $\mathbb{R}^3_{x,y,t}$ by giving the following necessary condition:

Theorem 1.2 ([7], [8]). *Suppose that the differential equation*

$$P(X,\partial)u = f$$

has a solution $u \in \mathcal{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$. Then we have

$$C_{2m-1}(X,\xi) = 0 \text{ if } P_m(X,\xi) = 0, \; X \in \Omega, \; \xi \in \mathbb{R}^d,$$

where we use the following notations:

- $\Omega \subset \mathbb{R}^d$: an open set,
- $P(X,\partial) = \sum_{|\alpha| \le m} a_\alpha(X)\partial^\alpha, \; a_\alpha(X) \in C^\infty(\Omega)$,
- $P_m(X,\xi) = \sum_{|\alpha|=m} a_\alpha(X)\xi^\alpha, \; \bar{P}_m(X,\xi) = \sum_{|\alpha|=m} \overline{a_\alpha(X)}\xi^\alpha$,
- $P_m^{(j)}(X,\xi) = \frac{\partial}{\partial \xi_j} P_m(X,\xi), \; P_{m,j}(X,\xi) = \frac{\partial}{\partial X_j} P_m(X,\xi)$,
- $C_{2m-1}(X,\xi) = \sum_{j=1}^{d} i(P_m^{(j)}(X,\xi)\bar{P}_{m,j}(X,\xi) - P_{m,j}(X,\xi)\bar{P}_m^{(j)}(X,\xi))$.

In fact, by (1.1), we have

- $X = (x,y,t) \in \mathbb{R}^3, \; \xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$,
- $P_1(X,\xi) = \sum_{|\alpha|=1} a_\alpha(X)\xi^\alpha = -\xi_1 - i\xi_2 + 2i(x+iy)\xi_3$,
- $\bar{P}_1(X,\xi) = \sum_{|\alpha|=1} \overline{a_\alpha(X)}\xi^\alpha = -\xi_1 + i\xi_2 - 2i(x-iy)\xi_3$.

Hence, we obtain that

$$C_1(X,\xi) = -8\xi_3 \neq 0 \text{ if } P_1(X,\xi) = 0 \text{ as } \xi_1 = -2y, \; \xi_2 = 2x, \; \xi_3 = 1.$$

Therefore we can see that the Lewy equation does not satisfy this necessary condition.

Besides, in [14] and [15], P. Schapira showed that the Lewy equation (1.1) has no solution in the space of Sato's hyperfunctions in any open non-void subset of $\mathbb{R}^3_{x,y,t}$ by proving that, at least, for first order PDE with analytic coefficients, nonsolvability in the distribution sense implies nonsolvability in the space of Sato hyperfunctions.

Now we have one question, "when can we always solve the Lewy equation for $f \in C_0^\infty(\mathbb{R}^d)$?". To this question, we can find that in [1], S.-Y. Chung, D. Kim and S. K. Kim showed that the Lewy equation for $f \in C_0^\infty(\mathbb{R}^3_{x,y,t})$ has a solution in the space $C^\infty(\mathbb{R}^2_{x,y}; \mathcal{G}'(\mathbb{R}_t))$. As a remark, the space $\mathcal{G}'(\mathbb{R})$ is a space of the Fourier ultrahyperfunctions. This space is a kind of the space of analytic functionals (see [1] and [13]).

Our motivation is to catch the smaller space in which Korean group's result holds than the space $\mathcal{G}'(\mathbb{R}_t)$ with respect to t variable and we obtained the following result:

Main Theorem 1.1 ([12]). *The Lewy equation has a solution in $C^1(\mathbb{R}^2_{x,y}; (\mathcal{G}^1)'(\mathbb{R}_t))$ for $f \in C^1_0(\mathbb{R}^2_{x,y}; L^1(\mathbb{R}_t))$.*

As a remark, the space $(\mathcal{G}^1)'(\mathbb{R})$ is a space of the tempered ultrahyperfunctions which is smaller than the space $\mathcal{G}'(\mathbb{R})$ with respect to t variable.

This result implies that the following Corollary 1.3:

Corollary 1.3 ([12]). *The Lewy equation has a solution in $C^\infty(\mathbb{R}^2_{x,y}; (\mathcal{G}^1)'(\mathbb{R}_t))$ for $f \in C^\infty_0(\mathbb{R}^3_{x,y,t})$.*

In this paper, we will make a report our main result more precisely by giving some supplementations and examples than in [12].

Authors	Year	u	Space	f
H. Lewy	1957	No	$C^1(\Omega)$	$C^\infty_0(\mathbb{R}^3)$
L. Hörmander	1960	No	$\mathcal{D}'(\Omega)$	$C^\infty_0(\mathbb{R}^3)$
P. Schapira	1967	No	$\mathcal{B}(\Omega)$	$C^\infty_0(\mathbb{R}^3)$
Korean Group	1993	Yes	$C^\infty(\mathbb{R}^2_{x,y}; \mathcal{G}'(\mathbb{R}_t))$	$C^\infty_0(\mathbb{R}^3)$
Y. Oka and K. Yoshino	2012	Yes	$C^\infty(\mathbb{R}^2_{x,y}; (\mathcal{G}^1)'(\mathbb{R}_t))$	$C^\infty_0(\mathbb{R}^3)$

§ 2. The spaces $(\mathcal{G}_1)'(\mathbb{R}^d)$ and $(\mathcal{G}^1)'(\mathbb{R}^d)$

For $x \in \mathbb{R}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, where $\partial_{x_j}^{\alpha_j} = (\frac{\partial}{\partial x_j})^{\alpha_j}$ and $\alpha = (\alpha_1, \cdots, \alpha_d)$ so that $\alpha_i \in \mathbb{Z}$ and $\alpha_i \geq 0$.

Definition 2.1 ([5]). Let us denote by $\mathcal{S}_{1,A}(\mathbb{R}^d)$, $A = (A_1, A_2, \ldots, A_d) \in (0, \infty)^d$, the space $C^\infty(\mathbb{R}^d)$ satisfying the following condition: For any $\delta > 1$ and $\beta \in \mathbb{Z}^d_+$, there exists a constant $C_{\beta,\delta} > 0$ such that

$$|\partial^\beta \varphi(x)| \leq C_{\beta,\delta} \exp\left(-\sum_{j=1}^d a_{\delta_j} |x_j|\right), \varphi \in C^\infty(\mathbb{R}^d),$$

where $a_{\delta j} = \frac{1}{\delta A_j}$. The space $\mathcal{S}_{1,A}(\mathbb{R}^d)$ is a Fréchet space with the semi-norms

$$||\varphi||_{\beta,\delta} = \sup_{x \in \mathbb{R}^d} |\partial_x^\beta \varphi(x)| \exp\left(\sum_{j=1}^d a_{\delta_j} |x_j|\right).$$

for any $\varphi \in \mathcal{S}_{1,A}(\mathbb{R}^d)$. The space $\mathcal{G}_1(\mathbb{R}^d)$ is given by the projective limit

$$\mathcal{G}_1(\mathbb{R}^d) = \varprojlim_{A \to 0} \mathcal{S}_{1,A}(\mathbb{R}^d).$$

Example 2.2. $f(x) = e^{-x^2} \in \mathcal{G}_1(\mathbb{R})$.

Remark. The space given by the inductive limit

$$\varinjlim_{A \to \infty} \mathcal{S}_{1,A}(\mathbb{R}^d).$$

is the Gel'fand-Shilov space $\mathcal{S}_1(\mathbb{R}^d)$ (see [5]).

Definition 2.3. We denote by $(\mathcal{G}_1)'(\mathbb{R}^d)$ the dual space of the space $\mathcal{G}_1(\mathbb{R}^d)$.

In 1961, M. Hasumi obtained the following structure theorem:

Proposition 2.4 ([6]). *Let T be in $(\mathcal{G}_1)'(\mathbb{R}^d)$. Then $T \in (\mathcal{G}_1)'(\mathbb{R}^d)$ can be expressed by*

$$T(x) = \partial^\beta h(x), \ \beta \in \mathbb{Z}_+^d,$$

where a continuous function $h(x)$ satisfying that there exist positive constants A and B such that

$$|h(x)| \le Ae^{B|x|}.$$

Example 2.5. $f(x) = e^x \in (\mathcal{G}_1)'(\mathbb{R})$, $e^x \notin (\mathcal{S}_1)'(\mathbb{R})$.

Next we define the space $\mathcal{G}^1(\mathbb{R}^d)$. At first, we define the space $\mathcal{S}^{1,B}(\mathbb{R}^d)$, $B \in (0,\infty)^d$, as follows:

Definition 2.6 ([5]). Let us denote by $\mathcal{S}^{1,B}(\mathbb{R}^d)$ the space $C^\infty(\mathbb{R}^d)$ satisfying the following condition: For any $\rho > 1$ and $\alpha \in \mathbb{Z}_+^d$, there exists a constant $C_{\alpha,\rho} > 0$ so that

$$|x^\alpha \partial_x^\beta \varphi(x)| \le C_{\alpha,\rho}(\rho B)^\beta \beta!, \ \beta \in \mathbb{Z}_+^d.$$

The space $\mathcal{S}^{1,B}(\mathbb{R}^d)$ is a Fréchet space with the semi-norms

$$||\varphi||^{\alpha,\rho} = \sup_{\substack{x \in \mathbb{R}^d \\ \beta \in \mathbb{Z}_+^d}} \frac{|x^\alpha \partial_x^\beta \varphi(x)|}{(\rho B)^\beta \beta!}.$$

It is known that the following result holds:

Proposition 2.7 ([5]). *Let $B \in (0,\infty)^d$. Then the spaces $\mathcal{S}^{1,B}(\mathbb{R}^d)$ and $\mathcal{S}_{1,B}(\mathbb{R}^d)$ are topologically isomorphic via the Fourier transform.*

Thus, we can replace the space $\mathcal{S}^{1,B}(\mathbb{R}^d)$ with the space $\mathcal{S}_{1,B}(\mathbb{R}^d)$ via the Fourier transform.

Definition 2.8. The space $\mathcal{G}^1(\mathbb{R}^d)$ is given by the projective limit

$$\mathcal{G}^1(\mathbb{R}^d) = \varprojlim_{B\to 0} \mathcal{S}^{1,B}(\mathbb{R}^d).$$

Remark. The space given by the inductive limit

$$\varinjlim_{B\to\infty} \mathcal{S}^{1,B}(\mathbb{R}^d).$$

is the Gel'fand-Shilov space $\mathcal{S}^1(\mathbb{R}^d)$ (see [5]).

By Proposition 2.7, we immediately obtain the relationship between the spaces $\mathcal{G}^1(\mathbb{R}^d)$ and $\mathcal{G}_1(\mathbb{R}^d)$ as follows:

Proposition 2.9. *The spaces $\mathcal{G}^1(\mathbb{R}^d)$ and $\mathcal{G}_1(\mathbb{R}^d)$ are topologically isomorphic via the Fourier transform.*

Thus, we can replace the space $\mathcal{G}^1(\mathbb{R}^d)$ with the space $\mathcal{G}_1(\mathbb{R}^d)$ via the Fourier transform.

Remark. The spaces $\mathcal{G}^1(\mathbb{R}^d)$ and $\mathcal{G}_1(\mathbb{R}^d)$ are subspaces of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, respectively. Moreover the space $\mathcal{G}_1(\mathbb{R}^d)$ has the subspace $\mathcal{D}(\mathbb{R}^d)$ but the space $\mathcal{G}^1(\mathbb{R}^d)$ does not have the subspace $\mathcal{D}(\mathbb{R}^d)$ because of its analytic property.

Definition 2.10. We denote the dual space of $\mathcal{G}^1(\mathbb{R}^d)$ by $(\mathcal{G}^1)'(\mathbb{R}^d)$.

By Proposition 2.9, we obtain the following relationship between the spaces $(\mathcal{G}^1)'(\mathbb{R}^d)$ and $(\mathcal{G}_1)'(\mathbb{R}^d)$:

Proposition 2.11. *The spaces $(\mathcal{G}^1)'(\mathbb{R}^d)$ and $(\mathcal{G}_1)'(\mathbb{R}^d)$ are topologically isomorphic via the Fourier transform.*

Thus, we can replace the space $(\mathcal{G}^1)'(\mathbb{R}^d)$ with the space $(\mathcal{G}_1)'(\mathbb{R}^d)$ via the Fourier transform.

On the other hand, we prepare the following space to clear the analytic property of the space $\mathcal{G}^1(\mathbb{R}^d)$:

Definition 2.12 ([11]). Let O'' be an open set in $\mathbb{R}^d$ and $\tilde{K}$ be a compact set in O''. Then let us denote by $\mathfrak{h}(\mathbb{R}^d + iO'')$ the space $\mathcal{O}(\mathbb{R}^d + iO'')$ satisfying the following condition: For any $\tilde{K} \subset O''$ and $m \in \mathbb{Z}_+^d$,

$$\|\varphi\|_{\tilde{K}}^m = \sup_{\mathrm{Im}\zeta\in\tilde{K}} |\zeta^m \varphi(\zeta)| < \infty.$$

The space $\mathfrak{h}(\mathbb{R}^d + iO'')$ is the space of the test functions for the tempered ultrahyperfunctions (see [6] and [11]).

Example 2.13. Let $z = x + iy$ and $x > 0$ be fixed. If $|y| \to \infty$, then we have

$$\Gamma(x+iy) \sim \sqrt{2\pi}|y|^{x-(1/2)}e^{-\pi|y|/2}$$

(see [10]). Therefore we can see that

$$F(\zeta) = \Gamma(i\zeta) \in \mathfrak{h}(\mathbb{R} + iO'')$$

where $O'' = (B_1, B_2)$ for some positive constants B_1, B_2.

We have the following Proposition 2.14 (in [12], we did not prove this proposition directly. So we give the proof here):

Proposition 2.14. *Let $B \in (0,\infty)^d$. Then the spaces $\mathcal{S}^{1,B}(\mathbb{R}^d)$ and $\mathfrak{h}(\mathbb{R}^d + i\{|\mathrm{Im}\,\zeta| < 1/B\})$ are topologically isomorphic.*

Proof. Let $\varphi \in \mathcal{S}^{1,B}(\mathbb{R}^d)$. Then for any compact sets $\tilde{K} \subset \{|\mathrm{Im}\,\zeta| < 1/B\}$, we have for any $m \in \mathbb{Z}_+^d$,

$$\begin{aligned}\sup_{\mathrm{Im}\zeta\in\tilde{K}} |\zeta^m\varphi(\zeta)| &\le C_B \sum_{|\beta|=0}^{\infty} \frac{|\xi|^m|\partial^\beta\varphi(\xi)|}{\beta!}|i\eta|^\beta \\ &\le C'_B\|\varphi\|^{\alpha,\rho}\sum_{|\beta|=0}^{\infty}(\rho B|\eta|)^\beta \\ &\le C'_B\|\varphi\|^{m,\rho}(1-(\rho B\eta))^{-1}\end{aligned}$$

for $\zeta = \xi + i\eta \in \mathbb{R}^d + i\tilde{K}$. Hence we have

$$\|\varphi\|^m_{\tilde{K}} \le C\|\varphi\|^{m,\rho}$$

for some constant $C > 0$. Conversely, if $\varphi \in \mathfrak{h}(\mathbb{R}^d + i\{|\mathrm{Im}\,\zeta| < 1/B\})$, then by the Cauchy's integral formula, we have

$$|\xi|^m|\partial^\beta\varphi(\xi)| \le |\xi|^m\left|\frac{\beta!}{2\pi i}\int_{C_r}\frac{\varphi(\zeta)}{(\zeta-\xi)^{\beta+1}}d\zeta\right| \le \beta!\left(\frac{1}{r}\right)^{|\beta|}\sup_{|\mathrm{Im}\zeta|\le r}|\zeta^m\varphi(\zeta)|,$$

where $C_r = \{\zeta \in \mathbb{C}^d \mid |\zeta - \xi| = r\}$ for any $0 < r < 1/B$. Hence we obtain

$$\|\varphi\|^{m,\rho} \le C\|\varphi\|^m_{\tilde{K}_r}.$$

□

Now the space $\mathfrak{h}(\mathbb{C}^d)$ is given by the projective limit

$$\mathfrak{h}(\mathbb{C}^d) = \varprojlim_{B\to 0} \mathfrak{h}(\mathbb{R}^d + i\{|\mathrm{Im}\ \zeta| < 1/B\}).$$

Then by Proposition 2.14, we immediately obtain the following Proposition 2.15:

Proposition 2.15. *The spaces $\mathcal{G}^1(\mathbb{R}^d)$ and $\mathfrak{h}(\mathbb{C}^d)$ are topologically isomorphic.*

Definition 2.16. We denote by $\mathfrak{h}'(\mathbb{C}^d)$ the dual space of the space $\mathfrak{h}(\mathbb{C}^d)$ called the space of the tempered ultrahyperfunctions in [11].

Remark. The space $\mathfrak{h}'(\mathbb{C}^d)$ of the tempered ultrahyperfunctions is a subspace of the space $Q'(\mathbb{C}^d)$ of the Fourier ultrahyperfunctions. Hence the space $\mathfrak{h}'(\mathbb{C}^d)$ is a kind of the space of the analytic functionals (see [11] and [13]).

By Proposition 2.15, we immediately obtain the following Proposition 2.17:

Proposition 2.17. *The spaces $(\mathcal{G}^1)'(\mathbb{R}^d)$ and $\mathfrak{h}'(\mathbb{C}^d)$ are topologically isomorphic.*

Remark. We can consider the $\mathfrak{h}'(\mathbb{C}^d)$ as $(\mathcal{G}^1)'(\mathbb{R}^d)$ below.

Example 2.18. Since the function e^x is in $(\mathcal{G}_1)'(\mathbb{R})$, the Fourier transform of e^x

$$\mathcal{F}[e^x](\zeta) = \delta(\zeta + i)$$

is in $\mathfrak{h}'(\mathbb{C}^d)$.

§ 3. The proof of Main Theorem and the example

We have showed the proof of Main Theorem 1.1 in [12]. Therefore we give the abbreviated proof (we refer to [12]). Moreover we give the example of our result.

Proof. Let us denote by $\mathcal{F}_3$ the Fourier transform for the third variable. Then by the Fourier transform for the third variable of Lewy equation, we have

$$\text{(3.1)}\quad \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\mathcal{F}_3 u)(x,y,\omega) + 2(x+iy)\omega(\mathcal{F}_3 u)(x,y,\omega)$$
$$= -(\mathcal{F}_3 f)(x,y,\omega),\ f \in C_0^1(\mathbb{R}^2_{x,y}; L^1(\mathbb{R}_t)),$$

where supp $(\mathcal{F}_3 f) \subset \{\sqrt{x^2+y^2} \le M,$ for some $M > 0\} \times \mathbb{R}_\omega$. By (3.1), we can see that

$$\text{(3.2)}\quad \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left\{e^{\omega(x^2+y^2)}(\mathcal{F}_3 u)(x,y,\omega)\right\} = -\frac{1}{2}e^{\omega(x^2+y^2)}(\mathcal{F}_3 f)(x,y,\omega).$$

Since the function $1/(x+iy)$ is the fundamental solution of the Cauchy-Riemann operator, we obtain

$$(3.3)\quad (\mathcal{F}_3 u)(x,y,\omega) = \frac{e^{-\omega(x^2+y^2)}}{4\pi^2} \iint_{\mathbb{R}^2} \frac{1}{x'+iy'}(-\mathcal{F}_3 f)(x-x',y-y',\omega)\, e^{\omega((x-x')^2+(y-y')^2)}dx'dy'.$$

By (3.3), with respect to x, y variables, we can see that $(\mathcal{F}_3 u)(\cdot,\cdot,\omega) \in C^1$ and with respect to ω variable, we can see that the function $(\mathcal{F}_3 u)(x,y,\cdot)$ is a continuous function satisfying the following estimate:

$$|(\mathcal{F}_3 u)(x,y,\omega)| \leq Ce^{M^2|\omega|}$$

for some constants $C > 0$ and $M^2 > 0$. By Proposition 2.4 and Proposition 2.11, the solution u belongs to the space $C^1(\mathbb{R}^2_{x,y}; (\mathcal{G}^1)'(\mathbb{R}_t))$. □

Example 3.1. If f is in C_0^∞ with respect to x, y variables and 0 with respect to t variable for Lewy equation, then we have the solution

$$u(x,y,t) = U(x,y) \otimes H(t+i),$$

where $U(x,y) \in C^\infty(\mathbb{R}^2_{x,y})$ and $H(t+i)$ defined by

$$H(t+i) = H_i(\zeta) := \begin{cases} 1, & \zeta \in (0,\infty)+i \\ 0, & \zeta \notin (0,\infty)+i \end{cases}$$

is in $\mathfrak{h}'(\mathbb{C})$.

Now we have for any $\varphi \in \mathfrak{h}(\mathbb{C})$,

$$\begin{aligned} \langle H'(t+i),\varphi\rangle &= \langle H_i'(\zeta),\varphi\rangle = -\langle H_i(\zeta),\varphi'\rangle \\ &= -\int_0^\infty \varphi'(\xi+i)d\xi = -[\varphi(\xi+i)]_0^\infty \\ &= \varphi(i) = \langle \delta(\zeta-i),\varphi\rangle . \end{aligned}$$

Thus we obtain

$$H'(t+i) = \delta(\zeta-i) := \begin{cases} \infty, & \zeta = i \\ 0, & \zeta \neq i \end{cases}, \quad \zeta \in \mathbb{C}.$$

Therefore we can see that $H(t+i)$ and its derivative $H'(t+i)$ are identically 0 on a real line $\mathbb{R}_t$.

Finally, we can see that the solvability of the Lewy equation holds in the space of the tempered ultrahyperfunctions smaller than the space of the Fourier ultrahyperfunctions. (In [12], we also show the solvability of the Lewy equation with non-homogeneous term f in an another space.) But we have not known whether the space of the tempered ultrahyperfunctions is the smallest yet. Therefore we will be would like to obtain the smallest space in which the Lewy equation always has a solution in the future.

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