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N- Fractional Calculus of the Function $f(z)=\log((z-b)^{3}-c)$ and Identities

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Abstract

In this article, N- fractional calculus of the logarithmic function

$$f(z)=\log((z-b)^{3}-c)\ ,\qquad (z-b)^{3}-c\neq 0,1\ ,$$

is discussed and some identities derived from them are reported.
That is, it is discussed in the manner below, for example.

$$(\log((z-b)^{3}-c))_{\gamma}=\left((\log((z-b)^{3}-c))_{1}\right)_{\gamma-1}$$

$$=-3e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(3k+3)}T^{k}$$

$$\times\{\Gamma(3k+2+\gamma)-2(\gamma-1)\Gamma(3k+1+\gamma)+(\gamma-1)(\gamma-2)\Gamma(3k+\gamma)\}$$

$$(|\Gamma(\gamma)|<\infty)$$

where

$$T=\frac{c}{(z-b)^{3}},\qquad |T|<1,\qquad \Gamma(\cdots)\ ;\text{Gamma Function}$$

and

$$[\lambda]_{k}=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda+k)/\Gamma(\lambda)\ \textit{with}\ [\lambda]_{0}=1.$$

(Notation of Pochhammer)

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C .)

Moreover, let $f = f(z)$ be a regular function in D $(z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin Z^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in Z^+), \tag{2}$$

where $-\pi \le \arg(\zeta - z) \le \pi$ for C_-, $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+,

$\zeta \ne z$, $z \in C$, $\nu \in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z, of the function f, if $|(f)_\nu| < \infty$.

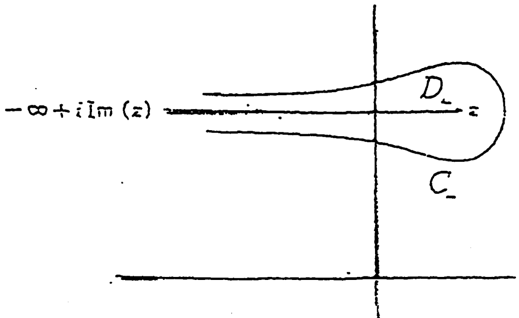

Fig. 1.

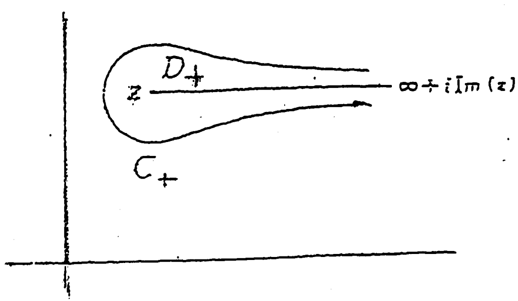

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu = n$ $(\in Z^+)$ and is reduced to the famous Cauchy's integral for $\nu = 0$. That is, (1) is an extention of Cauchy's integral and of Goursat's one, conversely Cauchy's and Goursat's ones are special cases of (1).

Moreover, notice that (1) is the representation which unifies the derivatives and integrations.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right)\qquad(\nu\notin Z^{-}),\quad [\text{Refer to (1)}] \tag{3}$$

with

$$N^{-m}=\lim_{\nu\to -m}N^{\nu}\qquad(m\in Z^{+}), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in \mathbf{R}), \tag{5}$$

then the set

$$\{N^{\nu}\}=\{N^{\nu}\mid \nu\in \mathbf{R}\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^{\nu})^{-1}=N^{-\nu}$ to the fractional calculus operator N^{ν}, for the function f such that $f\in F=\left\{f_{\nu};\ 0\neq |f_{\nu}|<\infty,\ \nu\in \mathbf{R}\right\}$, where $f=f(z)$ and $z\in \mathbf{C}$. (*vis.* $-\infty<\nu<\infty$).

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index ν " for the set of F.* (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S:=\{\pm N^{\nu}\}\cup\{0\}=\{N^{\nu}\}\cup\{-N^{\nu}\}\cup\{0\}\quad(\nu\in \mathbf{R}). \tag{7}$$

Then the set S is a commutative ring for the function $f\in F$, when the identity

$$N^{\alpha}+N^{\beta}=N^{\gamma}\qquad(N^{\alpha},N^{\beta},N^{\gamma}\in S) \tag{8}$$

holds. [5]

(III) **Lemma.** We have [1]

(i) $$((z-c)^{b})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right),$$

(ii) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad(|\Gamma(\alpha)|<\infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad(|\Gamma(\alpha)|<\infty),$$

where $z-c\neq 0$ for (i) and $z-c\neq 0,1$ for (ii), (iii),

(iv) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\begin{pmatrix}u=u(z),\\ v=v(z)\end{pmatrix}.$$

§ 1. Preliminary

[I] The theorem D below is reported by K. Nishimoto already (cf. J. Frac. Calc.Vol.29, May (2006), p.37). [12]

Theorem D. We have

(i) $$(((z-b)^{\beta}-c)^{\alpha})_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\,\Gamma(\beta k-\alpha\beta)}T^k \qquad \left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right|<\infty\right) \tag{1}$$

and

(ii) $$(((z-b)^{\beta}-c)^{\alpha})_{n}=(-1)^n(z-b)^{\alpha\beta-n}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}T^k \qquad \left(n\in Z_0^+\right). \tag{2}$$

where

$$T=\frac{c}{(z-b)^{\beta}}, \qquad |T|<1, \tag{3}$$

and

$$[\lambda]_k=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda-k)/\Gamma(\lambda)\ \textit{with}\ [\lambda]_0=1\,. \tag{4}$$

(Notation of Pochhammer

[II] The theorem E below for the fractional calculus of a logarithmic function is reported by K. Nishimoto already (cf. J. Frac. Calc.Vol.29, May (2006), p.40).[12]

Theorem E. We have

(i) $$(\log((z-b)^{\beta}-c))_{\gamma}=-e^{-i\pi\gamma}\beta(z-b)^{-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}T^k \qquad \left(\left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\right|<\infty\right) \tag{5}$$

and.

(ii) $$(\log((z-b)^{\beta}-c))_{n}=(-1)^{n+1}\beta(z-b)^{-n}$$

$$\times\sum_{k=0}^{\infty}[\beta k+1]_{n-1}T^k \qquad \left(n\in Z^+\right) \tag{6}$$

with (3), where

$$(z-b)^{\beta}-c\neq 0,1\,. \tag{7}$$

§ 2. N-Fractional Calculus of The Function in Title

Theorem 1. *Let be*

$$(z-b)^3 - c \neq 0, 1 \tag{1}$$

we have then

(i) $$(\log((z-b)^3 - c))_\gamma = -e^{-i\pi\gamma} 3 (z-b)^{-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)} T^k \qquad (|\Gamma(3k+\gamma)| < \infty) \tag{2}$$

and.

(ii) $$(\log((z-b)^3 - c))_n = (-1)^{n+1} 3 (z-b)^{-n}$$

$$\times \sum_{k=0}^{\infty} [3k+1]_{n-1} T^k \qquad (n \in Z^+) \tag{3}$$

(n-th derivatives)

where

$$T = \frac{c}{(z-b)^3}, \qquad |T| < 1 \tag{4}$$

Proof. Set $\beta = 3$ in Theorem E.

Corollary 1. *Let be*

$$z - b \neq 0, 1 \tag{5}$$

we have then

(i) $$(\log(z-b)^3)_\gamma = -e^{-i\pi\gamma} 3 \Gamma(\gamma)(z-b)^{-\gamma} \qquad (|\Gamma(\gamma)| < \infty) \tag{6}$$

and

(ii) $$(\log(z-b)^3)_n = (-1)^{n+1} 3 \Gamma(n)(z-b)^{-n} \qquad (n \in Z^+) \tag{7}$$

(n-th derivatives)

Proof. Set $c = 0$ in Theorem 1.

Note. We can obtain (6) and (7) from Lemma (ii) directly.

Theorem 2. *Let be*

$$(z-b)^{3/2} \pm \sqrt{c} \neq 0, 1, \tag{8}$$

we have then

(i) $$(\log((z-b)^3-c))_\gamma = -e^{-i\pi\gamma}\frac{3}{2}(z-b)^{-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\gamma)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \qquad \left(|\Gamma(\tfrac{3}{2}k+\gamma)|<\infty\right) \qquad (9)$$

and

(ii) $$(\log((z-b)^3-c))_n = (-1)^{n+1}\frac{3}{2}(z-b)^{-n}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+n)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \qquad (n\in Z^+) \qquad (10)$$

(n-th derivatives)

where

$$S=\frac{\sqrt{c}}{(z-b)^{3/2}}, \qquad |S|<1. \qquad (11)$$

Proof of (i). We have

$$(\log((z-b)^3-c))_\gamma = (\log((z-b)^{3/2}-\sqrt{c}))_\gamma + (\log((z-b)^{3/2}+\sqrt{c}))_\gamma . \qquad (12)$$

Now we have

$$(\log((z-b)^{3/2}-\sqrt{c}))_\gamma = -e^{-i\pi\gamma}\frac{3}{2}(z-b)^{-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\gamma)}{\Gamma(\frac{3}{2}k+1)}S^k \qquad \left(|\Gamma(\tfrac{3}{2}k+\gamma)|<\infty\right) \qquad (13)$$

from Theorem E. (i), setting $\beta=3/2$.

Therefore, we obtain (9) from (12), applying (13).

Proof of (ii). Set $\gamma=n$ in (9).

Note. We can obtain (6) and (7) from (9) and (10), setting $c=0$, clearly.

Theorem 3. *Let be* (1), *we have then*

(i) $$(\log((z-b)^3-c))_\gamma = -3e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}T^k$$

$$\times \{\Gamma(3k+2+\gamma)-2(\gamma-1)\Gamma(3k+1+\gamma)+(\gamma-1)(\gamma-2)\Gamma(3k+\gamma)\}$$

$$(|\Gamma(\gamma)|<\infty) \qquad (14)$$

and

(ii) $$(\log((z-b)^3-c))_n = (-1)^{n+1} 3(z-b)^{-n} \sum_{k=0}^{\infty} \frac{1}{\Gamma(3k+3)} T^k$$

$$\times \{\Gamma(3k+2+n)-2(n-1)\Gamma(3k+1+n)+(n-1)(n-2)\Gamma(3k+n)\}.$$

$$(n\text{-th derivarives}) \quad (n \in Z^+) \qquad (15)$$

where T is the one given by (4).

Proof of (i). We have

$$(\log((z-b)^3-c))_\gamma = ((\log((z-b)^3-c))_1)_{\gamma-1} \qquad (16)$$

$$= 3\,(((z-b)^3-c)^{-1}\cdot(z-b)^2)_{\gamma-1} \qquad (17)$$

$$= 3\sum_{s=0}^{\infty} \frac{\Gamma(\gamma)}{m!\,\Gamma(\gamma-m)}(((z-b)^3-c)^{-1})_{\gamma-1-m}((z-b)^2)_m \qquad (18)$$

(by Lemma (iv))

$$= 3\sum_{s=0}^{2} \frac{\Gamma(\gamma)}{m!\,\Gamma(\gamma-m)}\{e^{-i\pi(\gamma-1-m)}(z-b)^{-3-\gamma+1+m} \sum_{k=0}^{\infty} \frac{[1]_k\Gamma(3k+3+\gamma-1-m)}{k!\,\Gamma(3k+3)} T^k\}$$

$$\times \{e^{-i\pi m}\frac{\Gamma(m-2)}{\Gamma(-2)}(z-b)^{2-m}\} \qquad (19)$$

$$= -3\,e^{-i\pi\gamma}(z-b)^{-\gamma} \sum_{m=0}^{2} \frac{\Gamma(\gamma)\Gamma(m-2)}{m!\,\Gamma(\gamma-m)\Gamma(-2)} \sum_{k=0}^{\infty} \frac{\Gamma(3k+2+\gamma-m)}{\Gamma(3k+3)} T^k \,, \qquad (20)$$

$$= -3\,e^{-i\pi\gamma}(z-b)^{-\gamma}\{\sum_{k=0}^{\infty} \frac{\Gamma(3k+2+\gamma)}{\Gamma(3k+3)} T^k - 2(\gamma-1)\sum_{k=0}^{\infty} \frac{\Gamma(3k+1+\gamma)}{\Gamma(3k+3)} T^k$$

$$+(\gamma-1)(\gamma-2)\sum_{k=0}^{\infty} \frac{\Gamma(3k+\gamma)}{\Gamma(3k+3)} T^k\}\,, \qquad (21)$$

applying

$$\text{Theorem D. (i) to} \quad (((z-b)^3-c)^{-1})_{\gamma-1-m}$$

and Lemma (i) to $((z-b)^2)_m$.

We have then (14) from (21), clearly.

Proof of (ii). Set $\gamma = n$ in (i).

Theorem 4. *Let be* (1), *we have then*

$$\text{(i)} \qquad (\log((z-b)^3-c))_\gamma = -3e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}T^k$$

$$\times\{(1+2T)\Gamma(3k+4+\gamma)-(\gamma-2)(4+2T)\Gamma(3k+3+\gamma)$$

$$+6(\gamma-2)(\gamma-3)\Gamma(3k+2+\gamma)-4(\gamma-2)(\gamma-3)(\gamma-4)\Gamma(3k+1+\gamma)$$

$$+(\gamma-2)(\gamma-3)(\gamma-4)(\gamma-5)\Gamma(3k+\gamma)\} \qquad (|\Gamma(\gamma)|<\infty) \tag{22}$$

and

$$\text{(ii)} \qquad (\log((z-b)^3-c))_n = (-1)^{n+1}3(z-b)^{-n}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}T^k$$

$$\times\{(1+2T)\Gamma(3k+4+n)-(n-2)(4+2T)\Gamma(3k+3+n)$$

$$+6(n-2)(n-3)\Gamma(3k+2+n)-4(n-2)(n-3)(n-4)\Gamma(3k+1+n)$$

$$+(n-2)(n-3)(n-4)(n-5)\Gamma(3k+n)\} \qquad (n\in Z^+\geq 2) \tag{23}$$

(n-th derivatives)

where T *is the one given by* (4).

Proof of (i). We have

$$(\log((z-b)^3-c))_\gamma = ((\log((z-b)^3-c))_2)_{\gamma-2} \tag{24}$$

$$= 3(((z-b)^3-c)^{-1}\cdot(z-b)^2)_1)_{\gamma-2} \tag{25}$$

$$= -3(((z-b)^3-c)^{-2}\cdot((z-b)^4+2c(z-b)))_{\gamma-2} \tag{26}$$

$$= -3\sum_{m=0}^{\infty}\frac{\Gamma(\gamma-1)}{m!\,\Gamma(\gamma-1-m)}(((z-b)^3-c)^{-2})_{\gamma-2-m}((z-b)^4+2c(z-b))_m \tag{27}$$

$$= -3e^{-i\pi\gamma}(z-b)^{-4-\gamma}\sum_{m=0}^{4}\frac{\Gamma(\gamma-1)}{m!\,\Gamma(\gamma-1-m)}\{e^{i\pi m}(z-b)^m\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+4+\gamma-m)}{k!\,\Gamma(3k+6)}T^k\}$$

$$\times((z-b)^4+2c(z-b))_m \tag{28}$$

$$= -3e^{-i\pi\gamma}(z-b)^{-4-\gamma}[\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+4+\gamma)}{k!\,\Gamma(3k+6)}T^k\{(z-b)^4+2c(z-b)\}$$

$$-(\gamma-2)(z-b)\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+3+\gamma)}{k!\,\Gamma(3k+6)}T^k\{4(z-b)^3+2c\}$$

$$+\frac{1}{2}(\gamma-2)(\gamma-3)(z-b)^2\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+2+\gamma)}{k!\,\Gamma(3k+6)}T^k\{12(z-b)^2\}$$

$$-\frac{1}{3!}(\gamma-2)(\gamma-3)(\gamma-4)(z-b)^3\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+1+\gamma)}{k!\,\Gamma(3k+6)}T^k\{24(z-b)\}$$

$$+\frac{1}{4!}(\gamma-2)(\gamma-3)(\gamma-4)(\gamma-5)(z-b)^4\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+\gamma)}{k!\,\Gamma(3k+6)}T^k\cdot 24], \tag{29}$$

applying

Theorem D. (i) to $(((z-b)^3-c)^{-2})_{\gamma-2-m}$

and

Lemma (i) to $((z-b)^4+2c(z-b))_m=((z-b)^4)_m+2c(z-b)_m$.

We have then (22) from (29).

Proof of (ii). Set $\gamma=n$ in (i).

§ 3. Identities

Theorem 5. *Let be* § **2**.(2) *and* § **2**.(9) ,*we have then the identities as follows.*

(i) $$\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)}T^k=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\gamma)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \tag{1}$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}T^k\{\Gamma(3k+2+\gamma)-2(\gamma-1)\Gamma(3k+1+\gamma)$$

$$+(\gamma-1)(\gamma-2)\Gamma(3k+\gamma)\}\qquad(|\Gamma(\gamma)|<\infty) \tag{2}$$

$$= \sum_{k=0}^{\infty} \frac{[2]_k}{k!\,\Gamma(3k+6)} T^k \{(1+2T)\Gamma(3k+4+\gamma) - (\gamma-2)(4+2T)\Gamma(3k+3+\gamma)$$

$$+ 6(\gamma-2)(\gamma-3)\Gamma(3k+2+\gamma) - 4(\gamma-2)(\gamma-3)(\gamma-4)\Gamma(3k+1+\gamma)$$

$$+(\gamma-2)(\gamma-3)(\gamma-4)(\gamma-5)\Gamma(3k+\gamma)\} \qquad (|\Gamma(\gamma)|<\infty) \quad (3)$$

and

(ii) $$\sum_{k=0}^{\infty} \frac{\Gamma(3k+n)}{\Gamma(3k+1)} T^k = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2}k+n)}{\Gamma(\frac{3}{2}k+1)} \{S^k + (-S)^k\} \quad (4)$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(3k+3)} T^k \{\Gamma(3k+2+n) - 2(n-1)\Gamma(3k+1+n)$$

$$+(n-1)(n-2)\Gamma(3k+n)\} \qquad (n \in Z^+) \quad (5)$$

$$= \sum_{k=0}^{\infty} \frac{[2]_k}{k!\,\Gamma(3k+6)} T^k \{(1+2T)\Gamma(3k+4+n) - (n-2)(4+2T)\Gamma(3k+3+n)$$

$$+ 6(n-2)(n-3)\Gamma(3k+2+n) - 4(n-2)(n-3)(n-4)\Gamma(3k+1+n)$$

$$+(n-2)(n-3)(n-4)(n-5)\Gamma(3k+n)\} \qquad (n \in Z^+ \geq 2) \quad (6)$$

where

$$T = \frac{c}{(z-b)^3}, \quad |T|<1, \quad \textit{and} \quad S = \frac{\sqrt{c}}{(z-b)^{3/2}}, \quad |S|<1,$$

respectively.

Proof I. It is clear from Theorem 1, 2, 3 and 4.

Proof II. We have

$$\text{RHS of (1)} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2}\cdot 2k+\gamma)}{\Gamma(\frac{3}{2}\cdot 2k+1)} 2S^{2k} \quad (7)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)} T^k = \text{LHS of (1)}. \quad (8)$$

Proof III. We have

$$\text{RHS of (2)} = -\sum_{k=0}^{\infty}\frac{\Gamma(3k+2+\gamma)}{\Gamma(3k+3)}T^k - 2(\gamma-1)\sum_{k=0}^{\infty}\frac{\Gamma(3k+1+\gamma)}{\Gamma(3k+3)}T^k$$

$$+(\gamma-1)(\gamma-2)\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+3)}T^k \tag{9}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)}T^k\left\{\frac{(3k+1+\gamma)(3k+\gamma)}{(3k+2)(3k+1)} - 2(\gamma-1)\frac{(3k+\gamma)}{(3k+2)(3k+1)}\right.$$

$$\left.+(\gamma-1)(\gamma-2)\frac{1}{(3k+2)(3k+1)}\right\} \tag{10}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)}T^k = \text{LH of (1)}. \tag{11}$$

§ 4. Semi Derivatives and Integrals

Set $\gamma = 1/2$ and $-1/2$ in (i) of Theorem1 ~ 4, we have then the semi-derivatives and semi-integrals as follows.

(I) Semi - derivatives ;

1. $$(\log((z-b)^3-c))_{1/2} = 3i(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(3k+\frac{1}{2})}{\Gamma(3k+1)}\left(\frac{c}{(z-b)^3}\right)^k, \tag{1}$$

2. $$(\log((z-b)^3-c))_{1/2} = i\frac{3}{2}(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\frac{1}{2})}{\Gamma(\frac{3}{2}k+1)}$$

$$\times\left\{\left(\frac{\sqrt{c}}{(z-b)^{3/2}}\right)^k+\left(\frac{-\sqrt{c}}{(z-b)^{3/2}}\right)^k\right\}, \tag{2}$$

3. $$(\log((z-b)^3-c))_{1/2} = i3(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{\Gamma(3k+\tfrac{5}{2})+\Gamma(3k+\tfrac{3}{2})+\tfrac{3}{4}\Gamma(3k+\tfrac{1}{2})\}. \tag{3}$$

4. $$(\log((z-b)^3-c))_{1/2} = i\,3(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{(1+2T)\Gamma(3k+\tfrac{9}{2})+\tfrac{3}{2}(4+2T)\Gamma(3k+\tfrac{7}{2})+6(\tfrac{3\cdot 5}{2^2})\Gamma(3k+\tfrac{5}{2})$$

$$+4(\tfrac{3\cdot 5\cdot 7}{2^3})\Gamma(3k+\tfrac{3}{2})+(\tfrac{3\cdot 5\cdot 7\cdot 9}{2^4})\Gamma(3k+\tfrac{1}{2})\} \quad (T=\frac{c}{(z-b)^3}) \quad . \qquad (4)$$

(I) Semi - integrals;

1. $$(\log((z-b)^3-c))_{-1/2} = -3i(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(3k-\frac{1}{2})}{\Gamma(3k+1)}\left(\frac{c}{(z-b)^3}\right)^k , \qquad (5)$$

2. $$(\log((z-b)^3-c))_{-1/2} = -i\frac{3}{2}(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k-\frac{1}{2})}{\Gamma(\frac{3}{2}k+1)}$$

$$\times\left\{\left(\frac{\sqrt{c}}{(z-b)^{3/2}}\right)^k+\left(\frac{-\sqrt{c}}{(z-b)^{3/2}}\right)^k\right\} , \qquad (6)$$

3. $$(\log((z-b)^3-c))_{-1/2} = -i\,3(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{\Gamma(3k+\tfrac{3}{2})+3\Gamma(3k+\tfrac{1}{2})+\tfrac{15}{4}\Gamma(3k-\tfrac{1}{2})\} , \qquad (7)$$

4. $$(\log((z-b)^3-c))_{-1/2} = -i3(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{(1+2T)\Gamma(3k+\tfrac{7}{2})+\tfrac{5}{2}(4+2T)\Gamma(3k+\tfrac{5}{2})+6(\tfrac{5\cdot 7}{2^2})\Gamma(3k+\tfrac{3}{2})$$

$$+4(\tfrac{5\cdot 7\cdot 9}{2^3})\Gamma(3k+\tfrac{1}{2})+(\tfrac{5\cdot 7\cdot 9\cdot 11}{2^4})\Gamma(3k-\tfrac{1}{2})\} \quad (T=\frac{c}{(z-b)^3}) \quad . \qquad (8)$$

§ 5. Example

1. **Examples for Theorem 1. (ii) and Theorem 2. (ii).**

(I) When $n=1$, we obtain the below from Theorem 1. (ii) ;

$$(\log((z-b)^3-c))_1 = 3\,(z-b)^{-1}\sum_{k=0}^{\infty} T^k \qquad (1)$$

$$=3(z-b)^{-1}\sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k \qquad (T=\frac{c}{(z-b)^3}) \qquad (2)$$

$$=3(z-b)^{-1}(1-T)^{-1} \qquad (3)$$

$$=3(z-b)^2((z-b)^3-c)^{-1}. \qquad (4)$$

This result coincides with the obtained one by classical calculus

$$\frac{d}{dz}\log((z-b)^3-c)\ .$$

(II) When $n=2$, we obtain the below from Theorem 2. (ii) ;

$$(\log((z-b)^3-c))_2 = -\frac{3}{2}\,(z-b)^{-2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+2)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \qquad (5)$$

$$=-\frac{3}{2}\,(z-b)^{-2}\ \sum_{k=0}^{\infty}\frac{[1]_k(\frac{3}{2}k+1)}{k!}\{S^k+(-S)^k\} \qquad (S=\frac{\sqrt{c}}{(z-b)^{3/2}})\ . \qquad (6)$$

Now we have

$$\sum_{k=0}^{\infty}\frac{[1]_k}{k!}\{S^k+(-S)^k\}=(1-S)^{-1}+(1+S)^{-1} \qquad (7)$$

$$=\frac{2}{1-S^2}=2(z-b)^3((z-b)^3-c)^{-1}\ , \qquad (8)$$

and

$$\sum_{k=0}^{\infty}\frac{[1]_k(\frac{3}{2}k)}{k!}\{S^k+(-S)^k\}=\frac{3}{2}\sum_{k=0}^{\infty}\frac{[1]_{k+1}}{k!}\{S^{k+1}+(-S)^{k+1}\} \qquad (9)$$

$$=\frac{3}{2}\ \sum_{k=0}^{\infty}\frac{[2]_k}{k!}\{S^{k+1}+(-S)^{k+1}\} \qquad (10)$$

$$=\frac{3}{2}\{S(1-S)^{-2}+(-S)(1+S)^{-2}\} \tag{11}$$

$$=6c(z-b)^3((z-b)^3-c)^{-2}. \tag{12}$$

Therefore, we obtain

$$(6)=-\frac{3}{2}(z-b)^{-2}[2(z-b)^3((z-b)^3-c)^{-1}+6c(z-b)^3((z-b)^3-c)^{-2}] \tag{13}$$

$$=-3(z-b)((z-b)^3-c)^{-2}((z-b)^3+2c) \tag{14}$$

using (8) and (12).

This result coincides with the obtained one by the Classical calculus

$$\frac{d^2}{dz^2}\log((z-b)^3-c) .$$

2. Examples for the identities.

(I) When $n=2$ we have

$$\sum_{k=0}^{\infty}(3k+1)T^k=\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k(1+2T), \qquad (T=\frac{c}{(z-b)^3}) \tag{15}$$

from §3. (6).

Indeed we have

$$\text{LHS of (15)}=3\sum_{k=0}^{\infty}kT^k+\sum_{k=0}^{\infty}T^k=3\sum_{k=0}^{\infty}\frac{[1]_k k}{k!}T^k+\sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k \tag{16}$$

$$=3T(1-T)^{-2}+(1-T)^{-1} \tag{17}$$

$$=(1-T)^{-2}(2T+1). \tag{18}$$

And

$$\text{RHS of (15)}=(1+2T)\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k=(1+2T)(1-T)^{-2} . \tag{19}$$

That is, the identity (15) holds true.

(II) When $n=3$ we have

$$\sum_{k=0}^{\infty}(3k+2)(3k+1)T^k=\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k\{(1+2T)(3k+6)-(4+2T)\}, \tag{20}$$

from § 3. (6).

Indeed we have

$$\text{LHS of (20)} = 9\sum_{k=0}^{\infty}\frac{[1]_k k^2}{k!}T^k + 9\sum_{k=0}^{\infty}\frac{[1]_k k}{k!}T^k + 2\sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k \qquad (21)$$

$$= 9(1-T)^{-3}(T^2+T) + 9T(1-T)^{-2} + 2(1-T)^{-1} \qquad (22)$$

$$= 2(1-T)^{-3}(T^2+7T+1) \ , \qquad (23)$$

and

$$\text{RHS of (20)} = 3\sum_{k=0}^{\infty}\frac{[2]_k k}{k!}T^k + 6T\sum_{k=0}^{\infty}\frac{[2]_k k}{k!}T^k$$

$$+ 2\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k + 10T\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k \qquad (24)$$

$$= 6T(1-T)^{-3} + 12T^2(1-T)^{-3} + 2(1-T)^{-2} + 10T(1-T)^{-2} \qquad (25)$$

$$= 2(1-T)^{-3}(T^2+7T+1) = (23) = \text{LHS of (20)}. \qquad (26)$$

That is, the identity (20) holds true.

Note. We have

1. $$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}T^k = (1-T)^{-\lambda}. \qquad (27)$$

2. $$\sum_{k=0}^{\infty}\frac{[2]_k k}{k!}T^k = \sum_{k=1}^{\infty}\frac{[2]_k}{(k-1)!}T^k = \sum_{k=0}^{\infty}\frac{[2]_{k+1}}{k!}T^{k+1} \qquad (28)$$

$$= T\sum_{k=0}^{\infty}\frac{\frac{2\Gamma(3+K)}{2\Gamma(2)}}{k!}T^k = 2T\sum_{k=0}^{\infty}\frac{[3]_k}{k!}T^k = 2T(1-T)^{-3}. \qquad (29)$$

3. $$\sum_{k=0}^{\infty}\frac{[1]_k k^2}{k!}T^k = \sum_{k=1}^{\infty}\frac{[1]_k k}{(k-1)!}T^k = T\sum_{k=0}^{\infty}\frac{[1]_{k+1}(k+1)}{k!}T^k \qquad (30)$$

$$= T\sum_{k=0}^{\infty}\frac{[2]_k(k+1)}{k!}T^k = T\{\sum_{k=1}^{\infty}\frac{[2]_k k}{k!}T^k + \sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k\}. \qquad (31)$$

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5 (1996), Descartes Press, Koriyama Japan.

[2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus(Calculus of the 21st Century) ; Integrals and Differentiations of Arbitrary Order (1991), Descartes Press , Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol.6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field produced from The Set of N-Fractional Calculus Operator, J. Frac. Calc. Vol. 24, Nov. (2003), 29 - 36.

[6] K. Nishimoto ; N- Fractional Calculus of Products of Some Power Functions , J. Frac. Calc. Vol.27, May (2005), 83 - 88.

[7] K. Nishimoto ; N- Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol.29, May (2006), 35 - 44..

[8] K. Nishimoto ; N- Fractional Calculus of Some Composite AlgebraicFunctions, J. Frac. Calc. Vol.31, May (2007), 11 - 23.

[9] K. Nishimoto ; N- Fractional Calculus of Some Logarithmic Functions, J. Frac. Calc. Vol.32, Nov. (2007), 17 - 28.

[10] K. Nishimoto ; N- Fractional Calculus of Some Functions, and Their n-th Derivatives and Semi Differintegrations,J. Frac. Calc. Vol.32, Nov. (2007), 1 - 16.

[11] Shy-Der Lin, K. Nishimoto ,Tsuyako Miyakoda and H.M. Srivastava Some Differintegral Formulas for Power, Composite and Rational Functions, J. Frac. Calc. Vol.32, Nov. (2007), 87 - 98.

[12] Susana S. de Romero and K. Nishimoto ; N- Fractional Calculus and$n(\in Z^+)th$ Derivatives of Some Logarithmic Functions,J. Frac. Calc. Vol.32, Nov.(2007), 99 - 108.

[13] K. Nishimoto ; N- Fractional Calculus of Some Functions, which Have A Root Sign, J. Frac. Calc. Vol.33, May (2008), 1 - 12.

[14] K. Nishimoto ; N- Fractional Calculus of Some Funçtions,which Have Multiple Root Signs, J. Frac. Calc. Vol.33, May (2008),35 - 46.

[15] K. Nishimoto,Josefina Matera, Marleny Fuenmayor and Susana S. de Romero ; N- Fractional Calculus of Some Functions,which Have A Root Sign, J.Frac. Calc. Vol.33, May (2008), 59 - 70.

[16] K. Nishimoto ; N- Fractional Calculus of Some Functions and Dirichlet Interals J. Frac. Calc. Vol.34, Nov. (2008), 1 - 10.

[17] K. Nishimoto and T. Miyakoda ; N- Fractional Calculus of Some Multiple Power Functions and Some Identities,J.Frac.Calc.Vol.34, Nov. (2008),11-22.

[18] Tsuyako Miyakoda ; N- Fractional Calculus and $n(\in Z^+)th$ Derivati ves of Some Functions, J. Frac. Calc. Vol.34, Nov. (2008), 23 - 33.

[19] Marleny Fuenmayor, Josefina Matera and Susana S. de Romero.; N-Fractional Calculus of Functions $\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)^{-1}$, J. Frac.Calc.Vol.34, Nov. (2008), 87 - 98.

[20] K. Nishimoto ; N- Fractional Calculus of Some Exponential Functions, J.Frac. Calc. Vol.35, May (2009), 1 - 10.

[21] K. Nishimoto ; N- Fractional Calculus of Some Logarithmic Functions and Some Identities Derived from Them, J. Frac. Calc.Vol.35,May (2009),27 -42.

[22] K. Nishimoto, Ding-Kuo Chyan and Shih-Tong Tu ; N- Fractional Calculus of A Function Which Contains $(z-b)^{1/m}$ $(m(\in Z^{+})$ (I) J. Frac. Calc. Vol.35, May (2009), 81 - 94.

[23] Shy-Der Lin , Shih-Tong Tu and Ding-Kuo Chyan ; N- Fractional Calculus of Functions $\left(\sqrt[n]{\frac{1}{\sqrt[m]{z-b}}-c-d}\right)^{2}$ $(m,n \in Z^{+})$, J. Frac.Calc.Vol.35, May (2009), 107 - 117.

[24] K. Nishimoto, Susana S. de Romèro, Marleny Fuenmayor and Josefina Matera ; N- Fractional Calculus of Some Functions which Have Double Root Signs, J. Frac. Calc. Vol.36, Nov. (2009), 31 - 48.

[25] K. Nishimoto ; N- Fractional Calculus of Functions $((z-b)^{4}-c)^{-1}$ and Some Identities, J. Frac. Calc. Vol.37, May (2010), 1 - 13.

[26] Susana S. de Romero ,M.Fuenmayor and K. Nishimoto ; N-Frational Calculus of Some Logarithmic Functions Which Contain $(z-b)^{1/m}$ $(m(\in Z^{+})$, J. Frac. Calc. Vol.37, May (2010), 67 - 77.

[27] Susana S. de Romero ; SomeIdentities Derived from The N- Frational Calculus of Functions $f(z)=(z-b-c)^{-1}$, J. Frac. Calc. Vol.38, Nov. (2010),63 - 76.

[28] K. Nishimoto ; N- Fractional Calculus of Certain Generalized Logarithmic Functions. J. Frac. Calc. Vol.39, May (2011), 1 - 14.

[29] K; Nishimoto ; N- Fractional Calculus of the Irrational Functions $\sqrt{(z-b)^{2}-c}$ and Some Identities , J. Frac. Calc. Vol.39, May (2011), 15 - 27.

[30] K. Nishimoto ; N-Fractional Calculus of Some Algebraic Funtions and Identities. J. Frac. Calc. Vol.39, May (2011), 59 - 74.

[31] Chen-Te Yen, Ding-Kuo Chyan,Shih-Tong Tu and K. Nishimoto; N-Fractional Calculus of Some Functions and Their $n-th$ Derivatives $(n \in Z^{+})$, J. Frac. Calc. Vol.40, Nov. (2011), 1 - 13.

[32] K. Nishimoto ; N-Fractional Calculus of The Functions $f(z)=((z-b)^{2}-c)^{-2}$ and Identities (Part I), J. Frac. Calc. Vol. 40, Nov. (2011), 15 - 27.

[33] K. Nishimoto ; N-Fractional Calculus of The Functions $f(z)=((z-b)^{2}-c)^{-2}$ and Identities (Part II), J. Frac. Calc. Vol. 40, Nov. (2011), 29 - 36.

[34] T. Miyakoda and K. Nishimoto ; N-Fractional Calculus of Some Multiple Power Functions and Some dentities (Continue), J. Frac.Calc.Vol. 40, Nov. (2011), 37 - 48.

[35] K. Nishimoto ; On The Set $\{N^{\nu}\}$, Wher e N^{ν} is The N- Fractional Calculus Operator of Order ν, J. Frac. Calc. Vol. 40, Nov. (2011), 49 - 59.

[36] K. Nishimoto ; N-Fractional Calculus of Some Algebraic Functions and Identities (Continue), J. Frac. Calc. Vol. 40, Nov. (2011), 89 - 99.

[37] K. Nishimoto ; N-Fractional Calculus of The Function $f(z)=((z-b)^{2}-c)^{-3}$ and Identities , J. Frac. Calc. Vol. 41, May (2012), 25 - 38.

[38]. K. Nishimoto ; N-Fractional Calculus of The Function $f(z)=((z-a)(z-b)(z-c))^{-1}$ and Identities , J. Frac. Calc. Vol. 41, May (2012), 39 - 53.

[39] Susana S. de Romero,Josefina Matera, Ana I. Prieto and K. Nishimoto ; N-Fractional Calculus of The Function $f(z)=((z-b)^2-c)^{-1/3}$ and Identities , J. Frac. Calc. Vol. 41, May (2012), 55 - 74.

[40] Susana S.de Romero ; N-Fractional Calculus of Some Irrational Cubic Functions and Identities, J. Frac. Calc. Vol. 41, May (2012), 75 - 86.

[41] K. Nishimoto ; N-Fractional Calculus of The Function $f(z)=((z-b)^2-c)^{-3}$ and Identities (Continue), J. Frac. Calc. Vol. 42, Nov. (2012), 21 - 40.

[42] K. Nishimoto ; N-Fractional Calculus of The Function $f(z)=\log((z-b)^2-c)$ and Identities , J. Frac. Calc. Vol. 42, Nov. (2012), 59 - 74.

[43] K. Nishimoto ; N-Fractional Calculus of The Algebraic Function $f(z)=((z-b)^2-c)^{-p}$, $(p \in Z^+)$ and Identities , J. Frac. Calc. Vol. 42, Nov. (2012), 75 - 99.

[44] Saad Naji Al-Azawi ; Some Results in Fractional Calculus (2011), Lap Lambert, Saarbrucken, Germany.

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