数理解析研究所講究録
第1893巻 2014年 115-121

ヒルベルトC*-加群上のSelberg不等式について

Selberg type inequalities on Hilbert C^*-modules

大阪教育大学数学教育講座　瀬尾祐貴 (Yuki Seo)
Department of Mathematics Education, Osaka Kyoiku University

1. Introduction

This paper is based on [15].

We briefly review the Selberg inequality and its generalization in a Hilbert space.

Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. The Selberg inequality [2, 17] states that if $y_1, y_2, \ldots, y_n$ and x are nonzero vectors in H, then

$$\sum_{i=1}^{n}\frac{|\langle y_i,x\rangle|^2}{\sum_{j=1}^{n}|\langle y_j,y_i\rangle|}\leq\|x\|^2. \tag{1}$$

Moreover, Furuta [10] posed conditions enjoying the equality: The equality in (1) holds if and only if $x=\sum_{i=1}^{n}a_iy_i$ for some scalars $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that for arbitrary $i\neq j$

$$\langle y_i,y_j\rangle=0 \quad \text{or} \quad |a_i|=|a_j| \ \text{ with } \ \langle a_iy_i,a_jy_j\rangle\geq 0, \tag{2}$$

also see [7]. Note that the Selberg inequality is simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality. As a matter of fact, if $n=1$ and $y=y_1$, then we have the Cauchy-Schwarz inequality $|\langle y,x\rangle|\leq\|y\|\|x\|$. If $\{y_i\}$ is an orthonormal system, then we have the Bessel inequality $\sum_{i=1}^{n}|\langle y_i,x\rangle|^2\leq\|x\|^2$.

Fujii and Nakamoto [9] showed a refinement of the Selberg inequality (1): If $\langle y,y_i\rangle=0$ for given nonzero vectors $y_1,\ldots,y_n\in H$, then

$$|\langle x,y\rangle|^2+\sum_{i=1}^{n}\frac{|\langle x,y_i\rangle|^2}{\sum_{j=1}^{n}|\langle y_j,y_i\rangle|}\|y\|^2\leq\|x\|^2\|y\|^2 \tag{3}$$

holds for all $x\in H$. Also, Bombieri [1] showed the following generalization of the Bessel inequality: If $x, y_1, \ldots, y_n$ are nonzero vectors in H, then

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2\leq\|x\|^2\max_{1\leq i\leq n}\sum_{j=1}^{n}|\langle y_j,y_i\rangle|. \tag{4}$$

Moreover, Mitrinović, Pecǎrić and Fink [17, Theorem 5 in pp394] mentioned the following inequality equivalent to Bombieri's type (4): If $x, y_1, \ldots, y_n$ are nonzero vectors in H and $a_1, \ldots, a_n \in \mathbb{C}$, then

$$|\sum_{i=1}^{n}a_i\langle x,y_i\rangle|^2\leq\|x\|^2\sum_{i=1}^{n}|a_i|^2\sum_{j=1}^{n}|\langle y_j,y_i\rangle|. \tag{5}$$

In this paper, from a viewpoint of the operator theory, we propose a Selberg type inequality in a Hilbert C^*-module, which ia simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality in a Hibert C^*-module. As applications, we show Hilbert C^*-module versions of Fujii-Nakamoto type (3), Bombieri type (4) and Mitrinović, Pecǎrić and Fink type (5).

2. Preliminaries

Let $\mathscr{A}$ be a unital C^*-algebra with the unit element e. An element $a \in \mathscr{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathscr{A}$, the operator geometric mean of a and b is defined by

$$a \sharp b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{\frac{1}{2}} a^{\frac{1}{2}}$$

for invertible a. If a and b are non invertible, then $a \sharp b$ belongs to the double commutant $\mathscr{A}''$ of $\mathscr{A}$ in general. In fact, since $a \sharp b$ satisfies the upper semicontinuity, it follows that $a \sharp b = \lim_{\varepsilon \to +0}(a + \varepsilon e) \sharp (b + \varepsilon e)$ in the strong operator topology. If $\mathscr{A}$ is monotone complete in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have $a \sharp b \in \mathscr{A}$, see [12]. The operator geometric mean has the symmetric property: $a \sharp b = b \sharp a$. In the case that a and b commute, we have $a \sharp b = \sqrt{ab}$. For more details on the operator geometric mean, see [11, 8].

A complex linear space $\mathscr{X}$ is said to be an inner product $\mathscr{A}$-module (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y \rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle \quad (x, y, x \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a \quad (x, y \in \mathscr{X}, a \in \mathscr{A})$,
(iii) $\langle y, x \rangle = \langle x, y \rangle^* \quad (x, y \in \mathscr{X})$,
(iv) $\langle x, x \rangle \geq 0 \ (x \in \mathscr{X})$ and if $\langle x, x \rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathscr{A}$ and $\mathscr{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y \rangle = a^* \langle x, y \rangle$ for all $x, y \in \mathscr{X}, a \in \mathscr{A}$. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a semi-inner product $\mathscr{A}$-module.

In this case, we write $\| x \| := \sqrt{\| \langle x, x \rangle \|}$, where the latter norm denotes the C^*-norm of $\mathscr{A}$. If an inner-product $\mathscr{A}$-module $\mathscr{X}$ is complete with respect to its norm, then $\mathscr{X}$ is called a *Hilbert C^*-module.* In [6], from a viewpoint of operator theory, we presented the following Cauchy-Schwarz inequality in the framework of a semi-inner product C^*-module over a unital C^*-algebra: If $x, y \in \mathscr{X}$ such that the inner product $\langle x, y \rangle$ has a polar decomposition $\langle x, y \rangle = u|\langle x, y \rangle|$ with a partial isometry $u \in \mathscr{A}$, then

$$|\langle x, y \rangle| \ \leq \ u^* \langle x, x \rangle u \sharp \langle y, y \rangle. \tag{6}$$

Under the assumption that $\mathscr{X}$ is an inner product $\mathscr{A}$-module and $\langle y, y \rangle$ is invertible, the equality in (6) holds if and only if $xu = yb$ for some $b \in \mathscr{A}$. As applications of the Cauchy-Schwarz inequality (6), we cite [5, 18].

An element x of a Hilbert C^*-module $\mathscr{X}$ is called nonsingular if the element $\langle x, x \rangle \in \mathscr{A}$ is invertible. The set $\{x_i\} \subset \mathscr{X}$ is called orthonormal if $\langle x_i, x_j \rangle = \delta_{ij} e$. For more details on Hilbert C^*-modules, see [16].

3. Main theorem

Fiest of all, we show the following Selberg type inequality in a Hilbert C*-module.

Theorem 1. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \dots, y_n$ are nonsingular, then*

$$\sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|\right)^{-1} \langle y_i, x\rangle \leq \langle x, x\rangle. \tag{7}$$

The equality in (7) holds if and only if $x = \sum_{i=1}^n y_i a_i$ for some $a_i \in \mathscr{A}$ and $i = 1, \dots, n$ such that for arbitrary $i \neq j$ $\langle y_i, y_j\rangle = 0$ or $|\langle y_j, y_i\rangle| a_i = \langle y_i, y_j\rangle a_j$.

Theorem 1 is simultaneous extensions of the Bessel inequality [4] and the Cauchy-Schwarz inequality [6] in a Hilbert C^*-module. As a matter of fact, if $\{y_1, \dots, y_n\}$ is orthonormal in Theorem 1, then we have the Bessel inequality:

$$\sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \leq \langle x, x\rangle$$

holds for all $x \in \mathscr{X}$. If $n = 1$ and $y = y_1$ in Theorem 1 and $\langle x, y\rangle$ has a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then we have $u|\langle x, y\rangle| \langle y, y\rangle^{-1} |\langle y, x\rangle| u^* \leq \langle x, x\rangle$ and hence

$$|\langle x, y\rangle| = |\langle x, y\rangle| \langle y, y\rangle^{-1} |\langle y, x\rangle| \;\sharp\; \langle y, y\rangle \leq u^* \langle x, x\rangle u \;\sharp\; \langle y, y\rangle.$$

This implies the Cauchy-Schwarz inequality (6).

To prove Theorem 1, we need the following two lemmas:

Lemma 2. *If $a \in \mathscr{A}$, then the operator matrix on $\mathscr{A} \oplus \mathscr{A}$*

$$A = \begin{pmatrix} |a^*| & -a \\ -a^* & |a| \end{pmatrix}$$

is positive, and $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathrm{N}(A)$ if and only if $|a^*|\xi = a\eta$, where $N(A)$ is the kernel of A.

Lemma 3. *For any $y_1, y_2, \dots, y_n \in \mathscr{X}$*

$$\begin{pmatrix} \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ & \ddots & \\ \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n\rangle \end{pmatrix} \leq \begin{pmatrix} \sum_{j=1}^n |\langle y_j, y_1\rangle| & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^n |\langle y_j, y_n\rangle| \end{pmatrix}.$$

Proof of Theorem 1 For each $i = 1, \ldots, n$, put $c_i = \sum_{j=1}^{n} |\langle y_j, y_i\rangle|$. Since y_i is nonsingular, it follows that c_i is invertible in $\mathscr{A}$. It follows from Lemma 3 that

$$
\begin{aligned}
&\sum_{i=1}^{n} \langle x, y_i\rangle c_i^{-1}\langle y_i, y_j\rangle c_j^{-1}\langle y_j, x\rangle \\
&= (\langle x, y_1\rangle c_1^{-1} \cdots \langle x, y_n\rangle c_n^{-1}) \begin{pmatrix} \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ & \ddots & \\ \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n\rangle \end{pmatrix} \begin{pmatrix} c_1^{-1}\langle y_1, x\rangle \\ \cdots \\ c_n^{-1}\langle y_n, x\rangle \end{pmatrix} \\
&\le (\langle x, y_1\rangle c_1^{-1} \cdots \langle x, y_n\rangle c_n^{-1}) \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \begin{pmatrix} c_1^{-1}\langle y_1, x\rangle \\ \cdots \\ c_n^{-1}\langle y_n, x\rangle \end{pmatrix} \\
&= \sum_{i=1}^{n} \langle x, y_i\rangle c_i^{-1}\langle y_i, x\rangle
\end{aligned}
$$

and this implies

$$
\begin{aligned}
0 &\le \langle x - \sum_{i=1}^{n} y_i c_i^{-1}\langle y_i, x\rangle, x - \sum_{i=1}^{n} y_i c_i^{-1}\langle y_i, x\rangle\rangle \\
&= \langle x, x\rangle - 2\sum_{i=1}^{n} \langle x, y_i\rangle c_i^{-1}\langle y_i, x\rangle + \sum_{i=1}^{n} \langle x, y_i\rangle c_i^{-1}\langle y_i, y_j\rangle c_j^{-1}\langle y_j, x\rangle \\
&\le \langle x, x\rangle - \sum_{i=1}^{n} \langle x, y_i\rangle c_i^{-1}\langle y_i, x\rangle.
\end{aligned}
$$

Hence we have the desired inequality (7).

The equality in (7) holds if and only if the following (8) and (9) are satisfied:

$$x = \sum_{i=1}^{n} y_i c_i^{-1}\langle y_i, x\rangle \tag{8}$$

and for arbitrary $i \neq j$

$$(\langle x, y_i\rangle c_i^{-1}\langle x, y_j\rangle c_j^{-1}) \begin{pmatrix} |\langle y_j, y_i\rangle| & -\langle y_i, y_j\rangle \\ -\langle y_j, y_i\rangle & |\langle y_i, y_j\rangle| \end{pmatrix} \begin{pmatrix} c_i^{-1}\langle y_i, x\rangle \\ c_j^{-1}\langle y_j, x\rangle \end{pmatrix} = 0. \tag{9}$$

Put $A = \begin{pmatrix} |\langle y_j, y_i\rangle| & -\langle y_i, y_j\rangle \\ -\langle y_j, y_i\rangle & |\langle y_i, y_j\rangle| \end{pmatrix}$ and it follows that the condition (9) holds if and only if

$$A^{1/2} \begin{pmatrix} c_i^{-1}\langle y_i, x\rangle \\ c_j^{-1}\langle y_j, x\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff A \begin{pmatrix} c_i^{-1}\langle y_i, x\rangle \\ c_j^{-1}\langle y_j, x\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence it follows from Lemma 2 that the condition (9) is equivalent to the following (10) and (11): For arbitrary $i \neq j$

$$\langle y_i, y_j\rangle = 0 \tag{10}$$

or

$$|\langle y_j, y_i\rangle| c_i^{-1}\langle y_i, x\rangle = \langle y_i, y_j\rangle c_j^{-1}\langle y_j, x\rangle. \tag{11}$$

Conversely, suppose that $x = \sum_{i=1}^n y_i a_i$ for some $a_i \in \mathscr{A}$ and for $i \neq j$ $\langle y_i, y_j\rangle = 0$ or $|\langle y_j, y_i\rangle| a_i = \langle y_i, y_j\rangle a_j$. Then

$$\begin{aligned}
\sum_{i=1}^n \langle x, y_i\rangle \left(\sum_{j=1}^n |\langle y_j, y_i\rangle|\right)^{-1} \langle y_i, x\rangle &= \sum_{i=1}^n \langle x, y_i\rangle \left(\sum_{j=1}^n |\langle y_j, y_i\rangle|\right)^{-1} \sum_{j=1}^n \langle y_i, y_j\rangle a_j \\
&= \sum_{i=1}^n \langle x, y_i\rangle \left(\sum_{j=1}^n |\langle y_j, y_i\rangle|\right)^{-1} \sum_{j=1}^n |\langle y_j, y_i\rangle| a_i \\
&= \sum_{i=1}^n \langle x, y_i\rangle \left(\sum_{j=1}^n |\langle y_j, y_i\rangle|\right)^{-1} \left(\sum_{j=1}^n |\langle y_j, y_i\rangle|\right) a_i \\
&= \sum_{i=1}^n \langle x, y_i\rangle a_i \\
&= \langle x, x\rangle.
\end{aligned}$$

Whence the proof is complete. □

Remark 4. (1) In the case that $\mathscr{X}$ is a Hilbert space, the equality condition $|\langle y_j, y_i\rangle| a_i = \langle y_i, y_j\rangle a_j$ in Theorem 1 implies the condition (2) in Introduction. In fact, for some scalars $a_i, a_j \in \mathbb{C}$, it follows that $\langle a_i y_i, a_j y_j\rangle = a_i^* \langle y_i, y_j\rangle a_j = a_i^* |\langle y_j, y_i\rangle| a_i \geq 0$, and $|\langle y_j, y_i\rangle| = |\langle y_j, y_i\rangle^*|$ implies $|a_i| = |a_j|$.
(2) In the Hilbert space setting, K. Kubo and F. Kubo [14] showed another proof of Selberg's inequality (1) using Geršgorin's location of eigenvalues [13, Theorem 6.1.1] and a diagonal domination theorem of positive semidefinite matrix.

4. Applications

In [4], Dragomir, Khosravi and Moslehian showed a version of the Bessel inequality and some generalizations of this inequality in the framework of Hilbert C^*-modules. Moreover, in [3], Bounader and Chahbi showed a type and refinement of Selberg inequality in Hilbert C^*-modules. In this section, by using Theorem 1, we consider several Hilbert C^*-module versions of the Selberg inequality and the Bessel inequality.

Bounader and Chahbi in [3, Theorem 3.1] showed that if $\mathscr{X}$ is an inner product C^*-module and $y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$, and $x \in \mathscr{X}$, then

$$\sum_{i=1}^n \frac{|\langle y_i, x\rangle|^2}{\sum_{j=1}^n \| \langle y_j, y_i\rangle \|} \leq \langle x, x\rangle. \tag{12}$$

By Theorem 1, we have the following corollary, which is an improvement of (12):

Corollary 5. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then*

$$\sum_{i=1}^n \frac{|\langle y_i, x\rangle|^2}{\| \sum_{j=1}^n |\langle y_j, y_i\rangle| \,\|} \leq \langle x, x\rangle.$$

Moreover, Bounader and Chahbi showed a Hilbert C^*-module version of Fujii-Nakamoto type (3), which is a refinement of (12): If y and $y_1, \ldots, y_n$ are nonzero vectros in $\mathscr{X}$ such that $\langle y, y_i \rangle = 0$ for $i = 1, \ldots, n$, and $x \in \mathscr{X}$, then

$$|\langle y, x\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle y_i, x\rangle|^2}{\sum_{j=1}^{n} \| \langle y_i, y_j\rangle \|} \| \langle y, y\rangle \| \leq \| \langle y, y\rangle \| \langle x, x\rangle. \tag{13}$$

We show a Hilbert C^*-module version of a refinement of the Selberg inequality due to Fujii and Nakamoto, which is another version of (13):

Theorem 6. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \ldots, y_n$ are nonsingular, $\langle y, y_i\rangle = 0$ for $i = 1, \cdots, n$ and $\langle x, y\rangle = u|\langle x, y\rangle|$ is a polar decomposition in $\mathscr{A}$, i.e., $u \in \mathscr{A}$ is a partial isometry, then*

$$|\langle y, x\rangle| \leq u^*\langle y, y\rangle u \sharp \left(\langle x, x\rangle - \sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle| \right)^{-1} \langle y_i, x\rangle \right)$$

$$\left(\leq u^*\langle y, y\rangle u \sharp \langle x, x\rangle \right).$$

In [3, Corollary 3.5], Bounader and Chahbi showed a Hilbert C^*-module version of Bombieri type (4): If $y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ and $x \in \mathscr{X}$, then

$$\sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \leq \langle x, x\rangle \max_{1\leq i\leq n} \sum_{j=1}^{n} \| \langle y_i, y_j\rangle \| . \tag{14}$$

We show a Hilbert C^*-module version of Bombieri type, which is an improvement of (14):

Theorem 7. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then*

$$\sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \leq \langle x, x\rangle \max_{1\leq i\leq n} \| \sum_{j=1}^{n} |\langle y_j, y_i\rangle| \| .$$

As a corollary, we have the following Boas-Bellman type inequality [3, Corollary 3.6]:

Corollary 8. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then*

$$\sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \leq \langle x, x\rangle \left(\max_{1\leq i\leq n} \| \langle y_i, y_i\rangle \| + (n-1) \max_{j\neq i} \| \langle y_j, y_i\rangle \| \right).$$

Finally, we show a Mitrinović-Pečarić-Fink type inequality [17, Theorem 5 in pp394] in Hilbert C^*-modules, which is another version of [4, Theorem 3.8]:

Theorem 9. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algbera $\mathscr{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ and $a_1, \cdots, a_n \in \mathscr{A}$ such that $y_1, \ldots, y_n$ are nonsingular and $\langle x, \sum_{i=1}^{n} y_i a_i\rangle = u|\langle x, \sum_{i=1}^{n} y_i a_i\rangle|$ is a polar decomposition in $\mathscr{A}$, i.e., $u \in \mathscr{A}$ is a partial isometry, then*

$$|\sum_{i=1}^{n} \langle x, y_i\rangle a_i| \leq u^*\langle x, x\rangle u \sharp \left(\sum_{i=1}^{n} a_i^* \left(\sum_{j=1}^{n} |\langle y_j, y_i\rangle| \right) a_i \right).$$

References

[1] E. Bombieri, *A note on the large sieve*, Acta. Arith., **18** (1971), 401–404.

[2] E. Bombieri, *Le Grand Grible dans la Theórie Analytique des Nombres*, Asterisque 18, Societe Mathematique de France, 1974.

[3] N. Bounader and A. Chahbi, *Selberg type inequalities in Hilbert C^*-modules*, Int. J. Analy., **7** (2013), 385–391.

[4] S.S. Dragomir, M. Khosravi and M.S. Moslehian, *Bessel type inequalities in Hilbert C^*-modules*, Linear Multilinear Algebra, **58** (2010), 967–975.

[5] J.I. Fujii, M. Fujii and Y. Seo, *Operator inequalities on Hilbert C^*-modules via the Cauchy-Schwarz inquality*, to appear in Math. Ineq. Appl.

[6] J.I. Fujii, M. Fujii, M.S. Moslehian and Y. Seo, *Cauchy-Schwarz inequality in semi-inner product C^*-modules via polar decomposition*, J. Math. Anal. Appl., **394** (2012), 835-840.

[7] M. Fujii, K. Kubo and S. Otani, *A graph theoretical observation on the Selberg inequality*, Math. Japon., **35** (1990), 381–385.

[8] M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 4, Element, Zagreb, 2012.

[9] M. Fujii and R. Nakamoto, *Simultaneous extensions of Selberg inequality and Heinz-Kato-Furuta inequality*, Nihonkai Math. J., **9** (1998), 219–225.

[10] T. Furuta, *When does the equality of a generalized Selberg inequality hold?*, Nihonkai Math. J., **2** (1991), 25–29.

[11] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[12] M. Hamana, *Partial $*$-automorphisms, normalizers, and submodules in monotome complete C^*-algebras*, Canad. J.Math., **58** (2006), 1144–1202.

[13] R.A. Horn and C.A. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[14] K. Kubo and F. Kubo, *Diagonal matrix that dominates a positive semidefinite matrix*, Technical note, 1988.

[15] K. Kubo, F. Kubo and Y. Seo, *Selberg type inequalities in a Hilbert C^*-module and its applications*, to appear in Sci. Math. Jpn.

[16] E.C. Lance, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, 1995.

[17] D.S. Mitrinović, J. Pecărić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[18] Y. Seo, *Hölder type inequalities on Hilbert C^*-modules and its reverses*, Ann. Funct. Anal., **5** (2014), 1–9.

Department of Mathematics Education, Osaka Kyoiku University,
4-698-1 Asahigaoka, Kashiwara, Osaka 582-8582 JAPAN.
E-mail address : `yukis@cc.osaka-kyoiku.ac.jp`

大阪教育大学・数学教育講座　　瀬尾　祐貴