Abstract approach to Dirac equation with time dependent potential

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1. Introduction and statement of the result

Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space and

$$H_1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3); (1+|x|^2)^{1/2}u \in L^2(\mathbb{R}^3)\}, \quad \Sigma^1(\mathbb{R}^3) := H^1(\mathbb{R}^3) \cap H_1(\mathbb{R}^3).$$

In this paper we consider the Cauchy problem for Dirac equation in $L^2(\mathbb{R}^3)^4$:

(DE) $$\qquad i\frac{\partial u}{\partial t} = H_0 u + V(t,x)u, \quad H_0 := \alpha \cdot D + m\beta$$

with $u(\cdot, 0) = u_0 \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$. Here $u : [0,T] \times \mathbb{R}^3 \to \mathbb{C}^4$ is an unknown function and $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, $\beta := \alpha_4$ are 4×4 Hermitian matrices satisfying

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}I,$$

where δ_{jk} is Kronecker symbol, I is the identity matrix and $D = i^{-1}(\partial_1, \partial_2, \partial_3)$, $m \geq 0$. $V(t,x)$ is a 4×4 Hermitian matrix-valued potential.

We shall show the existence of a unique classical solution under some conditions on potential V. It seems that this problem is associated with Kato and Yajima [5].

Now we want to state our main theorem.

Theorem 1.1. *Set $V = V_0(t,x) + q(t,x)I$, where Hermitian matrix-valued function V_0 and real valued function q satisfy the conditions:*

(V1) $$\qquad |V_0(t,x)| \leq a|x|^{-1} + b,$$

(V2) $$\qquad \left|\frac{\partial}{\partial t}V_0(t,x)\right| \leq (a|x|^{-1} + b)\sigma_1(t),$$

(q) $$\qquad (1+|x|^2)^{-1/2}q \in C(0,T; L^\infty(\mathbb{R}^3)) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}^3)),$$

where $|V|$ denotes the operator norm of V, a, b are nonnegative constants with $a < 1/2$ and $\sigma \in L^1(0,T)$. Then for every initial value $u_0 \in \Sigma^1(\mathbb{R}^3)^4$ problem (DE) has a unique (classical) solution

$$u(\cdot) \in C^1([0,T]; L^2(\mathbb{R}^3)^4) \cap C([0,T]; \Sigma^1(\mathbb{R}^3)^4).$$

Remark 1. Conditions (V1) and (V2) imply $(a|x|^{-1}+b)^{-1}V_0 \in W^{1,1}(0,T; L^\infty(\mathbb{R}^3)^{4\times 4})$.

To prove Theorem 1.1 we employ an abstract approach. This approach becomes a little simple than [10].

Let $\{A(t); t \in [0,T]\}$ be a family of closed linear operators in a complex Hilbert space X. Then we consider the abstract Cauchy problem for linear evolution equations of the form

(ACP)
$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t), & t \in [0,T], \\ u(0) = u_0. \end{cases}$$

Here the initial value u_0 is selected as follows.

To introduce our assumption on $\{A(t); t \in [0,T]\}$ we need one more family $\{S(t); t \in [0,T]\}$ of auxiliary operators in X.

Assumption on $\{S(t)\}$. The family $\{S(t)\}$ satisfies the following three conditions:

(S1) For every $t \in [0,T]$, $S(t)$ is positive selfadjoint in X and

$$(u, S(t)u) \geq \|u\|^2 \quad \text{for } u \in D(S(t)).$$

Let Y_t be the Hilbert space $D(S(t)^{1/2})$ with new inner product $(\cdot, \cdot)_{Y_t}$ and norm $\|\cdot\|_{Y_t}$ for $t \in [0,T]$ and $u, v \in Y_t$:

$$(u, v)_{Y_t} := (S(t)^{1/2}u, S(t)^{1/2}v), \qquad \|u\|_{Y_t} := (u, u)_{Y_t}^{1/2},$$

embedded continuously and densely in X. In particular, $Y := Y_0$ plays the roll of the space of initial values as in Theorem 1.3 (see below).

(S2) For $t \in [0,T]$, $Y_t = Y$, and $S(\cdot)^{1/2} \in C_*([0,T]; B(Y,X))$, where $B(Y,X)$ is the space of all bounded linear operators on Y to X, with norm $\|\cdot\|_{B(Y,X)}$, while the subscript $_*$ is used to refer the strong operator topology in $B(Y,X)$ (for this notation see Kato [4]).

(S3) There exists a nonnegative function $\sigma \in L^1(I)$ such that

$$\left| \|S(t)^{1/2}v\| - \|S(s)^{1/2}v\| \right| \leq \left| \int_s^t \sigma(r)\,dr \right| \max_{r \in \{s,t\}} \|S(r)^{1/2}v\|, \quad v \in Y, \ t, s \in [0,T].$$

In connection with the symbol $B(Y,X)$ we shall also use the abbreviation: $B(X) := B(X,X)$, $B(Y) := B(Y,Y)$.

Let $\{S(t)\}$ be as defined above. Then we may introduce the following

Assumption on $\{A(t)\}$. The family $\{A(t)\}$ satisfies the following four conditions:

(A1) There exists a constant $\alpha \geq 0$ such that

$$|\text{Re } (A(t)v,v)| \leq \alpha\|v\|^2, \quad v \in D(A(t)), \ t \in [0,T].$$

(**A2**) $Y \subset D(A(t))$, $t \in [0, T]$.

(**A3**) There exists a constant $\beta \geq \alpha$ such that

$$| \operatorname{Re} \, (A(t)u, S(t)u)| \leq \beta \|S(t)^{1/2} u\|^2, \quad u \in D(S(t)) \subset Y, \ t \in [0, T].$$

(**A4**) $A(\cdot) \in C_*([0, T]; B(Y, X))$.

Theorem 1.2. *Suppose that* Assumptions on $\{A(t)\}$ *and* $\{S(t)\}$ *are satisfied. Then there exists a unique evolution operator* $\{U(t, s); (t, s) \in \Delta_+\}$ *for* (ACP)*, where* $\Delta_+ := \{(t, s); 0 \leq s \leq t \leq T\}$*, having the following properties*:

(**i**) $U(\cdot, \cdot)$ *is strongly continuous on* Δ_+ *to* $B(X)$*, with*

$$\|U(t, s)\|_{B(X)} \leq e^{\alpha(t-s)}, \quad (t, s) \in \Delta_+.$$

(**ii**) $U(t, r)U(r, s) = U(t, s)$ *on* Δ_+ *and* $U(s, s) = 1$ *(the identity).*

(**iii**) $U(t, s)Y \subset Y$ *and* $U(\cdot, \cdot)$ *is strongly continuous on* Δ_+ *to* $B(Y)$*, with*

$$(1.1) \qquad \|U(t, s)\|_{B(Y_s, Y_t)} \leq \exp\left(\int_s^t \widetilde{\sigma}(r) \, dr\right), \quad (t, s) \in \Delta_+,$$

$$(1.2) \qquad \|U(t, s)\|_{B(Y)} \leq \exp\left(\int_0^s \sigma(r) \, dr\right) \exp\left(2 \int_s^t \widetilde{\sigma}(r) \, dr\right), \quad (t, s) \in \Delta_+,$$

where $\widetilde{\sigma}(t) := \beta + \sigma(t)$.

Furthermore, let $v \in Y$. *Then* $U(\cdot, \cdot)v \in C^1(\Delta_+; X)$*, with*

(**iv**) $(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v$, $(t, s) \in \Delta_+$, *and*

(**v**) $(\partial/\partial s)U(t, s)v = U(t, s)A(s)v$, $(t, s) \in \Delta_+$.

The equation in (ACP) is naturally interpreted if the solution has an additional property $u(\cdot) \in C([0, T]; Y)$. In fact, it is guaranteed by condition (**A2**) that $u(t) \in Y \subset D(A(t))$ for every $t \in [0, T]$.

Theorem 1.3. *Let* $\{U(t, s)\}$ *be the evolution operator for* (ACP) *as in Theorem 1.2 above. For* $u_0 \in Y$ *and* $f(\cdot) \in C([0, T]; X) \cap L^1(0, T; Y)$ *define* $u(\cdot)$ *as*

$$u(t) := U(t, 0)u_0 + \int_0^t U(t, s)f(s) \, ds.$$

Then (ACP) *has a unique (classical) solution*

$$u(\cdot) \in C^1([0, T]; X) \cap C([0, T]; Y).$$

Remark 2. The assertion of Theorem 1.3 is same as in [**10**]. However, we can simplify assumption on $\{S(t)\}$ and the proof of Theorem 1.2 a little than [**10**].

2. Preliminaries

Let X be a (complex) Hilbert space. In this section we prepare some useful lemmas.

2.1. Lemmas on time-independent operators

In this subsection we consider a pair $\{A, S\}$ of closed linear operators in X. Let A be *quasi-accretive* in the sense of Kato [1, Section V.3.10]:

$$(2.1) \qquad \operatorname{Re}(Av, v) \geq -\alpha \|v\|^2, \quad v \in D(A),$$

for some constant $\alpha \geq 0$; in other words, $\alpha + A$ is accretive. Let S be a positive-definite selfadjoint operator in X, with $D(S) \subset D(A)$. Now we can state a condition connecting A and S: assume that there exist a constant $\beta \geq 0$ such that

$$(2.2) \qquad \operatorname{Re}(Au, Su) \geq -\beta(u, Su), \quad u \in D(S) \subset D(A).$$

Lemma 2.1. *Let A and S be as in* (2.1) *and* (2.2), *respectively. Then*

(a) $\alpha + A$ *is m-accretive in* X; (b) $D(S)$ *is a core for* A.

(a) was first proved by Okazawa [6], while (b) was later noted by Kato [3] (for a complete proof see Tanabe [13, Section 7.7] or Ouhabaz [11, Section 1.3.3]).

Given the pair $\{A, S\}$ as in Lemma 2.1, let $\{A_n; n > \alpha\}$ and $\{S_\varepsilon; \varepsilon > 0\}$ be Yosida approximations of A and S, respectively:

$$(2.3) \qquad A_n := AJ_n = n(1 - J_n), \qquad J_n := (1 + n^{-1}A)^{-1},$$

$$(2.4) \qquad S_\varepsilon := SJ_\varepsilon = \varepsilon^{-1}(1 - J_\varepsilon), \qquad J_\varepsilon := (1 + \varepsilon S)^{-1}.$$

Then the pair $\{A_n, S_\varepsilon\}$ satisfies conditions in Lemma 2.1 with α and β replaced with $\alpha(1 - n^{-1}\alpha)^{-1}$ and $\beta(1 - n^{-1}\beta)^{-1}$, respectively.

(2.1) and (2.2) are invariant under taking their Yosida approximation.

Lemma 2.2 (Okazawa [7, Lemma 2.2]). *Given A as in* Lemma 2.1, *let $\{A_n\}$ and $\{J_n\}$ be as in* (2.3). *Then $\|J_n\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$ $(n > \alpha)$ and $\|A_n\|_{B(X)} \leq n$ $(n \geq 2\alpha)$, with*

$$(2.1)_n \qquad \operatorname{Re}(A_n w, w) \geq -\alpha(1 - n^{-1}\alpha)^{-1}\|w\|^2, \quad w \in X, \quad n > \alpha.$$

Lemma 2.3 ([9, Lemmas 2.7]). *Given the pair $\{A, S\}$ as in* Lemma 2.1, *let $\{A_n\}$ and $\{S_\varepsilon\}$ be as in* (2.3) *and* (2.4). *Then*

$$(2.2)_{n,\varepsilon} \qquad \operatorname{Re}(A_n w, S_\varepsilon w) \geq -\beta(1 - n^{-1}\beta)^{-1}\|S_\varepsilon^{1/2} w\|^2, \quad w \in X, \quad n > \beta \geq \alpha.$$

2.2. Lemmas on time-dependent operators

Since we need conditions (**A1**), (**A3**) and (**S3**) as a whole only in the last step of the proof of Theorem 1.2, we may introduce a set of weaker conditions:

(**A1**)$_+$ There exists a constant $\alpha \geq 0$ such that

$$\text{Re } (A(t)v, v) \geq -\alpha \|v\|^2, \quad v \in D(A(t)), \ t \in [0, T].$$

(**A3**)$_+$ There exists a constant $\beta \geq \alpha$ such that

$$\text{Re } (A(t)v, S(t)v) \geq -\beta \|S(t)^{1/2}v\|^2, \quad v \in D(S(t)) \subset Y, \ t \in [0, T].$$

(**S3**)$_+$ There exists a nonnegative function $\sigma \in L^1(0, T)$ such that

$$\|S(t)^{1/2}u\| - \|S(s)^{1/2}u\| \leq \left(\int_s^t \sigma(r) \, dr \right) \|S(s)^{1/2}u\|, \quad u \in Y, \ 0 \leq s \leq t \leq T.$$

By virtue of Lemma 2.1, it follows from conditions (**A1**)$_+$, (**A2**) and (**A3**)$_+$ that $\alpha + A(t)$ is m-accretive in X. We can obtain following

Proposition 2.4. *Let* $\{S(t)\}$ *be a family of selfadjoint operators in* X, $\{A(t)\}$ *a family of closed linear quasi-accretive operators in* X, *satisfying conditions* (**S1**), (**S2**), (**S3**)$_+$ *for* $\{S(t)\}$, *and conditions* (**A1**)$_+$, (**A2**), (**A3**)$_+$, (**A4**) *for* $\{A(t)\}$. *Then there exist Yosida approximations* $\{A_n(t); n > \alpha\}$ *and* $\{S_\varepsilon(t); \varepsilon > 0\}$ *of* $\{A(t)\}$ *and* $\{S(t)\}$, *respectively:*

$$(2.5) \qquad A_n(t) := A(t)J_n(t) = n(1 - J_n(t)), \qquad J_n(t) := (1 + n^{-1}A(t))^{-1},$$

$$(2.6) \qquad S_\varepsilon(t) := S(t)J_\varepsilon(t) = \varepsilon^{-1}(1 - J_\varepsilon(t)), \qquad J_\varepsilon(t) := (1 + \varepsilon S(t))^{-1}.$$

Put

$$(2.7) \qquad \alpha_n := \alpha(1 - n^{-1}\alpha)^{-1}, \qquad \beta_n := \beta(1 - n^{-1}\beta)^{-1} \quad (n > \beta \geq \alpha).$$

Then the pair of $\{A_n(t)\}$ *and* $\{S_\varepsilon(t)\}$ *satisfies* Assumptions *on* $\{A(t)\}$ *and* $\{S(t)\}$ *with* Y, α *and* β *replaced with* X, α_n *and* β_n, *respectively:*

(**S$_\varepsilon$1**) $D(S_\varepsilon(t)) = X$, $t \in [0, T]$.

(**S$_\varepsilon$2**) $S_\varepsilon(\cdot) \in C_*([0, T]; B(X))$.

(**S$_\varepsilon$3**)$_+$ *For* $w \in X$,

$$\|S_\varepsilon(t)^{1/2}w\|^2 - \|S_\varepsilon(s)^{1/2}w\|^2 \leq \left[\left(1 + \int_s^t \sigma(r) \, dr \right)^2 - 1 \right] \|S_\varepsilon(s)^{1/2}w\|^2, \quad 0 \leq s \leq t \leq T.$$

(**A$_n$1**)$_+$ *For* $n > \alpha$, $t \in [0, T]$, $\|J_n(t)\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$ *and*

$$\text{Re } (A_n(t)w, w) \geq -\alpha_n \|w\|^2, \quad w \in X.$$

(**A$_n$2**) $D(A_n(t)) = X$, $t \in [0, T]$.

(**A$_n$3**)$_+$ *For* $n > \beta$, $t \in [0, T]$, $\text{Re } (A_n(t)w, S_\varepsilon(t)w) \geq -\beta_n \|S_\varepsilon(t)^{1/2}w\|^2$, $w \in X$.

(**A$_n$4**) $A_n(\cdot) \in C_*([0, T]; B(X))$ $(n > \beta)$, *with* $\|A_n(t)\|_{B(X)} \leq n$ $(n \geq 2\alpha, \ t \in [0, T])$.

($S_\varepsilon 1$) and ($A_n 2$) are trivial. ($A_n 1$)$_+$ and ($A_n 3$)$_+$ follow from Lemmas 2.2 and 2.3, respectively. ($A_n 4$) and ($S_\varepsilon 2$) were proved in [**7**, Lemma 3.1(a), (b)] and [**10**, Proposition 2.5], respectively. We show only the proof of ($S_\varepsilon 3$)$_+$. The first half of the proof is same as in [**10**, Proposition 2.5].

Proof of ($S_\varepsilon 3$)$_+$. First we remind of the definition of $S_\varepsilon(\cdot)$:

$$w = J_\varepsilon(t)w + \varepsilon S_\varepsilon(t)w, \quad w \in X.$$

Then the symmetry and positivity of $J_\varepsilon(\cdot)$ yield that

(2.8) $\quad (w, S_\varepsilon(t)w - S_\varepsilon(s)w)$

$$= \big(w, S_\varepsilon(t)(J_\varepsilon(s) + \varepsilon S_\varepsilon(s))w - (J_\varepsilon(t) + \varepsilon S_\varepsilon(t))S_\varepsilon(s)w\big)$$

$$= \big(J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big) + \big((J_\varepsilon(t) - J_\varepsilon(s))w, (S(t) - S(s))J_\varepsilon(s)w\big)$$

$$= \big(J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big)$$

$$- \varepsilon\big(J_\varepsilon(t)(S(t) - S(s))J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big)$$

$$\leq \big(J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big)$$

$$= \|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2.$$

On the other hand, it follows from condition ($S3$)$_+$ that

(2.9) $$\|S(t)^{1/2}J_\varepsilon(s)w\| \leq \Big(1 + \int_s^t \sigma(r)\,dr\Big)\|S_\varepsilon(s)^{1/2}w\|,$$

where we have used $\|J_\varepsilon(s)^{1/2}\|_{B(X)} \leq 1$ ($\varepsilon > 0$). We see from (2.9) that

(2.10) $$\|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2 \leq \Big[\Big(1 + \int_s^t \sigma(r)\,dr\Big)^2 - 1\Big]\|S_\varepsilon(s)^{1/2}w\|^2.$$

Combining (2.10) with (2.8), we obtain ($S_\varepsilon 3$)$_+$. $\qquad\square$

The next proposition is essential in the proof of Theorem 1.2 in which it is required to differentiate $S_\varepsilon(\cdot)$. For that purpose $\{S_\varepsilon(t); \ t \in [0, T]\}$ is replaced with a new family

(2.11) $$S_\varepsilon^h(t) := \frac{1}{h}\int_t^{t+h} S_\varepsilon(s)\,ds, \quad h > 0, \ t \in [0, T],$$

where we define as $S_\varepsilon(s) := S_\varepsilon(T)$ ($s > T$). Then, in view of ($S_\varepsilon 2$), we have

(2.12) $$\frac{d}{dt}S_\varepsilon^h(t)w = \frac{1}{h}(S_\varepsilon(t + h)w - S_\varepsilon(t)w),$$

(2.13) $$\text{s-}\lim_{h\downarrow 0} S_\varepsilon^h(t)w = S_\varepsilon(t)w, \quad w \in X.$$

Proposition 2.5. *Assume that* $\{S(t)\}$ *satisfies conditions* **(S1)**, **(S2)** *and* **(S3)**$_+$. *Let* $\{S_\varepsilon^h(t); \; t \in [0,T]\}$ *be as in* (2.11). *Then* $S_\varepsilon^h(\cdot) \in C_*^1([0,T]; B(X))$, *with for* $w \in X$,

$$(\mathbf{S}_\varepsilon^h\mathbf{3})_+ \qquad \left(w, \frac{d}{ds}S_\varepsilon^h(s)w\right) \leq \frac{1}{h}\left[\left(1 + \int_s^{s+h} \sigma(r)\,dr\right)^2 - 1\right]\|S_\varepsilon(s)^{1/2}w\|^2, \qquad s \in [0,T],$$

where one defines as $\sigma(s) := 0$ $(s > T)$. *Moreover, for* $w \in X$ *one has*

$$(2.14) \qquad \limsup_{h \downarrow 0} \int_{t_0}^t \left(w, \frac{d}{ds}S_\varepsilon^h(s)w\right)ds \leq 2\int_{t_0}^t \sigma(s)\|S_\varepsilon(s)^{1/2}w\|^2\,ds, \qquad t_0 \leq t.$$

With regard to the proof of Proposition 2.6, $(\mathbf{S}_\varepsilon^h\mathbf{3})_+$ follows from (2.12) and $(\mathbf{S}_\varepsilon\mathbf{3})_+$. On the other hand, (2.14) is a consequence of following:

Lemma 2.6. *Let* $\varphi \in L^1(0,T)$ *and* $\psi \in L^\infty(0,T)$. *Then*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^t \left[\left(1 + \int_s^{s+h} \varphi(r)\,dr\right)^2 - 1\right]\psi(s)\,ds = 2\int_0^t \varphi(s)\psi(s)\,ds, \qquad 0 \leq t \leq T,$$

where one define as $\varphi(s) := 0$ $(s > T)$.

Proof. We also define that $\varphi(s) := 0$ $(s < 0)$. Since

$$\left(1 + \int_s^{s+h} \varphi(r)\,dr\right)^2 - 1 = \left(\int_s^{s+h} \varphi(r)\,dr\right)\left(2 + \int_s^{s+h} \varphi(\tau)\,d\tau\right),$$

by integrating by parts, we obtain

$$\int_0^t \left[\left(1 + \int_s^{s+h} \varphi(r)\,dr\right)^2 - 1\right]\psi(s)\,ds$$

$$= \int_0^{t+h} \varphi(r)\left[\int_{r-h}^r \left(2 + \int_s^{s+h} \varphi(\tau)\,d\tau\right)\widetilde{\psi}(s)\,ds\right]dr$$

$$= \int_0^{t+h} \varphi(r)\left[\int_{r-h}^{r+h} \left(2\overline{\psi}(s) + \varphi(\tau)\int_{\tau-h}^\tau \overline{\psi}(s)\,ds\right)d\tau\right]dr,$$

where $\widetilde{\psi}, \overline{\psi} \in L^1(\mathbb{R})$ are defined as

$$\widetilde{\psi}(s) := \begin{cases} \psi(s), & 0 < s < t, \\ 0, & \text{otherwise}, \end{cases} \qquad \overline{\psi}(s) := \begin{cases} \widetilde{\psi}(s), & r - h < s < r, \\ 0, & \text{otherwise}. \end{cases}$$

Set

$$\Phi_h(r) := \frac{1}{h}\varphi(r)\left[\int_{r-h}^{r+h} \left(2\overline{\psi}(s) + \varphi(\tau)\int_{\tau-h}^\tau \overline{\psi}(s)\,ds\right)d\tau\right].$$

Then we obtain $\lim_{h \downarrow 0} \Phi_h(r) = 2\varphi(r)\widetilde{\psi}(r)$ (a.a. $r \in (0,t)$) and

$$|\Phi_h(r)| \leq |\varphi(r)|\left(2 + \int_{r-h}^{r+h} |\varphi(\tau)|\,d\tau\right)\|\psi\|_{L^\infty(0,T)}$$

$$\leq |\varphi(r)|\left(2 + \|\varphi\|_{L^1(0,T)}\right)\|\psi\|_{L^\infty(0,T)} \in L^1(0,T).$$

Thus the assertion follows from the Lebesgue convergence theorem. $\qquad\square$

3. Construction of evolution operators

In this section we shall prove Theorem 1.2. The major part of the assertions in Theorem 1.2 is contained in the following

Theorem 3.1. *Let $\{S(t)\}$ be a family of selfadjoint operators in X, $\{A(t)\}$ a family of closed linear quasi-accretive operators in X, satisfying conditions* **(S1)**, **(S2)**, **(S3)**$_+$ *for $\{S(t)\}$, and conditions* **(A1)**$_+$, **(A2)**, **(A3)**$_+$, **(A4)** *for $\{A(t)\}$. Then there exists a unique two-parameter family $\{U(t,s); (t,s) \in \Delta_+\}$ in $B(X)$, having the properties:*

(iii)$_w$ $U(t,s)Y \subset Y$, *with* (1.1);

(iv)$_w$ $U(\cdot,\cdot)v \in W^{1,1}(\Delta_+; X)$, *with*

$$(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v, \quad \text{a.a. } t \in (s,T), \ v \in Y;$$

in addition to properties **(i)**, **(ii)** *and* **(v)** *in Theorem 1.2.*

Therefore the first purpose of this section is to prove Theorem 3.1. To replace **(iii)$_w$** and **(iv)$_w$** with **(iii)** and **(iv)** in the final step we need the whole conditions **(S1)**–**(S3)** and **(A1)**–**(A4)** in Theorem 1.2.

To prove Theorem 3.1 we consider the approximate problem:

$$(\text{ACP})_n \qquad \begin{cases} (d/dt)u_n(t) + A_n(t)u_n(t) = 0, & t \in [s,T], \\ u_n(s) = w \in X, \end{cases}$$

where $\{A_n(t); n > \alpha\}$ is the Yosida approximation as in (2.5).

According to Pazy [**12**, Theorems 5.1.1 and 5.1.2] (in which $A_n(\cdot) \in C([0,T]; B(X))$ is assumed, however, it can be replaced by $A_n(\cdot) \in C_*([0,T]; B(X))$ (condition **(A$_n$4)**) with appropriate modification of the conclusion), we obtain the following

Proposition 3.2. *Let $s \in [0,T)$ and $n > 2\beta$, where β is defined in* **(A3)**$_+$. *Then the approximate problem* $(\text{ACP})_n$ *has a unique classical solution $u_n(\cdot) \in C^1([s,T]; X)$. Accordingly, there exists a unique evolution operator $\{U_n(t,s); (t,s) \in \Delta_+\}$ for* $(\text{ACP})_n$ *having the following properties:*

(i)$_n$ $U_n(\cdot,\cdot)$ *is strongly continuous on Δ_+ to $B(X)$, with*

$$\|U_n(t,s)\|_{B(X)} \leq e^{n(t-s)}, \quad (t,s) \in \Delta_+.$$

(ii)$_n$ $U_n(t,r)U_n(r,s) = U_n(t,s)$ *on Δ_+ and $U_n(s,s) = 1$ (the identity).*

(iii)$_n$ $U_n(t,s)$ *is uniformly continuous on Δ_+.*

(iv)$_n$ $(\partial/\partial t)U_n(t,s)w = -A_n(t)U_n(t,s)w, \quad w \in X, \ (t,s) \in \Delta_+.$

(v)$_n$ $(\partial/\partial s)U_n(t,s)v = U_n(t,s)A_n(s)v, \quad w \in X, \ (t,s) \in \Delta_+.$

For the limiting procedure for $\{U_n(t,s)\}$ we need several estimates of $\{U_n(t,s)\}$ which are independent of n.

Lemma 3.3. *Let* $\{U_n(t,s)\}$ *be as in Proposition 3.2,* α_n, β_n *as in (2.7) and* σ *as in condition* $(\mathbf{S3})_+$. *If* $n > 2\beta$, *then for* $(t,s) \in \Delta_+$,

(a) $\|U_n(t,s)\|_{B(X)} \le e^{\alpha_n(t-s)}$.

(b) $U_n(t,s)Y \subset Y$ *and with*

$$\|S(t)^{1/2}U_n(t,s)v\| \le e^{\beta_n(t-s)}\exp\left(\int_s^t \sigma(r)\,dr\right)\|S(s)^{1/2}v\|, \quad v \in Y.$$

(c) *There exists a constant* $c \ge 0$ *such that*

(3.1) $$\|A(t)v\| \le c\|S(t)^{1/2}v\|, \quad v \in Y,$$

and hence

(3.2) $$\|A_n(t)U_n(t,s)v\| \le M\|v\|_Y, \quad v \in Y,$$

where $M := 2c\exp(2\beta T + \|\sigma\|_{L^1(0,T)})$.

Proof. (a) We see from property $(\mathbf{iv})_n$ and $(\mathbf{A}_n\mathbf{1})_+$ that for $w \in X$,

$$(\partial/\partial r)\|U_n(r,s)w\|^2 = -2\,\mathrm{Re}\,(A_n(r)U_n(r,s)w, U_n(r,s)w)$$
$$\le 2\alpha_n\|U_n(r,s)w\|^2, \quad s \le r \le t.$$

Integrating this inequality, we obtain the assertion.

(b) Let $\{S_\varepsilon(t)\}$ and $\{S_\varepsilon^h(t)\}$ be as in (2.6) and (2.11). Since $S_\varepsilon^h(t)^{1/2}$ is bounded and symmetric on X, it follows from property $(\mathbf{iv})_n$ that for $v \in Y$,

(3.3) $$(\partial/\partial r)\|S_\varepsilon^h(r)^{1/2}U_n(r,s)v\|^2 = -2\,\mathrm{Re}\,\left(A_n(r)U_n(r,s)v, S_\varepsilon^h(r)U_n(r,s)v\right)$$
$$+ \left(U_n(r,s)v, ((d/dr)S_\varepsilon^h(r))U_n(r,s)v\right).$$

Integrating this equality on $[s,t]$, we see from $(\mathbf{S}_\varepsilon^h\mathbf{3})_+$ that

$$\|S_\varepsilon^h(t)^{1/2}U_n(t,s)v\|^2 \le \|S_\varepsilon^h(s)^{1/2}v\|^2 - 2\int_s^t \mathrm{Re}\,\left(A_n(r)U_n(r,s)v, S_\varepsilon^h(r)U_n(r,s)v\right)dr$$
$$+ \int_s^t \frac{1}{h}\left[\left(1 + \int_r^{r+h}\sigma(\tau)\,d\tau\right)^2 - 1\right]\|S_\varepsilon(r)^{1/2}U_n(r,s)v\|^2\,dr.$$

Passing to the limit as $h \downarrow 0$, we see from (2.13), (2.14) and $(\mathbf{A}_n\mathbf{3})_+$ that

$$\|S_\varepsilon(t)^{1/2}U_n(t,s)v\|^2 \le \|S_\varepsilon(s)^{1/2}v\|^2 - 2\int_s^t \mathrm{Re}\,\left(A_n(r)U_n(r,s)v, S_\varepsilon(r)U_n(r,s)v\right)dr$$
$$+ 2\int_s^t \sigma(r)\|S_\varepsilon(r)^{1/2}U_n(r,s)v\|^2\,dr$$
$$\le \|S(s)^{1/2}v\|^2 + 2\int_s^t \left(\beta_n + K\sigma(r)\right)\|S_\varepsilon(r)^{1/2}U_n(r,s)v\|^2\,dr.$$

Applying the Gronwall lemma, we obtain

$$\|S_\varepsilon(t)^{1/2}U_n(t,s)v\|^2 \le \exp\Big(2\int_s^t \big(\beta_n + \sigma(r)\big)\,dr\Big)\|S(s)^{1/2}v\|^2.$$

Passing to the limit as $\varepsilon \to 0$, we obtain the assertion.

(c) The existence of such a constant c is guaranteed by conditions (**A2**) and (**A4**). Therefore we see from $(\mathbf{A}_n\mathbf{1})_+$ and (3.1) that

$$\|A_n(t)v\| \le (1 - n^{-1}\alpha)^{-1}\|A(t)v\| \le 2c\|S(t)^{1/2}v\|, \quad v \in Y.$$

Noting that

$$(3.4) \qquad \|S(s)^{1/2}v\| \le \Big(1 + \int_0^s \sigma(r)\,dr\Big)\|S(0)^{1/2}v\| \le \exp\Big(\int_0^s \sigma(r)\,dr\Big)\|v\|_Y,$$

we obtain (3.2) as a consequence of (b). $\qquad\square$

Next two lemmas guarantee the existence and uniqueness of evolution operator.

Lemma 3.4 ([10, Lemma 3.4]). *Let* $\{U_n(t,s)\}$ *be the evolution operator for* $(\mathrm{ACP})_n$. *Then there exists a new family* $\{U(t,s); (t,s) \in \Delta_+\}$ *such that* $U(t,s) := \mathrm{s\text{-}lim}_{n\to\infty} U_n(t,s)$, *where the convergence is uniform on* Δ_+, *and has properties* (**i**) *and* (**ii**) *in* Theorem 1.2, *with*

$$\|U(t,s)v - U_n(t,s)v\| \le \sqrt{\frac{2T}{n - 2\alpha}}Me^{2\alpha T}\|v\|_Y, \quad v \in Y,\ n > 2\beta.$$

Lemma 3.5 ([10, Lemma 3.4]). *Let* $\{U(t,s)\}$ *be as in* Lemma 3.4 *and* $v \in Y$. *Then*

(a) $U(t,s)Y \subset Y$ *and*

$$(3.5) \qquad \|S(t)^{1/2}U(t,s)v\| \le \exp\Big(\int_s^t \widetilde{\sigma}(r)\,dr\Big)\|S(s)^{1/2}v\|, \quad (t,s) \in \Delta_+,\ v \in Y,$$

where $\widetilde{\sigma}$ *is defined as in* Theorem 1.2 (**iii**).

(b) $U(\cdot,\cdot)v \in W^{1,\infty}(\Delta_+; X)$, *with properties* (**iv**)$_\mathbf{w}$ *and* (**v**).

(c) $\{U(t,s); (t,s) \in \Delta_+\}$ *is unique:* $U(t,s) \equiv V(t,s)$ *on* Δ_+ *if* $\{V(t,s); (t,s) \in \Delta_+\}$ *is another family in* $B(X)$ *with properties* (**i**), (**ii**) *and* (**v**).

This completes the proof of Theorem 3.1. The purpose of the second half in this section is to prove properties (**iii**) and (**iv**) in Theorem 1.2.

Lemma 3.6. *Let* $\{U(t,s)\}$ *be as in* Lemma 3.4 *and* $v \in Y$. *Assume that* $S(t)$ *satisfies conditions* (**S1**)–(**S3**). *Then*

(a) $S(t)^{1/2}U(t,s)v$ *is weakly continuous on* Δ_+.

(b) $S(t)^{1/2}U(t,s)v \to S(t_0)^{1/2}v$ *as* $(t,s) \to (t_0,t_0)$.

(c) *For* $t \in (0,T]$, $S(t)^{1/2}U(t,\cdot)v \in C([0,t]; X)$.

Proof. (a) Since $S_\varepsilon(\cdot)^{1/2} \in C_*([0,T]; B(X))$ (see ($\mathbf{S}_\varepsilon\mathbf{2}$)), $S_\varepsilon(t)^{1/2}U(t,s)v$ is continuous on Δ_+. For $w \in Y$, we see that

$$
\begin{aligned}
&\left|(S(t)^{1/2}U(t,s)v, w) - (S_\varepsilon(t)^{1/2}U(t,s)v, w)\right| \\
&= \left|(S(t)^{1/2}U(t,s)v, [1 - (1 + \varepsilon S(t))^{-1/2}]w)\right| \\
&\leq \exp(\|\widetilde{\sigma}\|_{L^1(0,T)}) \|S(0)^{1/2}v\| \cdot \left\|w - (1 + \varepsilon S(t))^{-1/2}w\right\| \\
&\leq \varepsilon^{1/2} \exp(\|\widetilde{\sigma}\|_{L^1(0,T)}) \|v\|_Y \|w\|_Y,
\end{aligned}
$$

where we have used (1.2) and (3.4). Passing to $\varepsilon \downarrow 0$, since Y is dense in X, we can conclude (a).

(b) It is follows from (a) that

$$
\|S(t_0)^{1/2}v\| \leq \liminf_{(t,s) \to (t_0,t_0)} \|S(t)^{1/2}U(t,s)v\|.
$$

On the other hand, it follows from (3.5) that

$$
\limsup_{(t,s) \to (t_0,t_0)} \|S(t)^{1/2}U(t,s)v\| \leq \|S(t_0)^{1/2}v\|.
$$

Combining these estimates and (a), we obtain the assertion.

(c) Let $r \in [s,t]$. Then we see from (3.5) that

$$
\|S(t)^{1/2}U(t,r)v - S(t)^{1/2}U(t,s)v\| \leq \exp(\|\widetilde{\sigma}\|_{L^1(0,T)}) \|S(r)^{1/2}(1 - U(r,s))v\|.
$$

Therefore the assertion follows from (b). $\qquad\square$

Now we are in a position to prove (**iii**) and (**iv**) of Theorem 1.2. As is easily seen, the proof of (**iii**) and (**iv**) is based on Lemmas 3.7 and 3.8 below. In other words, we need the whole assumptions on $\{A(t)\}$ and $\{S(t)\}$.

Lemma 3.7 ([7, Lemma 3.9]). *Let $\{A(t)\}$ and $\{S(t)\}$ be as in Theorem 1.2. Assume that conditions* (**A1**), (**A2**) *and* (**A3**) *are satisfied. Then*

$$(3.6) \qquad |\operatorname{Re}(A(t)v, S_\varepsilon(t)v)| \leq \beta \|S_\varepsilon(t)^{1/2}v\|^2, \quad v \in Y, \quad t \in [0,T].$$

Under conditions (**S1**)–(**S3**) Proposition 2.5 is modified as follows.

Lemma 3.8. *Assume that $\{S(t)\}$ satisfies conditions* (**S1**)–(**S3**). *Let $\{S_\varepsilon^h(t)\}$ be as in Proposition 2.5. Then condition* ($\mathbf{S}_\varepsilon^h\mathbf{3}$)$_+$ *is replaced with*

$$(\mathbf{S}_\varepsilon^h\mathbf{3}) \qquad \left|\left(w, \frac{d}{ds}S_\varepsilon^h(s)w\right)\right| \leq \frac{1}{h}\left[\left(1 + \int_s^{s+h} \sigma(r)\,dr\right)^2 - 1\right] \max_{r \in \{s, s+h\}} \|S_\varepsilon(r)^{1/2}w\|^2$$

for $w \in X$, $h > 0$ and $s \in [0,T]$. Consequently, for t, $t_0 \in [0,T]$ one has

$$(3.7) \qquad \limsup_{h \downarrow 0} \left|\int_{t_0}^t \left|\left(w, \frac{d}{ds}S_\varepsilon^h(s)w\right)\right| ds\right| \leq 2\left|\int_{t_0}^t \sigma(s)\|S_\varepsilon(s)^{1/2}w\|^2\,ds\right|.$$

The next lemma completes the proof of Theorem 1.2.

Lemma 3.9. *For $\{A(t)\}$ and $\{S(t)\}$ as in Theorem 1.2 let $\{U(t,s)\}$ be as defined in Lemma 3.4 and $v \in Y$. Then*

(a) $U(t, \cdot)v \in C([0,t]; Y)$, $t \in (0,T]$.

(b) $S(\cdot)^{1/2}U(\cdot, s)v \in C([s,T]; X)$, $s \in [0,T)$.

(c) $U(\cdot, s)v \in C([s,T]; Y)$, $s \in [0,T)$.

(d) $U(\cdot, \cdot)v \in C(\Delta_+; Y)$; *this establishes property* (**iii**) *in Theorem 1.2.*

(e) $U(\cdot, \cdot)v \in C^1(\Delta_+; X)$, *with property* (**iv**) *in Theorem 1.2.*

Proof. We follow the idea in [**7**, Lemma 3.10].

(a) We see from condition (**S3**) that for $(t,s) \in \Delta_+$,

$$(3.8) \qquad \|S(s)^{1/2}w\| \le \exp\left(\int_s^t \sigma(r)\,dr\right)\|S(t)^{1/2}w\|, \quad w \in Y.$$

Hence we obtain

$$\|U(t,r)v - U(t,s)v\|_Y \le \exp(\|\sigma(r)\|_{L^1(0,T)})\|S(t)^{1/2}U(t,r)v - S(t)^{1/2}U(t,s)v\|.$$

Therefore the assertion follows from Lemma 3.6 (c).

(b) By virtue of Lemma 3.6 (a), it suffices to show that

$$(3.9) \qquad \|S(\cdot)^{1/2}U(\cdot, s)v\| \in C[s,T].$$

We trace the proof of Lemma 3.3 (b). Let us t, $t_0 \in [s,T]$. Integrating the inequality (3.3) from $r = t_0$ to $r = t$ and passing to the limit as $n \to \infty$, we have

$$\|S_\varepsilon^h(t)^{1/2}U(t,s)v\|^2 - \|S_\varepsilon^h(t_0)^{1/2}U(t_0,s)v\|^2$$
$$= -2\int_{t_0}^t \text{Re}\,\left(A(r)U(r,s)v, S_\varepsilon^h(r)U(r,s)v\right)\,dr$$
$$+ \int_{t_0}^t \left(U(r,s)v, \left(\frac{d}{dr}S_\varepsilon^h(r)\right)U(r,s)v\right)\,dr.$$

Passing to the limit as $h \downarrow 0$, we see from (3.7) that

$$\left|\|S_\varepsilon(t)^{1/2}U(t,s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0,s)v\|^2\right|$$
$$\le 2\left|\int_{t_0}^t \left|\text{Re}\,\left(A(r)U(r,s)v, S_\varepsilon(r)U(r,s)v\right)\right|\,dr\right|$$
$$+ 2\left|\int_{t_0}^t \sigma(r)\|S_\varepsilon(r)^{1/2}U(r,s)v\|^2\,dr\right|.$$

with domain $D(\mathcal{H}(t)) = C_0^\infty(\mathbb{R}^3)^4$, where H_0 is the free Dirac operator, $V_0(t,x)$ and $q(t,x)$ are satisfying condition (V1), (V2) and (q). Since $\mathcal{H}(t)$ is symmetric, $\mathcal{H}(t)$ is closable. Then we take as $H(t)$ the closure $\widetilde{\mathcal{H}}(t)$ of $\mathcal{H}(t)$, i.e., $H(t) = \widetilde{\mathcal{H}}(t)$. Set

$$S(t) := (H_0 + V_0(t))^2 + (1 + |x|^2)I,$$
$$D(S(t)) := \{u \in L^2(\mathbb{R}^3)^4; S(t)u \in L^2(\mathbb{R}^3)^4\}.$$

Then we can show that $S(t)$ is positive selfadjoint on $D(S(t))$ (see [8, Lemma 5.4]) and $Y_t := D(S(t)^{1/2})$ is regarded as a Hilbert space, embedded continuously and densely in $L^2(\mathbb{R}^3)^4$, with inner product

$$(u, v)_{Y_t} = (S(t)^{1/2}u, S(t)^{1/2}v), \quad u, v \in Y_t.$$

We can show that $S(t)$ satisfies condition (**S1**) and the first half of condition (**S2**):

Lemma 4.1 (cf. [8, Lemma 6.1]). *Let $S(t)$ be as above. Then for $t \in [0, T]$,*

$$Y_t = Y = \Sigma^1(\mathbb{R}^3)^4$$

and there exist time independent positive constants c_1, c_2 such that

$$(4.1) \qquad c_1 \|S(t)^{1/2}u\|^2 \le \|u\|^2 + \|\nabla u\|^2 + \||x|u\|^2 \le c_2 \|S(t)^{1/2}u\|^2, \quad u \in Y.$$

Moreover the second half of condition (**S2**) follows from (V2), because (V2) implies that $V_0(\cdot, x)$ is continuous on $[0, T]$.

The following lemma guarantees that $S(t)$ satisfies conditon (**S3**):

Lemma 4.2. *Let $S(t)$ be as above. Then*

$$\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\| \le \left| \int_s^t \sigma_1(r)\,dr \right| \max_{r \in \{s,t\}} \|S(r)^{1/2}v\|, \quad v \in Y, \; t, s \in [0, T].$$

Proof. Since

$$\|S(t)^{1/2}v\|^2 = \|(H_0 + V_0(t))v\|^2 + \||x|v\|^2 + \|v\|^2,$$

we have

$$\frac{d}{dt}\|S(t)^{1/2}v\|^2 = 2\left(\frac{\partial}{\partial t}V_0(t)v, (H_0 + V_0(t))v\right).$$

Therefore we obtain from the Schwartz inequality and Hardy inequality that

$$\left|\frac{d}{dt}\|S(t)^{1/2}v\|^2\right| = 2\sigma(t)(\|2a\nabla v\| + \|v\|)\|(H_0 + V_0(t))v\|.$$

It follows from (4.1) that

$$\left|\frac{d}{dt}\|S(t)^{1/2}v\|\right| = (2a + b)c_2\sigma(t)\|S(t)^{1/2}v\|.$$

Thus the assertion follows by $\sigma_1(t) = (2a + b)c_2\sigma(t)$. $\qquad\qquad\square$

Now we shall verify conditions (**A1**)–(**A4**).

Lemma 4.3. *Let $A(t) = iH(t)$ and $S(t)$ be as above. Then for each $T > 0$*

(A1) $\operatorname{Re}(A(t)v, v) = 0$, $v \in D(A(t))$, a.a. $t \in (0, T)$.

(A2) $Y = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4 \subset D(A(t))$, a.a. $t \in (0, T)$.

(A3) *There exists a constant $\beta \geq 0$ such that*

$$\left|\operatorname{Re}(A(t)u, S(t)u)\right| \leq \beta \left\|S(t)^{1/2}u\right\|^2, \quad u \in D(S(t)), \text{ a.a. } t \in (0, T).$$

(A4) $A(\cdot) \in C_*\big(0, T; B(\Sigma^1(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4)\big)$.

Proof. Noting that $\operatorname{Re}(A(t)u, u) = -\operatorname{Im}(H(t)u, u)$ the assertion follows from symmetry of $H(t)$. Moreover the continuity of $A(t)$ follows from (V2) and (q). Therefore, it is sufficient to show that there exist $\beta > 0$ such that

$$\left|\operatorname{Im}(H(t)u, S(t)u)\right| \leq \beta(t)\left\|S(t)^{1/2}u\right\|^2, \quad u \in D(S), \text{ a.a. } t \in (0, T).$$

By integration by parts we have

$$\operatorname{Im}(H(t), S(t)u) = \operatorname{Im}\big((H_0 + V_0(t))u, |x|^2 u\big) + \operatorname{Im}\big(q(t)u, (H_0 + V_0(t))^2 u\big)$$
$$= \operatorname{Re}\big((\alpha \cdot x)u, u\big) - \operatorname{Re}\big((\alpha \cdot \nabla q(t))u, (H_0 + V_0(t))u\big).$$

We see from condition (q) that there exists a constant $c_q > 0$ such that

$$\|(1 + |x|^2)^{-1/2} \nabla q(t)\|_{L^\infty} \leq c_q.$$

Hence it follows from the Schwarz inequality and that

$$\left|\operatorname{Im}(H(t), S(t)u)\right| \leq \|\,|x|u\| \cdot \|u\| + \|\,|\nabla q(t)|u\| \cdot \|(H_0 + V_0(t))u\|$$
$$\leq \|\,|x|u\| \cdot \|u\| + c_q\|(1 + |x|^2)^{1/2}u\| \cdot \|(H_0 + V_0(t))u\|,$$

Therefore we obtain the desired inequality. $\qquad\square$

According to the above lemmas, we can obtain Theorem 1.1.

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