数理解析研究所講究録
第1879巻 2014年 97-106

A simplicial algorithm with ω-$k$sections and its convergence

Takahito Kuno*and Tomohiro Ishihama†

Graduate School of Systems end Information Engineering
University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan

Abstract

The simplicial algorithm is a kind of branch-and-bound method for solving convex maximization problems. The subdivision rules sufficient for convergence include ω-subdivision, ω-bisection and the conventional bisection. In this paper, we develop a new convergent subdivision rule interpolating ω-bisection and ω-subdivision.

Key words: Global optimization, convex maximization, branch-and-bound, simplicial algorithm, subdivision rule.

1 Introduction

Branch-and-bound is a potent tool for solving convex maximization problems, a typical class of multiextremal global optimization problems. Algorithms based on branch-and-bound can be categorized roughly into three types: the *conical algorithm* [13], which subdivides the feasible set using a set of cones in the branching step, and narrows down the cone containing an optimal solution with the help of concavity cuts in the bounding step; the *simplicial algorithm* [3], which uses simplices instead of cones in the branching step, and carries out the bounding steps using concave envelopes; and the *rectangular algorithm* [1], which uses rectangles in the branching step and can be applied to only separable objective function problems. Unlike discrete optimization problems, no matter how many times the branching step is repeated, continuous optimization problems never yield trivial subproblems in general. Therefore, the convergence analysis is essential in developing deterministic branch-and-bound algorithms for convex maximization. In each type of branch-and-bound algorithms, it is observed that the convergence is accelerated by using the ω-subdivision rule, which exploits a by-product of the bounding step to subdivide cones, simplices or rectangles in the branching step. The convergence when using this rule was proven by Jaumard-Meyer [5, 6], Locatelli [9] and Kuno-Ishihama [8] for the conical algorithm, by Locatelli-Raber [10, 11] and Kuno-Buckland

*The author was partially supported by the Grant-in-Aid for Scientific Research (C 25330022) from the Japan Society for the Promotion of Sciences. E-mail: takahito@cs.tsukuba.ac.jp
†Current affiliation: NS Solutions Corporation, Tokyo 104-8280, Japan.

[7] for the simplicial algorithm, and by Falk-Soland [1] and Soland [12] for the rectangle algorithm.

In [7, 8], they also proposed a new kind of subdivision rule, called ω-bisection, which is a hybrid of ω-subdivision and the conventional bisection. In this paper, we extend this idea and develop a rule for subdividing each simplex into a prescribed number of subsimplices in the branching step of the simplicial algorithm. In Section 2, we formulate the convex maximization problem and illustrate how the simplicial algorithm solves it. In Section 3, we modify the relaxed problem solved in the bounding step and review the convergence under the usual ω-subdivision rule. In Section 4, for a given number k, we develop a convergent rule for subdividing each simplex into k subsimplices in the branching step. If $k = 2$, then this subdivision rule coincides with ω-bisection; and if k is equal to the problem dimension, it is the usual ω-subdivision. In Section 5, we report the results of a numerical comparison among ω-bisection, trisection, quadsection, and the usual ω-subdivision.

2 Simplicial algorithm for convex maximization

The problem considered in this paper is as follows:

$$\left| \begin{array}{ll} \text{maximize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \end{array} \right. \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The objective function f is a nonlinear convex function defined on an open convex set, which is assumed to be large enough to contain an n-simplex Δ^1 enclosing the feasible set

$$D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

Also the following are assumed throughout the paper:

(A1) The set D is bounded and contains the origin $\mathbf{0} \in \mathbb{R}^n$ as an interior point.

(A2) The objective function value $f(\mathbf{x})$ is nonnegative and bounded for any $\mathbf{x} \in \Delta^1$.

The simplicial algorithm is a kind of branch-and-bound algorithm for solving the multi-extremal problem (1) to optimality. In the preprocessing, the feasible set D is enclosed in the n-simplex Δ^1, which is given as $\text{conv}\{\mathbf{v}_j^1 \mid j = 1, \ldots, n+1\}$, the convex hull of $n+1$ affinely independent vectors. In the branching step, Δ^1 is subdivided into a set of subsimplices Δ^i, $i \in \mathscr{I}$, satisfying

$$\Delta^1 = \bigcup_{i \in \mathscr{I}} \Delta^i; \quad \text{int}\Delta^r \cap \text{int}\Delta^s = \emptyset, \quad \text{if } r, s \in \mathscr{I} \text{ and } r \neq s,$$

where $\mathscr{I}$ is an (infinite) index set, and int $\cdot$ represents the set of interior points. Associated with each Δ^i is a subproblem:

$$\left| \begin{array}{ll} \text{maximize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in D \cap \Delta^i. \end{array} \right. \tag{2}$$

Since (2) is essentially the same problem as the original (1), it cannot be solved directly. In the bounding step, replacing f by its concave envelope g^i, a minimal concave function overestimating f on Δ^i, the following relaxed problem is solved:

$$\left| \begin{array}{ll} \text{maximize} & g^i(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in D \cap \Delta^i. \end{array} \right. \tag{3}$$

In our case, where f is convex, g^i is an affine function which agrees with f at the vertices of Δ^i. This implies that (3) is just a linear program and can be solved easily. Let $\boldsymbol{\omega}^i$ be an optimal solution of (3). Then we have

$$g^i(\boldsymbol{\omega}^i) \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in D \cap \Delta^i.$$

Therefore, if $g^i(\boldsymbol{\omega}^i) \leq f(\mathbf{x}^*)$ for the incumbent solution $\mathbf{x}^*$ of (1) obtained during the algorithm, we can conclude that Δ^i contains no solution better than $\mathbf{x}^*$ and discard it from further consideration. Otherwise, Δ^i is again subdivided into smaller subsimplices in the branching step. Unlike discrete optimization problems, no matter how many times the branching and bounding steps are repeated, (2) can have infinitely many feasible solutions. In that case, at least one infinite sequence of simplices is generated in a nested fashion:

$$\Delta^1 \supset \Delta^2 \supset \cdots \supset \Delta^i \supset \Delta^{i+1} \supset \cdots. \tag{4}$$

The convergence of the simplicial algorithm depends largely on how to subdivide the simplex $\Delta^i = \text{conv}\{\mathbf{v}^i_j \mid j = 1, \ldots, n+1\}$. The simplest subdivision rule is *bisection*, where the longest edge of Δ^i is cut at the midpoint. Then Δ^i is divided into two subsimplices, either of which is the successor Δ^{i+1} in the sequence (4). Under this rule, the sequence gradually shrinks to a single point. Since $\boldsymbol{\omega}^i \in \Delta^i$ for $i = 1, 2, \ldots$, we have $|g^i(\boldsymbol{\omega}^i) - f(\boldsymbol{\omega}^i)| \to 0$, as $i \to \infty$. This exhaustiveness guarantees that the incumbent $\mathbf{x}^*$ converges to a globally optimal solution of (1).

Although not exhaustive, an often-used alternative is *ω-subdivision*. The simplex Δ^i is subdivided into up to $n+1$ subsimplices, radially at $\boldsymbol{\omega}^i$. Let J^i be an index set such that $r \in J^i$ if $\boldsymbol{\omega}^i$ is affinely independent of $\mathbf{v}_j$'s for $j \neq r$. Then the children of Δ^i are given as

$$\Delta^{ir} = \text{conv}\{\mathbf{v}^i_1, \ldots, \mathbf{v}_{r-1}, \boldsymbol{\omega}^i, \mathbf{v}^i_{r+1}, \ldots, \mathbf{v}^i_{n+1}\}, \quad r \in J^i.$$

The ω-subdivision rule has been said to be empirically more efficient than bisection. The theoretical convergence was, however, an open question for some decades until Locatelli-Raber proved it in 2000 [10, 11]. In the succeeding section, we will outline an easier alternative proof given in [7], which provides a clue leading to another kind of convergent subdivision rule.

3 Relaxation of the relaxed subproblem

Introducing an auxiliary variable $\tau \geq 0$, let us relax the feasible set D a little bit into

$$D(\tau) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq (1+\tau)\mathbf{b}\}.$$

Since $\mathbf{b} > \mathbf{0}$ under the assumption (A1), it holds that

$$D = D(0) \subset D(\tau') \subset D(\tau''), \quad \text{if } 0 \leq \tau' \leq \tau''.$$

Definition 3.1. For a positive constant σ, a point $\mathbf{x}$ is referred to as a *σ-feasible solution* of (1) if $\mathbf{x} \in D(\sigma)$.

Let $\sigma > 0$ be a prescribed tolerance, and let

$$F^i = \max\{f(\mathbf{v}_j^i) \mid j = 1, \ldots, n+1\}, \quad i = 1, 2, \ldots.$$

Note that F^i is nonnegative under the assumption (A2), and besides nonincreasing in i. Let us select a number M to satisfy

$$M > F^1/\sigma,$$

and define a problem in $\mathbf{x}$ and τ:

$$(\mathrm{P}^i) \left| \begin{array}{ll} \text{maximize} & g^i(\mathbf{x}) - M\tau \\ \text{subject to} & \mathbf{x} \in D(\tau) \cap \Delta^i, \quad \tau \geq 0. \end{array} \right.$$

It is easy to see that (P^i) is equivalent to a linear program

$$\left| \begin{array}{ll} \text{maximize} & \mathbf{d}\boldsymbol{\lambda} - M\tau \\ \text{subject to} & \mathbf{A}\mathbf{V}\boldsymbol{\lambda} - \mathbf{b}\tau \leq \mathbf{b}, \quad \mathbf{e}\boldsymbol{\lambda} = 1, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \tau \geq 0, \end{array} \right. \tag{5}$$

where $\mathbf{e} \in \mathbb{R}^{n+1}$ is the all-ones vector, and

$$\mathbf{d} = [g^i(\mathbf{v}_1^i), \ldots, g^i(\mathbf{v}_{n+1}^i)] = [f(\mathbf{v}_1^i), \ldots, f(\mathbf{v}_{n+1}^i)], \quad \mathbf{V} = [\mathbf{v}_1^i, \ldots, \mathbf{v}_{n+1}^i].$$

The dual problem of (6) is written as

$$\left| \begin{array}{ll} \text{minimize} & \boldsymbol{\mu}\mathbf{b} + \nu \\ \text{subject to} & \boldsymbol{\mu}\mathbf{A}\mathbf{V} + \nu\mathbf{e} \geq \mathbf{d}, \quad \nu\mathbf{b} \leq M, \quad \boldsymbol{\mu} \geq \mathbf{0}. \end{array} \right. \tag{6}$$

Since (5) is always feasible and the objective function is bounded from above by F^i, both (5) and (6) have optimal solutions, denoted $(\boldsymbol{\lambda}^i, \tau^i)$ and $(\boldsymbol{\mu}^i, \nu^i)$, respectively. Obviously , an optimal solution of (P^i) is then given by $(\boldsymbol{\omega}^i, \tau^i)$ with $\boldsymbol{\omega}^i = \mathbf{V}\boldsymbol{\lambda}^i$.

Proposition 3.1. *If $D \cap \Delta^i \neq \emptyset$, then the optimal value of (P^i) is an upper bound on the subproblem (2), i.e.,*

$$g^i(\boldsymbol{\omega}^i) - M\tau^i \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in D \cap \Delta^i.$$

Proposition 3.2. *If $g^i(\boldsymbol{\omega}^i) - M\tau^i < 0$, then $D \cap \Delta^i = \emptyset$. Otherwise, $\boldsymbol{\omega}^i$ is a σ-feasible solution of (1).*

If $D \cap \Delta^i$ turns out to be empty, then Δ^i is discarded from further consideration. We may therefore assume that $g^i(\boldsymbol{\omega}^i) - M\tau^i \geq 0$ and $\boldsymbol{\omega}^i \in D(\sigma)$ for every i. Let

$$\Delta^i_+ = \mathrm{conv}\{\mathbf{v}^i_j \mid j \in J^i\}, \quad J^i = \{j \mid \lambda^i_j > 0\}.$$

Then Δ^i_+ is a $|J^i|$-face of Δ^i, on which $\boldsymbol{\omega}^i$ lies. Even more remarkable is the following:

Lemma 3.3. *It holds that*

$$g^i(\mathbf{x}) \leq \boldsymbol{\mu}^i \mathbf{A}\mathbf{x} + \nu^i, \quad \forall \mathbf{x} \in \Delta^i.$$

In particular,

$$g^i(\mathbf{x}) = \boldsymbol{\mu}^i \mathbf{A}\mathbf{x} + \nu^i, \quad \text{if } \mathbf{x} \in \Delta^i_+.$$

Lemma 3.3 suggests that we can adopt $\boldsymbol{\mu}^i\mathbf{A}$ and ν^i as the coefficients and write g^i explicitly as follows:

$$g^i(\mathbf{x}) = \boldsymbol{\mu}^i \mathbf{A}\mathbf{x} + \nu^i, \quad i = 1, 2, \ldots.$$

Then we can show the following:

Lemma 3.4. *The gradient of g^i is bounded, i.e., there exists a constant L such that*

$$\|\boldsymbol{\mu}^i \mathbf{A}\| \leq L, \quad i = 1, 2, \ldots.$$

Lemma 3.5. *It holds that*

$$g^k(\mathbf{x}) \leq g^i(\mathbf{x}), \quad \mathbf{x} \in \Delta^k_+, \quad i = 1, \ldots, k.$$

Lemma 3.6. *If Δ^{i+1} is generated by subdividing Δ^i radially at an arbitrary point $\mathbf{u}^i \in \Delta^i_+$ for each i, then there exists a subsequence $\{i_r \mid r = 1, 2, \ldots\}$ such that*

$$\mathbf{u}^{i_{2s-1}} \in \Delta^{i_{2s-1}}_+ \cap \Delta^{i_{2s}}_+, \quad s = 1, 2, \ldots.$$

Using these lemmas, together with the *bounded convergence principle* (see e.g., Lemma III.2 in [4]), we can derive a convergence result:

Theorem 3.7. *If Δ^{i+1} is generated by subdividing Δ^i radially at an arbitrary point $\mathbf{u}^i \in \Delta^i_+$ for each i, then*

$$\liminf_{i \to \infty} |g^i(\mathbf{u}^i) - f(\mathbf{u}^i)| = 0.$$

Since $\boldsymbol{\omega}^i \in \Delta^i_+$, the convergence of the algorithm with the usual ω-subdivision, where $\mathbf{u}^i = \boldsymbol{\omega}^i$ for each i, is just a corollary of Theorem 3.7:

Corollary 3.8. *If Δ^{i+1} is generated by subdividing Δ^i radially at $\boldsymbol{\omega}^i$ for each i, then*

$$\liminf_{i \to \infty} |g^i(\boldsymbol{\omega}^i) - f(\boldsymbol{\omega}^i)| = 0.$$

Theorem 3.7 is, however, not sufficient to ensure the convergence to an optimal solution or an approximately optimal solution of (1). For this purpose, we need to further restrict the selection of $\mathbf{u}^i$ for each i. If we select $\mathbf{u}^i = \boldsymbol{\omega}^i \in \Delta^i_+$ for each i, as in Corollary 3.8, and update the incumbent $\mathbf{x}^*$ with $\boldsymbol{\omega}^i$ appropriately, then $\mathbf{x}^*$ converges to an *optimal σ-feasible solution* which satisfies

$$\mathbf{x}^* \in D(\sigma) \quad \text{and} \quad f(\mathbf{x}^*) \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in D$$

Although it is a rather satisfactory result from a theoretical viewpoint, ω-subdivision rule has a serious weakness, i.e., Δ^i is subdivided into up to $n+1$ subsimplices for each i. This means that we need to solve $n+1$ linear programs, in the worst case, to update the upper bound on f over $D \cap \Delta^i$. To overcome this, we develop a new subdivision rule called *ω-ksection*, which holds the number of subsimplices generated at a time below a prescribed number $k \leq n+1$.

4 Simplicial algorithm based on ω-ksection

In establishing the ω-ksection rule, we need to make an additional assumption:

(A3) The objective function f of (1) is strictly convex.

Therefore, if $\mathbf{x}^1 \neq \mathbf{x}^2$, we assume

$$f((1-\lambda)\mathbf{x}^1 + \lambda\mathbf{x}^2) < (1-\lambda)f(\mathbf{x}^1) + \lambda f(\mathbf{x}^2), \quad \forall \lambda \in (0,1).$$

Let

$$k^i = \min\{k, |J^i|\}, \quad i = 1,2,\ldots.$$

In the ω-ksection rule, Δ^i is subdivided radially at a point in a k^i-face of Δ^i_+. Before giving the detail, let us see how the sequence (4) behaves under the assumption (A3) if such a subdivision rule is applied.

Lemma 4.1. *If Δ^{i+1} is generated by subdividing Δ^i radially at a point $\mathbf{u}^i$ in a k^i face of Δ^i_+ for each i, then $\{\mathbf{v}^i_j \mid i = 1,2,\ldots\}$ has an accumulation point $\mathbf{v}^0_j$ for $j = 1,\ldots,n+1$. Among $\mathbf{v}^0_j$'s, there exists an accumulation point $\mathbf{u}^0$ of $\{\mathbf{u}^i \mid i = 1,2,\ldots\}$.*

On the basis of this observation, let us develop the procedure for ω-ksection (also see Figure 1). For each subset $K \subset J^i$ with $|K| = k^i$, let

$$\mathbf{u}_K = \sum_{j \in K} \frac{\lambda^i_j}{\Lambda_K} \mathbf{v}^i_j, \quad \Lambda_K = \sum_{j \in K} \lambda^i_j.$$

Also let

$$\rho_K = \min\{\|\mathbf{v}^k_j - \mathbf{u}_K\| \mid j \in K\}.$$

Among $\mathbf{u}_K$'s we select as $\mathbf{u}^i$ the one with the largest ρ_K, say

$$\rho_{K^i} = \max\{\rho_K \mid K \subset J^i, |K| = k^i\}.$$

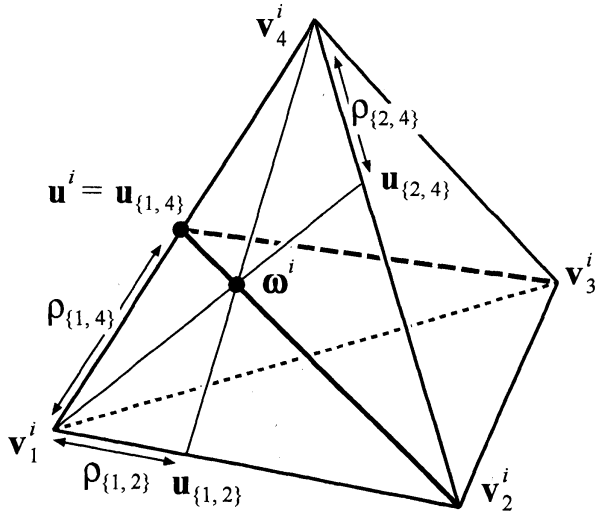

Figure 1: ω-Bisection when $J^i = \{1,2,4\}$.

For $\mathbf{u}^i = \mathbf{u}_{K^i}$, the children of Δ^i are given as

$$\Delta^{i_r} = \mathrm{conv}\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{r-1}^i, \mathbf{u}^i, \mathbf{v}_{r+1}^i, \ldots, \mathbf{v}_{n+1}^i\}, \quad r \in K^i,$$

some one of which is adopted as Δ^{i+1} in (4).

Lemma 4.2. *There exist a number $k^0 \in \{2,\ldots,k\}$ and a set $J^0 \subset \{1,\ldots,n+1\}$ such that $k^i = k^0$ and $J^i = J^0$ for infinitely many i. Moreover,*

(i) for each $K \subset J^0$ with $|K| = k^0$, the sequence $\{\mathbf{v}_K^i \mid i = 1,2,\ldots\}$ has an accumulation point $\mathbf{v}_K^0 \in \{\mathbf{v}_j^0 \mid j \in K\}$, and

(ii) for each $j \in J^0$, the sequence $\{\lambda_j^i \mid i = 1,2,\ldots\}$ has an accumulation point $\lambda_j^0 \geq 0$ such that $\sum_{j \in J^0} \lambda_j^0 = 1$.

Theorem 4.3. *If Δ^{i+1} is generated by subdividing Δ^i according to the ω-ksection rule for each i, then*

$$\liminf_{k \to \infty} |g^i(\boldsymbol{\omega}^i) - f(\boldsymbol{\omega}^i)| = 0. \tag{7}$$

Let us incorporate the ω-$k$section rule into the simplicial algorithm. For two prescribed tolerances $\varepsilon \geq 0$, $\sigma > 0$ and a number $k \in \{2,\ldots,n\}$, the algorithm can be described as follows:

algorithm simplicial_ω_$k$section

find an n-simplex $\Delta^1 = \mathrm{conv}\{\mathbf{v}_1^1, \ldots, \mathbf{v}_{n+1}^1\}$ enclosing D;
select a number $M > F^1/\sigma$ for $F^1 = \max\{f(\mathbf{v}_j^1) \mid j = 1,\ldots,n+1\}$;
define the convex envelope g^1 of f on Δ^1;
solve (P^1) of maximizing $g^1(\mathbf{x}) - M\tau$ subject to $\mathbf{x} \in D(\sigma) \cap \Delta^1$ and $\tau \geq 0$;
let $\alpha(\Delta^1) \leftarrow g^1(\boldsymbol{\omega}^1) - M\tau^1$ for an optimal solution $(\boldsymbol{\omega}^1, \tau^1)$ of (P^1);

$(\mathbf{x}^1, \beta^1) \leftarrow (\boldsymbol{\omega}^1, f(\boldsymbol{\omega}^1))$; $\mathscr{L} \leftarrow \{\Delta^1\}$; $i \leftarrow 1$;
while $\mathscr{L} \neq \emptyset$ do
  select a simplex $\Delta^i = \mathrm{conv}\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{n+1}^i\}$ with the largest value of α from $\mathscr{L}$;
  if $\alpha(\Delta^i) \leq (1+\varepsilon)\beta^i$ then
    $\mathscr{L} \leftarrow \emptyset$
  else
    let $\boldsymbol{\omega}^i$ denote the σ-feasible solution providing $\alpha(\Delta^i)$;
    identify the set $K^i \subset \{1, \ldots, n+1\}$ and the point $\mathbf{u}^i \in \Delta^i$ according to ω-*k*section;
    subdivide Δ^i into $\Delta^{ir} = \mathrm{conv}\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{r-1}^i, \mathbf{u}^i, \mathbf{v}_{r+1}^i, \ldots, \mathbf{v}_{n+1}^i\}$ for $r \in K^i$;
    for $r \in K^i$ do
      define the concave envelope g^{ir} of f on Δ^{ir};
      solve (P^{ir}) of maximizing $g^{ir}(\mathbf{x}) - M\tau$ subject to $\mathbf{x} \in D(\sigma) \cap \Delta^{ir}$ and $\tau \geq 0$;
      let $\alpha(\Delta^{ir}) \leftarrow g^{ir}(\boldsymbol{\omega}^{ir}) - M\tau^{ir}$ for an optimal solution $(\boldsymbol{\omega}^{ir}, \tau^{ir})$ of (P^{ir})
    end for
    $\mathbf{x}^{i+1} \leftarrow \arg\max\{f(\mathbf{x}) \mid \mathbf{x} = \mathbf{x}^i, \boldsymbol{\omega}^{ir}, r \in K^i\}$ and let $\beta^{i+1} \leftarrow f(\mathbf{x}^{i+1})$;
    $\mathscr{L} \leftarrow \mathscr{L} \cup \{\Delta^{ir} \mid r \in K^{ir}\} \setminus \{\Delta^i\}$; $i \leftarrow i+1$
  end if
end while
end.

Theorem 4.4. *Suppose $\varepsilon = 0$. If the algorithm simplicial_ω_ksection terminates after i iterations, then $\mathbf{x}^i$ is an optimal σ-feasible solution of (1). Even if not, every accumulation point of the sequence $\{\mathbf{x}^i \mid i = 1, 2, \ldots\}$ is an optimal σ-feasible solution.*

Let us introduce the following notion of solution to the convex maximization problem (1):

Definition 4.1. For two positive constants ε and σ, a point $\mathbf{x}^*$ is referred to as an (ε, σ)-*optimal solution* of (1) if

$$\mathbf{x}^* \in D(\sigma) \quad \text{and} \quad f(\mathbf{x}^*) \geq \frac{1}{1+\varepsilon} f(\mathbf{x}), \quad \forall \mathbf{x} \in D.$$

Corollary 4.5. *If $\varepsilon > 0$, then the algorithm simplicial_ω_ksection terminates after finitely many iterations and yields an (ε, σ)-optimal solution $\mathbf{x}^k$ of (1).*

5 Numerical results

Lastly, let us report numerical results of comparing the algorithm when using ω-bisection, ω-trisection, ω-quadsection, and the usual ω-subdivision. The test problem was a convex quadratic maximization problem of of the form

$$\left| \begin{array}{ll} \text{maximize} & f(\mathbf{x}) + 5.0\mathbf{dy} \\ \text{subject to} & \mathbf{Ax} + \mathbf{By} \leq \mathbf{b}, \quad (\mathbf{x}, \mathbf{y}) \geq \mathbf{0}, \end{array} \right. \tag{8}$$

Table 1: Average number of branchings and CPU time (in seconds).

$m \times n$	k	$q = 0.2n$		$q = 0.3n$		$q = 0.4n$	
		#	*time*	#	*time*	#	*time*
60×100	2	3.2	0.0819	11.2	0.2304	204.4	4.044
	3	3.0	0.0774	6.7	0.1526	272.2	4.459
	4	3.0	0.0797	6.8	0.1593	326.7	5.218
	$q+1$	3.0	0.0794	6.8	0.1594	296.2	4.528
60×150	2	3.6	0.1373	10.4	0.3861	19.6	0.8458
	3	3.6	0.1280	10.0	0.3319	24.5	0.8985
	4	3.9	0.1378	11.1	0.3556	25.0	0.8913
	$q+1$	3.9	0.1354	11.3	0.3401	21.2	0.7353
80×150	2	1.8	0.1458	5.4	0.3333	42.4	2.486
	3	1.8	0.1459	5.3	0.3161	112.2	5.556
	4	1.8	0.1492	5.5	0.3179	206.4	9.914
	$q+1$	1.8	0.1463	5.5	0.3117	134.6	6.118

where

$$f(\mathbf{x}) = 0.5\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}\mathbf{x}.$$

To make the feasible set bounded, the vector $\mathbf{b} \in \mathbb{R}^m$ was fixed to $[1,\ldots,1,n]^{\mathsf{T}}$ and all components of the last law of $\mathbf{A} \in \mathbb{R}^{m\times q}$ and $\mathbf{B} \in \mathbb{R}^{m\times(n-q)}$ were set to ones. Other entries of $\mathbf{A}$, together with components of $\mathbf{c} \in \mathbb{R}^q$ and $\mathbf{d} \in \mathbb{R}^{n-q}$, were generated randomly in the interval $[-0.5, 1.0]$, so that the percentages of zeros and negative numbers were about 20% and 10%, respectively. The matrix $\mathbf{Q} \in \mathbb{R}^{q\times q}$ was symmetric, tridiagonal, and the tridiagonal entries were random numbers in $[0.0, 1.0]$. Note that the objective function of (8) can be linearized simply by replacing the nonlinear part f with its concave envelope. Therefore, we may perform the branching procedure in the $\mathbf{x}$-space of dimension $q \leq n$, not in the whole space of dimension n. In accordance with this decomposition principle [4], we coded all computer programs in GNU Octave [2], and tested those on a single core of Intel Core i7 (3.20GHz). The values of the two tolerances ε and σ were fixed at 10^{-5} and 10^{-10}, respectively.

Table 1 shows the computational results. Three different sizes $(60,100)$, $(60,150)$ and $(80,150)$ were selected as (m,n), and q was set to 20%, 30% and 40% of the whole set of variables for each (m,n), i.e., the triple (m,n,q) was up to $(80,150,60)$. Ten instances were solved for each (m,n,q). The column labeled '#' lists the average number of iterations, and '*time*' lists the average computational time in seconds; and the rows $k = 2,3,4$, and $q+1$ represent the results of ω-bisection, ω-trisection, ω-quadsection, and the usual ω-subdivision, respectively.

We immediately see from Table 1 that both the number of iterations and the CPU time increase rather rapidly, as the percentage of nonlinear variables increases, for every subdivision rule. It is, however, a well-known characteristic common to deterministic global optimization algorithms. Instead, we should notice that there is a little difference in performance among the four subdivision rules. It is presumed that the number $|J^i|$ of $\mathbf{v}_j^i$'s spanning $\boldsymbol{\omega}^i$ is in general

rather small compared with q. As a result, the number k^i of children of Δ^i would be often defined by $|J^i|$. This finding needs to be confirmed in further studies. Anyway, we may conclude that the ω-$k$section rule performs at least as well as the usual ω-subdivision when $k = 2,3,4$.

References

[1] Falk, J.E., and R.M. Soland, "An algorithm for separable nonconvex programming problems", *Management Science* **15** (1969), 550-569.

[2] GNU Octave, http://www.gnu.org/software/octave/.

[3] Horst, R., "An algorithm for nonconvex programming problems", *Mathematical Programming* **10** (1976), 312-321.

[4] Horst, R. and H. Tuy, *Global Optimization: Deterministic Approaches*, 2nd ed., Springer-Verlag (Berlin, 1993).

[5] Jaumard, B., and C. Meyer, "A simplified convergence proof for the cone partitioning algorithm", *Journal of Global Optimization* **13** (1998), 407–416.

[6] Jaumard, B., and C. Meyer, "On the convergence of cone splitting algorithms with ω-subdivisions", *Journal of Optimization Theory and Applications* **110** (2001), 119–144.

[7] Kuno, T., and P.E.K. Buckland, "A convergent simplicial algorithm with ω-subdivision and ω-bisection strategies", *Journal of Global Optimization* **52** (2012), 371-390.

[8] Kuno, T., and T. Ishihama, "A convergent conical algorithm with ω-bisection for concave minimization", Technical Report CS-TR-13-20, University of Tsukuba (Ibaraki, 2013).

[9] Locatelli, M., "Finiteness of conical algorithm with ω-subdivisions", *Mathematical Programming* **85** (1999), 593-616.

[10] Locatelli, M., and U. Raber, "On convergence of the simplicial branch-and-bound algorithm based on ω-subdivisions", *Journal of Optimization Theory and Applications* **107** (2000), 69-79.

[11] Locatelli, M., and U. Raber, "Finiteness result for the simplicial branch-and-bound algorithm based on ω-subdivisions", *Journal of Optimization Theory and Applications* **107** (2000), 81-88.

[12] Soland, R.M., "Optimal facility location with concave costs", *Operations Research* **22** (1971), 373-382.

[13] Tuy, H., "Concave programming under linear constraints", *Soviet Mathematics* **5** (1964), 1437–1440.