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Two-dimensional Stochastic Navier-Stokes Equations derived from a certain Variational Problem

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We study the existence of weak solutions of stochastic Navier-Stokes equation on a two-dimensional torus, which appears in a variational problem. We construct its weak solutions due to an approximation by a sequence of solutions of equations with enlarged viscosity terms and then by showing an a priori estimate for them.

1 Introduction

Navier-Stokes equations perturbed by some random force, stochastic Navier-Stokes equations, have been studied by many authors. In this paper we discuss the initial value problem of the following type of stochastic Navier-Stokes equation for the velocity field $u=u(t,x)=(u^1(t,x),u^2(t,x))$, $t\geq 0$, $x=(x_1,x_2)$ and the pressure term $p=p(t,x)$ on a two-dimensional torus $\mathbb{T}^2=[0,2\pi]^2$:

$$\frac{\partial u^i}{\partial t}+\sum_{j=1}^{2}\left(u^j\frac{\partial u^i}{\partial x_j}+\sqrt{2\mu}\frac{\partial u^i}{\partial x_j}\dot{B}^j(t)\right)-\mu\Delta u^i+\frac{\partial p}{\partial x_i}=0,\quad t>0, i=1,2, \tag{1}$$

with the incompressibility condition:

$$\operatorname{div}u\equiv\sum_{j=1}^{2}\frac{\partial u^j}{\partial x_j}=0,\quad t>0, x\in\mathbb{T}^2, \tag{2}$$

under the initial condition:

$$u(0,x)=u_0(x),\quad x\in\mathbb{T}^2, \tag{3}$$

where $\mu>0$ is a constant and $\dot{B}(t)=\frac{d}{dt}B(t)$ is a formal derivative of the two dimensional Brownian motion $B(t)=(B^1(t),B^2(t))$. We solve the equation (1) - (3) in the class of u's satisfying $\int_{\mathbb{T}^2}u dx=0$. We assume that u_0 is a

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$\mathbf{V}$-valued deterministic function where $\mathbf{V} = \mathbf{W}^{1,2}(\mathbb{T}^2;\mathbb{R}^2) \cap \mathbf{H}$, $\mathbf{W}^{1,2}(\mathbb{T}^2;\mathbb{R}^2)$ denotes the usual $\mathbb{R}^2$-valued Sobolev space and $\mathbf{H}$ is the family of $\mathbb{R}^2$-valued square integrable functions on $\mathbb{T}^2$ which are of divergence free and have mean zero, that is,

$$\mathbf{H} = \left\{ u \in \mathbf{L}^2(\mathbb{T}^2;\mathbb{R}^2) \mid \operatorname{div} u = 0, \int_{\mathbb{T}^2} u dx = 0 \right\},$$

where $\operatorname{div} u$ is defined in a distribution sense.

The equation (1) - (3) appears in a certain variational problem (see [13]). The solution of the equation (1) - (3) will be defined in a weak sense. The aim of this paper is to show the existence of the weak solution of (1) - (3) under a suitable assumption on the initial condition, which will be described in later section.

Several authors have discussed the existence of solutions of stochastic Navier-Stokes equations which fulfill the coercivity condition ([4], [10], [14]). Note that (1) does not satisfy the coercivity condition. This means that we cannot directly apply their results to our equation. Also, note that (1) is derived from the following Stratonovich equation which is of Euler type:

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^{2} \left(u^j \frac{\partial u^i}{\partial x_j} + \sqrt{2\mu} \frac{\partial u^i}{\partial x_j} \circ \dot{B}^j \right) + \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2. \tag{4}$$

We use the following method to show existence of a solution. First we construct a solution of the equation (1) - (3) with the diffusion term $\mu \Delta u$ replaced by $\frac{2+\delta}{2} \mu \Delta u$ for each $\delta > 0$ by the Galerkin's method. This is possible since the modified equation satisfies the coercivity condition for each $\delta > 0$. In the second step, we take the limit $\delta \to 0$ to construct a weak solution of our equation by showing a uniform estimate which implies the tightness of the distributions $\mathcal{L}(u_n^\delta)$ of the solutions $(u_n^\delta)_{\delta \in (0,1], n \geq 1}$ of the approximating finite dimensional equation in a certain proper functional spaces. Similar approach can be found in a construction of weak solutions of two-dimensional stochastic Euler equations ([2], [3], [5], [6]). The cases of a bounded domain with Dirichlet boundary condition, an unbounded domain and the periodic boundary condition are discussed in [2], [5] and [6], respectively. The case with the stochastic term containing ∇u as in our equation is not studied in these papers. Our method does not directly apply for higher dimensional case. Indeed, by applying Itô's formula for $|u_n^\delta(t)|_{\mathbf{H}}^2$, we obtain the following estimate :

$$\sup_{n \geq 1, \delta > 0} \left\{ \mathbf{E} \left\{ |u_n^\delta(t)|_{\mathbf{H}}^2 \right\} + \delta \mu \int_0^t \mathbf{E} \left\{ ||u_n^\delta(s)||_{\mathbf{V}}^2 \right\} ds \right\} < \infty.$$

This does not imply that $(u_n^\delta(t))_{\delta > 0}$ has a strongly convergent subsequence in $\mathbf{H}$. However, on the two-dimensional torus, we can show such statement relying on the identity:

$$\sum_{j=1}^{2} \int_{\mathbb{T}^2} \frac{\partial (u \cdot \nabla u)}{\partial x_j} \cdot \frac{\partial u}{\partial x_j} dx = 0, \quad \text{for } u \in \mathbf{C}_\sigma^\infty, \tag{5}$$

where $\mathbf{C}_\sigma^\infty$ is a family of infinitely differentiable $\mathbb{R}^2$-valued functions which are of divergence free and have mean zero, that is,

$$\mathbf{C}_\sigma^\infty = \left\{ u \in \mathbf{C}^\infty(\mathbb{T}^2;\mathbb{R}^2) \mid \operatorname{div} u = 0, \int_{\mathbb{T}^2} u dx = 0 \right\}.$$

This equation has an important role on solving our problem. Concerning two-dimensional deterministic Euler equations, energy and enstrophy are conserved quantities (see also [1]). Let us remark that our equation is considered to have enstrophy conservation since the term which is in front of the noise of (4) is the first order differential operator. The key properties for showing existence are the equation (5) and the form of the stochastic term of (4).

2 Derivation of our equations

As stated in the last section, our equation is considered to be derived from a certain variational problem. In this section, we want to introduce this idea briefly. We denote by Diff($\mathbf{R}^n$) the family of volume preserving diffeomorphisms of $\mathbf{R}^n$. Let $\Phi(t) = (\Phi_1(t), \cdots, \Phi_n(t))$, $t \in [0,1]$ be an integral curve which takes values in Diff($\mathbf{R}^n$). Suppose that $\Psi^0, \Psi^1 \in$ Diff($\mathbf{R}^n$) are given. In this setting, let us consider the following action functional J:

$$J(\Phi) = \int_0^1 \int_{\mathbf{R}^n} \sum_{j=1}^n \left|\frac{\partial \Phi_j(t,x)}{\partial t}\right|^2 dx\, dt, \tag{6}$$

under the initial and terminal condition: $\Phi(0) = \Psi^0$ and $\Phi(1) = \Psi^1$, respectively. Then, it is known that the time derivative

$$u(t,x) = (u^1(t,x), \cdots, u^n(t,x)) = \left(\frac{\partial \widetilde{\Phi}_1}{\partial t}(t, \widetilde{\Phi}^{-1}(t,x)), \cdots, \frac{\partial \widetilde{\Phi}_n}{\partial t}(t, \widetilde{\Phi}^{-1}(t,x)\right),$$

of a stationary point $\widetilde{\Phi}(t,x) = (\widetilde{\Phi}_1(t,x), \cdots, \widetilde{\Phi}_n(t,x))$ of J satisfies the Euler equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, & t > 0,\ x \in \mathbf{R}^n, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbf{R}^n, \end{cases} \tag{7}$$

where $p = p(t,x)$ is the pressure term, see [13] for example.

Later, in [13], the case where the integral curve appearing above is affected by some random force is studied, that is, for an n-dimensional Brownian motion $B = (B^1, \cdots, B^n)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\Psi^0, \Psi^1 \in$ Diff($\mathbf{R}^n$), a.s., the following random action functional J_B is introduced:

$$J_B(\Phi) = \int_{\mathbf{R}^n} \int_0^1 \sum_{j=1}^n \left|\frac{\partial \Phi_j(t,x)}{\partial t} + \sqrt{2\mu}\frac{dB_t^j}{dt}\right|^2 dxdt, \tag{8}$$

where $\Phi(0,\omega) = \Psi^0(\omega)$ and $\Phi(1,\omega) = \Psi^1(\omega)$. By proceeding similarly to the deterministic case,

$$
\begin{aligned}
u(t,x,\omega) &= \left(u^1(t,x,\omega),\cdots,u^n(t,x,\omega)\right)\\
&= \left(\frac{\partial\bar{\Phi}_1^B}{\partial t}(t,\tilde{\phi}^{-1}(t,x),\omega),\cdots,\frac{\partial\bar{\Phi}_n^B}{\partial t}(t,\tilde{\phi}^{-1}(t,x),\omega)\right), \quad t>0,\ x\in\mathbf{R}^n,\ \omega\in\Omega,
\end{aligned}
$$

would satisfy

$$
\begin{cases}
\dfrac{\partial(u^i+\sqrt{2\mu}\dot{B}^i)}{\partial t}+\displaystyle\sum_{j=1}^n\left(u^j\frac{\partial u^i}{\partial x_j}+\sqrt{2\mu}\frac{\partial u^i}{\partial x_j}\circ\dot{B}^j\right)+\frac{\partial p}{\partial x_i}=0, \quad i=1,\cdots,n,\\
\operatorname{div} u = 0,
\end{cases}
\tag{9}
$$

where $\bar{\Phi}^B$ is the random stationary point of J_B:

$$
\begin{aligned}
&\bar{\Phi}^B(t,x,\omega)\\
&= (\bar{\Phi}_1^B(t,x,\omega)+\sqrt{2\mu}\dot{B}^1(t,\omega),\cdots,\bar{\Phi}_n^B(t,x,\omega)+\sqrt{2\mu}\dot{B}^n(t,\omega)).
\end{aligned}
$$

Note that the notation $\circ$ appearing in the stochastic term means the Stratonovich sense. It is seen in [13] that if there exists a weak solution $u(t,x,\omega)$ of the equation (9) with the initial value $u_0 \in \mathbf{W}^{1,2}(\mathbf{R^n};\mathbf{R^n})$ satisfying $\operatorname{div} u_0 = 0$, its expectation $\overline{u}(t,x) = \int_\Omega u(t,x,\omega)P(d\omega)$ satisfies the following Reynolds equation:

$$
\begin{cases}
\dfrac{\partial\overline{u}}{\partial t}-\mu\Delta\overline{u}+(\overline{u}\cdot\nabla)\overline{u}+\nabla p = -\overline{(u-\overline{u})\cdot\nabla)(u-\overline{u})}, & t>0,\ x\in\mathbf{R}^n,\\
\operatorname{div}\overline{u}=0, & t>0,\ x\in\mathbf{R}^n,
\end{cases}
$$

However, the existence of the weak solution of (9) is not shown in [13]. Note that a Stratonovich integral can be rewritten into an Itô integral by using the following well-known formula (see [13]):

$$
\int_0^t\frac{\partial u^i}{\partial x_j}(s)\circ dB^j(s)=\int_0^t\frac{\partial u^i}{\partial x_j}(s)dB^j(s)+\frac{1}{2}\langle\langle M_{\frac{\partial u^i}{\partial x_j}},B^j\rangle\rangle(t),
$$

where $M_{\frac{\partial u^i}{\partial x_j}}$ denotes the martingale part determined uniquely by the decomposition of the process $\frac{\partial u^i}{\partial x_j}$ and $\langle\langle M_{\frac{\partial u^i}{\partial x_j}},B^j\rangle\rangle$ the quadratic variation determined by two processes $M_{\frac{\partial u^i}{\partial x_j}}$ and B^j. Thus, by applying this formula with respect to the interchange between those two integrals to our equations, the following stochastic Navier-Stokes equation appears with an Itô integral: for each $i=1,\cdots,n$,

$$
\begin{cases}
\dfrac{\partial u^i}{\partial t}+\sqrt{2\mu}\ddot{B}_t^i+\displaystyle\sum_{j=1}^n\left(u^j\frac{\partial u^i}{\partial x_j}+\sqrt{2\mu}\frac{\partial u^i}{\partial x_j}\dot{B}_t^j\right)-\mu\Delta u^i+\frac{\partial p}{\partial x_i}=0,\\
\operatorname{div} u=0,
\end{cases}
\tag{10}
$$

In our argument, we only study the equation (10) on a two-dimensional torus $\mathbf{T}^2$, in which we disregard the term $\ddot{B}^i(t)$, $i=1,2$. This is reasonable because $\sum_{i=1}^{2}\int_{\mathbf{T}^2}\ddot{B}^i(t)\phi^i(x)dx$ formally vanishes under the assumption that if we consider the class of functions whose integral is equal to zero, that is, $\int_{\mathbf{T}^2}\phi(x)dx=0$ for any smooth vector field ϕ with div $\phi=0$.

Concerning equations derived from this type of variational problems, there is a result studied by [8]. They study a variational problem of diffusion processes with values in the group of volume preserving diffeomorphisms (which differs from ours) and show that Navier-Stokes equations by taking the expectation for associated random critical point. In our case and [13], stochastic Navier-Stokes equations is obtained.

3 Concept of solutions

In this section, we formulate our problem and prepare for some notations. We denote the inner product of $\mathbf{H}$ by $\langle\cdot,\cdot\rangle$, that is,

$$\langle u,v\rangle=\sum_{j=1}^{2}\int_{\mathbb{T}^2}u^j(x)v^j(x)dx,\quad u,v\in\mathbf{H},$$

and the norm of $\mathbf{H}$ by $|\cdot|_{\mathbf{H}}$. We also denote the inner product of $\mathbf{V}$ by $\langle\langle\cdot,\cdot\rangle\rangle$, that is,

$$\langle\langle u,v\rangle\rangle=\sum_{j=1}^{2}\langle\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_j}\rangle,\quad u,v\in\mathbf{V},$$

and the norm of $\mathbf{V}$ by $||\cdot||_{\mathbf{V}}$. Recall that $\mathbf{V}=\mathbf{W}^{1,2}(\mathbb{T}^2;\mathbb{R}^2)\cap\mathbf{H}$ and

$$\mathbf{W}^{1,2}(\mathbb{T}^2;\mathbb{R}^2)=\left\{u\in\mathbf{L}^2(\mathbb{T}^2;\mathbb{R}^2)\mid\frac{\partial u}{\partial x_j}\in\mathbf{L}^2(\mathbb{T}^2;\mathbb{R}^2),\quad j=1,2\right\},$$

where $\frac{\partial u}{\partial x_j}$, $j=1,2$ are defined in a distribution sense. Let A be the linear operator with domain $D(A)=\mathbf{W}^{2,2}(\mathbb{T}^2;\mathbb{R}^2)\cap\mathbf{V}$ such that

$$A:D(A)\to\mathbf{H},\quad Au=-\mu\mathbb{P}\Delta u,$$

where $\mathbf{W}^{2,2}(\mathbb{T}^2;\mathbb{R}^2)$ is the Sobolev space consisting of all $u\in\mathbf{L}^2(\mathbb{T}^2;\mathbb{R}^2)$ such that $\frac{\partial u}{\partial x_j}\in\mathbf{W}^{1,2}(\mathbb{T}^2;\mathbb{R}^2)$ for $j=1,2$ and $\mathbb{P}$ is the projection onto $\mathbf{H}$. Let us set $\mathbb{Z}_0^2=\mathbb{Z}^2\backslash\{(0,0)\}$. For each $k=(k_1,k_2)\in\mathbb{Z}_0^2$, we define

$$e_k(x)=\begin{cases}\frac{k^\perp}{\sqrt{2}\pi|k|}\cos(k\cdot x), & k\in(\mathbb{Z}_0^2)_+,\\ \frac{k^\perp}{\sqrt{2}\pi|k|}\sin(k\cdot x), & k\in(\mathbb{Z}_0^2)_-,\end{cases}$$

where $(\mathbb{Z}_0^2)_+=\{k\in\mathbb{Z}_0^2\,|\,k_1>0\}\cup\{k\in\mathbb{Z}_0^2\,|\,k_1=0,k_2>0\}$, $(\mathbb{Z}_0^2)_-=\mathbb{Z}_0^2\backslash(\mathbb{Z}_0^2)_+$ and $k^\perp=(k_2,-k_1)$. Then, $(e_k)_{k\in\mathbb{Z}_0^2}\subset\mathbf{C}_\sigma^\infty$ is a trigonometric basis in $\mathbf{H}$.

Furthermore, let us set $\lambda_k = \mu|k|^2$ for $k \in \mathbb{Z}_0^2$. A is a nonnegative self adjoint linear operator of $\mathbf{H}$ and has a compact resolvent. Note that its eigenvalues and the corresponding eigenfunctions are $(\lambda_k)_{k\in\mathbb{Z}_0^2}$ and $(e_k)_{k\in\mathbb{Z}_0^2}$, respectively. We define the bilinear operator B such that

$$B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}', \quad B(v,w) = \mathbb{P}(v\cdot\nabla)w,$$

where $\mathbf{V}'$ is the dual space of $\mathbf{V}$. The linear operator G is given by

$$G : \mathbf{V} \to \mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2;\mathbf{H}), \quad Gv = \sqrt{2\mu}\mathbb{P}\nabla v,$$

where $\mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2;\mathbf{H})$ denotes the family of Hilbert-Schmidt operators from $\mathbb{R}^2$ to $\mathbf{H}$. Note that the adjoint operator $(Gv)^*$ of Gv belongs to $\mathbf{L}_{\mathrm{H.S}}(\mathbf{H};\mathbb{R}^2)$ and

$$(Gv)^*\,\phi = -\sqrt{2\mu}\left(\langle\frac{\partial\phi}{\partial x_1}, v\rangle, \langle\frac{\partial\phi}{\partial x_2}, v\rangle\right) \quad \text{for } \phi \in \mathbf{C}_\sigma^\infty.$$

We denote by $\mathbf{C}_{\sigma,0}^\infty$ the family of vector fields u, whose form is of $u = \sum_{k\in\mathbb{Z}_0^2} u_k e_k$, where each u_k is a constant and $u_k = 0$ except finitely many for k. Clearly, $\mathbf{C}_{\sigma,0}^\infty$ is a linear subspace of $\mathbf{C}_\sigma^\infty$. We set $\mathbf{U} = \mathbf{W}^{k_0,2}(\mathbb{T}^2;\mathbb{R}^2)\cap\mathbf{H}$, $k_0 > 2$, equipped with its norm $||u||_{k_0,2} = \left(\sum_{k\in\mathbb{Z}_0^2}|\langle u, e_k\rangle|^2|k|^{2k_0}\right)^{\frac{1}{2}}$. We denote by $\mathbf{U}'$ the dual space of $\mathbf{U}$. The weak form of our equation is formulated as follows:

$$\sum_{i=1}^{2}\left(\int_{\mathbb{T}^2} u^i(t,x)\phi^i(x)dx - \int_{\mathbb{T}^2} u_0^i(x)\phi^i(x)dx\right) \tag{11}$$
$$= \sum_{i,j=1}^{2}\int_0^t\int_{\mathbb{T}^2} u^i(s,x)u^j(s,x)\frac{\partial\phi^i(x)}{\partial x_j}dsdx$$
$$+\sqrt{2\mu}\sum_{i,j=1}^{2}\int_0^t\left(\int_{\mathbb{T}^2} u^i(s,x)\frac{\partial\phi^i(x)}{\partial x_j}dx\right)dB_s^j + \mu\sum_{i=1}^{2}\int_0^t\left(\int_{\mathbb{T}^2} u^i(s,x)\Delta\phi^i(x)dx\right)ds,$$

for all $\phi \in \mathbf{C}_{\sigma,0}^\infty$ and $t \geq 0$. Note that the term containing p does not appear in (11) by the Helmholtz decomposition of $\mathbf{L}^2(\mathbb{T}^2;\mathbb{R}^2)$.

Our equation (11) is rewritten into

$$\begin{cases} du(t) + \{Au(t) + B(u(t),u(t))\}\,dt + Gu(t)\,dB_t = 0, & t > 0, \\ u(0) = u_0. \end{cases} \tag{12}$$

We give the definition of the weak solution of our equation (12).

Definition 3.1. *We say that $\{u(t), B(t)\}_{t\geq 0}$ is a weak solution of the stochastic Navier-Stokes equation (12) with the initial value u_0 if*

1. *$\{u(t)\}_{t\geq 0}$ is an $\mathcal{F}_t$-adapted process on a probability space $(\Omega,\mathcal{F},P)$.*
2. *$u \in L^2(0,T;\mathbf{V})\cap L^\infty(0,T;\mathbf{H})$, a.s. for $T > 0$.*

3. $\{B(t), \{\mathcal{F}_t\}\}_{t\geq 0}$ *is a two-dimensional Brownian motion on* $(\Omega, \mathcal{F}, P)$.

4. For every $T > 0$, $\phi \in \mathbf{C}_{\sigma,0}^{\infty}$ *and a.e.-*$t \in [0,T]$, *P-a.s.,*

$$\langle u(t), \phi\rangle - \langle u_0, \phi\rangle = \\ -\int_0^t \langle A^*\phi, u(s)\rangle ds + \int_0^t \langle B(u(s),\phi), u(s)\rangle ds - \int_0^t (Gu(s))^*\phi\, dB(s),$$

holds.

Let us give a slight remark about the above definition. The weak solution is meant of itself in a probabilistic meaning, that is, this type of solution is called a martingale solution as well.

We say the equation (12) satisfies the coercivity condition if the following Condition 3.1 holds:

Condition 3.1. *([10])* $G : \mathbf{V} \to \mathbf{L}_{H.S}(\mathbb{R}^2, \mathbf{H})$ *is continuous and*

$$2\langle Av, v\rangle - |Gv|^2_{\mathbf{L}_{H.S}(\mathbb{R}^2,\mathbf{H})} \geq \delta\mu||v||^2_{\mathbf{V}} - \lambda_0|v|^2_{\mathbf{H}} - \rho,$$

for all $v \in \mathbf{V}$ *and for some* $\delta \in (0, 2]$, $\lambda_0 \geq 0$ *and* $\rho \geq 0$.

Our equation (12) does not satisfy Condition 3.1, because

$$2\langle Au(t), u(t)\rangle - |Gu(t)|^2_{\mathbf{L}_{\mathbf{H}.\mathbf{S}}(\mathbb{R}^2,\mathbf{H})} = 0. \tag{13}$$

Namely, in our case, $\delta = 0$. Our main result is

Theorem 3.1 (Existence of the weak solution). *There exists a weak solution* $\{u(t), B(t)\}_{t\geq 0}$ *of the stochastic Navier-Stokes equation (12) with the initial value* $u_0 \in \mathbf{V}$.

4 Existence of solutions

The following lemma is essential to prove our theorem.

Lemma 4.1. *If* $u \in \mathbf{C}_{\sigma}^{\infty}$, *then* $\langle\langle u, u\cdot\nabla u\rangle\rangle = 0$ *holds.*

Its proof is based on a standard calculation.

Proof. For any $\mathbb{R}^2$-valued function u on $\mathbb{R}^2$ which is of divergence free, we can choose a C^2-function ϕ on $\mathbb{R}^2$ such that $u = \nabla^{\perp}\phi$ holds, where $\nabla^{\perp}\phi = (-\partial_2\phi, \partial_1\phi)$ and $\partial_k\phi$ denotes $\dfrac{\partial\phi}{\partial x_k}$, $k = 1, 2$, (see [1]). Then

$$\begin{aligned} &\langle\langle u, u\cdot\nabla u\rangle\rangle \\ =&\langle\partial_1 u, \partial_1(u\cdot\nabla u)\rangle + \langle\partial_2 u, \partial_2(u\cdot\nabla u)\rangle \\ =&\langle\partial_1 u, \partial_1 u\cdot\nabla u\rangle + \langle\partial_1 u, u\cdot\nabla(\partial_1 u)\rangle + \langle\partial_2 u, \partial_2 u\cdot\nabla u\rangle + \langle\partial_2 u, u\cdot\nabla(\partial_2 u)\rangle, \end{aligned} \tag{14}$$

holds. The second and the forth terms in the last line of (14) are equal to 0 due to the incompressibility condition. Therefore, it suffices to show $\langle \partial_1 u, \partial_1 u \cdot \nabla u\rangle + \langle \partial_2 u, \partial_2 u \cdot \nabla u\rangle = 0$ to prove this lemma. Indeed, this is obtained by the following calculation:

$$\begin{aligned}
&\langle \partial_1 u, \partial_1 u \cdot \nabla u\rangle + \langle \partial_2 u, \partial_2 u \cdot \nabla u\rangle \\
&= \sum_{j,k,l=1}^{2} \int_{\mathbb{T}^2} \partial_l \left(\nabla^\perp \phi(x)\right)^k \partial_l \left(\nabla^\perp \phi(x)\right)^j \partial_j \left(\nabla^\perp \phi(x)\right)^k dx = 0.
\end{aligned}$$

□

sketch of proof of Theorem 3.1.

Step 1: Approximation by finite dimensional S.D.E.s

We denote by $\mathbf{H}_n$ the linear subspace of $\mathbf{H}$ spanned by $\{e_k\}_{0<|k|\le n}$. Let us define the linear operator Π_n as $\Pi_n u = \sum_{0<|k|\le n}(u, e_k)e_k$, $u \in \mathbf{U}'$, where $(\cdot,\cdot)$ is the dual pairing between $\mathbf{U}'$ and $\mathbf{U}$. Its restriction to $\mathbf{H}$ is the orthogonal projection onto $\mathbf{H}_n$. We set

$$A_\delta = -\frac{2+\delta}{2}\mu \mathbb{P}\Delta, \quad \delta > 0.$$

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\ge 0})$ be a complete filtered probability space on which a two-dimensional $\mathcal{F}_t$-Brownian motion $\{B_t\}_{t\ge 0}$ is defined. We assume that $\mathcal{F}_0$ contains the P-null sets. Then, we consider the following finite dimensional stochastic differential equations on $\mathbf{H}_n$:

$$\begin{cases} du_n^\delta(t) + \left\{A_\delta u_n^\delta(t) + \Pi_n B(u_n^\delta(t), u_n^\delta(t))\right\} dt + \Pi_n G u_n^\delta(t)\, dB_t = 0, & t > 0, \\ u_n^\delta(0) = \Pi_n u_0, \end{cases} \tag{15}$$

By standard argument, we see that there exists a unique solution u_n^δ for any $\delta > 0$, $n \ge 1$ and $T > 0$.

Step 2: A priori estimate

We use Itô's formula for $||u_n^\delta(t)||_{\mathbf{V}}^2$ and Lemma 4.1. Then, if we assume that u_0 is a $\mathbf{V}$-valued function, we obtain the following uniform estimate:

$$\sup_{n\ge 1, \delta>0} \mathbf{E}^P \left\{ \int_0^T ||u_n^\delta(t)||_{\mathbf{V}}^2 dt \right\} < \infty, \tag{16}$$

for each $T > 0$. Similarly, by applying Itô's formula again, we see

$$\sup_{n\ge 1, \delta>0} \mathbf{E}^P \left\{ \sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^2 \right\} < \infty, \tag{17}$$

holds for each $T > 0$. As a result, we obtain

Lemma 4.2. *The following a priori estimates hold:*

$$\sup_{n\geq 1,\delta>0} \mathbf{E}^P \{||u_n^\delta(t)||_{\mathbf{V}}^2\} \leq ||u_0||_{\mathbf{V}}^2,$$

$$\sup_{n\geq 1,\delta>0} \mathbf{E}^P \left\{ \int_0^T ||u_n^\delta(t)||_{\mathbf{V}}^2 dt \right\} < \infty,$$

$$\sup_{n\geq 1,\delta>0} \mathbf{E}^P \left\{ \sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^2 \right\} < \infty.$$

Step 3: Compactness argument

We use the method of martingale problems to solve our equations. Let $(\delta_k)_{k\geq 1}$ a sequence satisfying $\delta_k \downarrow 0$ as $k \to \infty$ and consider the family of solutions $(u_k^{\delta_k})_{k\geq 1}$ of (15). Set

$$\Omega_T = C([0,T];\mathbf{U}'),$$

endowed with the sup-norm $||u||_{\Omega_T} = \sup_{t\in[0,T]} ||u(t)||_{\mathbf{U}'}$ and $\mathcal{B}$ the topological σ-field on Ω_T. We denote by $\mathcal{B}_t$ the sub-σ-field of $\mathcal{B}$ generated by $u(s)$, $0 \leq s \leq t$. Set

$$\mathbb{W}_T = \Omega_T \cap L^2(0,T;\mathbf{H}) \cap L_w^2(0,T;\mathbf{V}) \cap C([0,T];\mathbf{H}_\sigma).$$

where $L_w^2(0,T;\mathbf{V})$ is the space $L^2(0,T;\mathbf{V})$ with its weak topology and $C([0,T];\mathbf{H}_\sigma)$ the space $C([0,T];\mathbf{H})$ with its weak topology. Let us set P^k the probability law of $u_k^{\delta_k}$ on Ω_T. We denote by $\mathcal{D}$ the family of functions Ψ defined on Ω_T whose forms are of

$$\Psi(u) = \psi(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle),$$

for some $n \in \mathbb{N}$, where $\psi \in C_0^2(\mathbb{R}^n)$ and for all $\phi_i \in \mathbf{C}_{\sigma,0}^\infty$, $i = 1,\cdots,n$. We define a linear operator $\mathcal{L}_k$ on $\mathcal{D}$, $k = 1,2,\cdots$, as

$$\begin{aligned}\mathcal{L}_k\Psi(u) =& \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle)\left\{(-\Pi_k Gu)^*\phi_i\cdot((-\Pi_k Gu)^*\phi_j)^*\right\}\\ &+\sum_{i=1}^n \frac{\partial\psi}{\partial\alpha_i}(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle)\left\{\frac{2+\delta_k}{2}\mu\langle u,\Delta\phi_i\rangle + \langle\Pi_k(u\cdot\nabla)\Pi_k\phi_i,u\rangle\right\},\end{aligned}$$

for $\Psi \in \mathcal{D}$. In these settings, we formulate the martingale problem associated to our equations.

Definition 4.1. *We say that a probability measure P defined on $(\Omega_T,\mathcal{B})$ is a solution of $(\mathcal{L}_k,\mathcal{D})$-martingale problem starting at $u \in \mathbf{H}$ if*

1. $P(x(0) = u) = 1$,
2. $\Psi(x(t)) - \Psi(x(0)) - \int_0^t \mathcal{L}_k(\Psi(x(s))ds, \quad t \in [0,T]$,

is a $\mathcal{B}_t$-local martingale under P.

Since $(u_k^{\delta_k}, B)$ is a solution of (15) for each k, it follows that P^k is a solution of $(\mathcal{L}_k, \mathcal{D})$-martingale problem starting at $\Pi_k u_0$. We shall prove the following lemmas:

Lemma 4.3. *The family of probability measures $(P^k)_{k=1,2,\ldots}$ is relatively compact in $\mathbb{W}_T$.*

Suppose that Lemma 4.3 is proven, we denote by $\bar{P}$ its limit. Similarly to $\mathcal{L}_k$, $k = 1, 2, \cdots$, we define a linear operator $\mathcal{L}$ on $\mathcal{D}$ as

$$\begin{aligned}\mathcal{L}\Psi(u) =& \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle u,\phi_1\rangle, \cdots, \langle u,\phi_n\rangle)\left\{(-Gu)^*\phi_i \cdot ((-Gu)^*\phi_j)^*\right\} \\ &+\sum_{i=1}^{n} \frac{\partial\psi}{\partial\alpha_i}(\langle u,\phi_1\rangle, \cdots, \langle u,\phi_n\rangle)\left\{\mu\langle u, \Delta\phi_i\rangle + \langle (u\cdot\nabla)\phi_i, u\rangle\right\}.\end{aligned}$$

Then, the following lemma holds.

Lemma 4.4. *The probability measure $\bar{P}$ is a solution of $(\mathcal{L}, \mathcal{D})$-martingale problem starting at u_0.*

Let us mention that the proof of its lemma is based on the strategy using [4]. By Lemma 4.4, we see that

$$\begin{aligned}M^{\phi}(t,x) \equiv& \langle x(t), \phi\rangle - \langle u_0, \phi\rangle \\ &- \mu\int_0^t \langle x(s), \Delta\phi\rangle ds - \int_0^t \langle (x(s)\cdot\nabla)\phi, x(s)\rangle ds,\end{aligned}$$

and

$$M^{\phi}(t,x)^2 - \int_0^t (-Gx(u))^*\phi \cdot ((-Gx(u))^*\phi)^* du,$$

are local martingales. The remaining part of the proof is standard. It is solved by applying the representation theorem of martingale (see e.g. [9], Theorem 8.2).

□

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