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UNCONDITIONAL EXISTENCE OF DENSITIES FOR THE NAVIER-STOKES EQUATIONS WITH NOISE

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ABSTRACT. The first part of the paper contains a short review of recent results about the existence of densities for finite dimensional functionals of weak solutions of the Navier–Stokes equations forced by Gaussian noise. Such results are obtained for solutions limit of spectral Galerkin approximations.

In the second part of the paper we prove via a "transfer principle" that existence of densities is universal, in the sense that it does not depend on how the solution has been obtained, given some minimal and reasonable conditions of consistence under conditional probabilities and weak–strong uniqueness. A quantitative version of the transfer principle is also available for stationary solutions.

1. INTRODUCTION

When dealing with a stochastic evolution PDE, the solution depends not only on the time and space independent variables, but also on the "chance" variable, which plays a completely different role. The existence of a density for the probability distribution of the solution is thus a form of regularity with respect to this new variable.

In this paper we detail some results related to the existence of densities of finite dimensional projections of any solution of the Navier–Stokes equations

$$\begin{aligned} &\dot{u} + (u \cdot \nabla)u + \nabla p = \nu\Delta u + \eta, \\ &\operatorname{div} u = 0, \end{aligned} \tag{1.1}$$

with Dirichlet boundary conditions on a bounded domain, or with periodic boundary conditions on the torus. Here η is Gaussian noise. Most of the results have appeared in [DR14, Rom13], some additional results are in progress [Rom14b, Rom14c, Rom14a].

To be more precise, our result concerns the existence of densities for finite dimensional functionals of the solution, and one reason for this is that there is no canonical reference measure in infinite dimension, as is the Lebesgue measure in finite dimension. To understand the right reference measure is an open problem even in dimension two and for any suitable choice of the driving noise.

Our interest in the existence of densities stems from a series of mathematical motivations. The first and foremost is the investigation of the regularity properties of solutions of the Navier–Stokes equations.

On the other hand regularity is not the only open problem in the mathematical theory of the Navier–Stokes equations (either with random forcing, or without). The first obvious choice is the related problem of uniqueness. In the probabilistic framework we can deal with different notions of uniqueness, the weaker being the statistical uniqueness, that is the uniqueness of distributions. Although the results detailed in this paper

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are very far from any uniqueness result, we mention that the law of an infinite dimensional random variable can be characterized by the laws of its finite dimensional projections. By the results of [FR08], it is then sufficient to show that the laws of two solutions agree at every time. An even easier condition, following from the results of [Rom08], requires that we show agreement between the laws of the corresponding invariant measures, that is, if the processes agree at time $t = \infty$, then they agree for every time, including their time correlations.

An additional (rather vague though) "folkloristic" motivation for the interest in finite dimensional projections is that most of the real–life experiments to evaluate the velocity of a fluid are based on a finite number of samples in a finite number of points (Eulerian point of view), or by tracing some particles (smoke, etc...) moving according to the fluid velocity (Lagrangian point of view). The literature on experimental fluid dynamics is huge. Here we refer for instance to [Tav05] for some examples of design of experiments. Let us focus on the Eulerian point of view. To simplify, consider a torus, then sampling the velocity field means measuring the velocity in some space points $y_1, \dots, y_d$: $u \rightsquigarrow (u(t, y_1), \dots, u(t, y_d))$, and a bit of Fourier series manipulations shows that this is a "projection".

An interesting difficulty in proving regularity of the density emerges as a by–product of the (more general and fundamental!) problem of proving uniqueness and regularity of solutions of the Navier–Stokes equations. Indeed, a fundamental and classical tool is the Malliavin calculus, a differential calculus where the differentiating variable is the underlying noise driving the system. The Malliavin derivative $\mathcal{D}_H u(t)$, the derivative with respect to the variations of the noise perturbation, is given as

$$\mathcal{D}_H u = \lim_{\epsilon \downarrow 0} \frac{u(W + \epsilon \int H\,ds) - u(W)}{\epsilon},$$

where we have written the solution u as $u(W)$ to show the explicit dependence of u from the noise forcing. We point, for instance, to [Nua06] for further details and definitions, and we only notice that the Malliavin derivative $\mathcal{D}_H u$ of the solution u of (1.1), as a variation, satisfies the linearization around the solution, namely,

$$\frac{d}{dt}\mathcal{D}_H u - \nu\Delta\mathcal{D}_H u + (u \cdot \nabla)\mathcal{D}_H u + ((\mathcal{D}_H u) \cdot \nabla) u = SH,$$

and good estimates on $\mathcal{D}_H u(t)$ originate only from good estimates on the linearization of (1.1), which are not available so far. This settles the need of methods to prove existence and regularity of the density that do not rely on this calculus, as done in [DR14].

In this paper we tackle the problem of universality of the result obtained in [DR14], which are valid only for limits of Galerkin approximations. At the present time we do not know if the Navier–Stokes equations admit a unique distribution, therefore it might happen that solutions obtained by different means may have different properties. In a way this is reminiscent of the problem of suitable weak solutions introduced by [Sch77]. Only much later it has been proved that solutions obtained by the spectral Galerkin methods are suitable [Gue06] (under some non–trivial conditions though), and hence results of partial regularity are true for those solutions.

Our main theorem is a "transfer principle" (Theorem 4.1), that states that as long as we can prove existence of a density for a finite dimensional functional of the solution and for a weak solution that satisfies *weak–strong uniqueness*, then existence of a density holds for any other solution satisfying weak–strong uniqueness and a closure property with respect to conditional probabilities.

An important limitation of our transfer principle is that it applies only on "instantaneous" properties, namely to random variables depending only on one time, in particular, the results on time continuity of densities in [Rom14c] are still out of reach.

The transfer principle is qualitative in nature, as it may transfer only the existence. In general no quantitative information can be inherited. This seems to be mainly an artefact of the proof, that in turns depends on good moments of the solution in smoother spaces. Indeed, in the case of stationary solutions, we can prove a quantitative version of the principle (Theorem 4.2).

2. Weak solutions

We consider problem (1.1) with either periodic boundary conditions on the three–dimensional torus $\mathbb{T}_3 = [0, 2\pi]^3$ or Dirichlet boundary conditions on a smooth domain $\mathcal{O} \subset \mathbf{R}^3$. We will understand weak martingale solutions of (1.1) as probability measures on the path space. We will then define *legit* families of solutions as classes of solutions that are closed by conditional probability and for which weak–strong uniqueness holds.

2.1. **Preliminaries.** Let H be the standard space of square summable divergence free vector fields, defined as the closure of divergence free smooth vector fields satisfying the boundary condition, with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Define likewise V as the closure with respect to the H^1 norm. Let Π_L be the Leray projector, $A = -\Pi_L\Delta$ the Stokes operator, and denote by $(\lambda_k)_{k\geqslant1}$ and $(e_k)_{k\geqslant1}$ the eigenvalues and the corresponding orthonormal basis of eigenvectors of A. Define the bi-linear operator $B : V \times V \to V'$ as $B(u,v) = \Pi_L(u\cdot\nabla v)$, $u, v \in V$. We recall that $\langle u_1, B(u_2,u_3)\rangle = -\langle u_3, B(u_2,u_1)\rangle$. We refer to Temam [Tem95] for a detailed account of all the above definitions.

The noise $\dot\eta = \mathcal{S}\dot W$ in (1.1) is coloured in space by a covariance operator $\mathcal{S}^\star\mathcal{S} \in \mathscr{L}(H)$, where W is a cylindrical Wiener process (see [DPZ92] for further details). We assume that $\mathcal{S}^\star\mathcal{S}$ is trace–class and we denote by $\sigma^2 = \operatorname{Tr}(\mathcal{S}^\star\mathcal{S})$ its trace. Finally, consider the sequence $(\sigma_k^2)_{k\geqslant1}$ of eigenvalues of $\mathcal{S}^\star\mathcal{S}$, and let $(q_k)_{k\geqslant1}$ be the orthonormal basis in H of eigenvectors of $\mathcal{S}^\star\mathcal{S}$. For simplicity we may assume that the Stokes operator A and the covariance commute, so that

$$\dot\eta(t,y) = S\,dW = \sum_{k\in\mathbf{Z}^3_\star} \sigma_k\dot\beta_k(t)e_k(y).$$

2.2. **Weak and strong solutions.** With the above notations, we can recast problem (1.1) as an abstract stochastic equation,

$$du + (\nu Au + B(u))\,dt = \mathcal{S}\,dW, \tag{2.1}$$

with initial condition $u(0) = x \in H$. It is well–known [Fla08] that for every $x \in H$ there exist a martingale solution of this equation, that is a filtered probability space $(\widetilde\Omega, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}, \{\widetilde{\mathscr{F}}_t\}_{t\geqslant0})$, a cylindrical Wiener process $\widetilde W$ and a process u with trajectories in $C([0,\infty); D(A)')\cap L^\infty_{loc}([0,\infty), H)\cap L^2_{loc}([0,\infty); V)$ adapted to $(\widetilde{\mathscr{F}}_t)_{t\geqslant0}$ such that the above equation is satisfied with $\widetilde W$ replacing W.

We will describe, equivalently, a martingale solution as a measure on the path space (in other words via a martingale problem). Let $\Omega_{NS} = C([0,\infty); D(A)')$ and let $\mathscr{F}^{NS}$

be its Borel σ–algebra. Denote by $\mathscr{F}_t^{NS}$ the σ–algebra generated by the restrictions of elements of Ω_{NS} to the interval $[0,t]$ (roughly speaking, this is the same as the Borel σ–algebra of $C([0,t];D(A)'))$. Let ξ be the canonical process, defined by $\xi_t(\omega) = \omega(t)$, for $\omega \in \Omega_{NS}$

Definition 2.1 ([FR08]). A probability measure $\mathbb{P}$ on Ω_{NS} is a solution of the martingale problem associated to (2.1) with initial distribution μ if

- $\mathbb{P}[L^\infty_{loc}(\mathbf{R}^+, H) \cap L^2_{loc}(\mathbf{R}^+; V)] = 1$,
- for each $\phi \in D(A)$, the process

$$\langle \xi_t - \xi_0, \phi\rangle + \int_0^t \langle \xi_s, A\phi\rangle - \langle B(\xi_s, \phi), \xi_s\rangle \, ds$$

  is a continuous square summable martingale with quadratic variation $t\|\mathcal{S}\phi\|_H^2$ (hence a Brownian motion),
- the marginal of $\mathbb{P}$ at time 0 is μ.

The second condition in the definition above has a twofold meaning. On the one hand it states that the canonical process is a weak (in terms of PDEs) solution, on the other hand it identifies the driving Wiener process, and hence is a weak (in terms of stochastic analysis) solution.

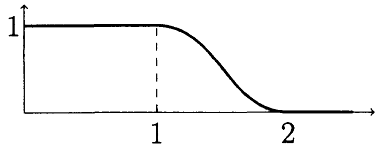
2.2.1. *Strong solutions.* It is also well–known that (2.1) admits local smooth solutions defined up to a random time (a stopping time, in fact) τ_∞ that corresponds to the (possible) time of blow–up in higher norms. To consider a quantitative version of the local smooth solutions, notice that τ_∞ can be approximated monotonically by a sequence of stopping times

$$\tau_R = \inf\{t > 0 : \|Au_R(t)\|_H \geqslant R\},$$

where u_R is a solution of the following problem,

$$du_R + \big(\nu A u_R + \chi(\|Au_R\|_H^2/R^2) B(u_R, u_R)\big)\, dt = \mathcal{S}\, dW,$$

with initial condition in $D(A)$, and where $\chi : [0,\infty) \to [0,1]$ is a suitable *cut–off* function, namely a non–increasing C^∞ function such that $\chi \equiv 1$ on $[0,1]$ and $\chi_R \equiv 0$ on $[2,\infty)$. The process u_R is a strong (in PDE sense) solution of the cut–off equation.



Moreover it is a strong solution also in terms of stochastic analysis, so it can be realized uniquely on any probability space, given the noise perturbation.

As it is well–known in the theory of Navier–Stokes equations, the regular solution is unique in the class of weak solutions that satisfy some form of the energy inequality. We will give two examples of such classes for the equations with noise.

Remark 2.2. The analysis of strong (PDE meaning) solutions can be done on larger spaces, up to $D(A^{1/4})$, which is a critical space with respect to the Navier–Stokes scaling. The extension is a bit technical though, see [Rom11].

2.2.2. *Solutions satisfying the almost sure energy inequality.* An almost sure version of the energy inequality has been introduced in [Rom08, Rom10]. Given a weak solution $\mathbb{P}$, choose $\phi = e_k$ as a test function in the second property of Definition 2.1, to get a one dimensional standard Brownian motion β^k. Since $(e_k)_{k\geqslant 1}$ is an orthonormal basis, the $(\beta^k)_{k\geqslant 1}$ are a sequence of independent standard Brownian motions. Then the process $W_{\mathbb{P}} = \sum_k \beta^k e_k$ is a cylindrical Wiener process[1] on H. Let $z_{\mathbb{P}}$ be the solution to the linearization at 0 of (2.1), namely $dz_{\mathbb{P}} + Az_{\mathbb{P}} = \mathcal{S}\, dW_{\mathbb{P}}$, with initial condition $z(0) = 0$. Finally, set $v_{\mathbb{P}} = \xi - z_{\mathbb{P}}$. It turns out that $v_{\mathbb{P}}$ is a solution of

$$\dot{v} + \nu A v + B(v + z_{\mathbb{P}}, v + z_{\mathbb{P}}) = 0, \qquad \mathbb{P}-\text{a.s.},$$

with initial condition $v(0) = \xi_0$. An energy balance functional can be associated to $v_{\mathbb{P}}$,

$$\mathcal{E}_t(v,z) = \frac{1}{2}\|v(t)\|_H^2 + \nu\int_0^t \|v(r)\|_V^2\, dr - \int_0^t \langle z(r), B(v(r) + z(r), v(r))\rangle_H\, dr.$$

We say that a solution $\mathbb{P}$ of the martingale problem associated to (2.1) (as in Definition 2.1) satisfies the *almost sure energy inequality* if there is a set $T_P \subset (0,\infty)$ of null Lebesgue measure such that for all $s \notin T_P$ and all $t \geqslant s$,

$$P[\mathcal{E}_t(v,z) \leqslant \mathcal{E}_s(v,z)] = 1.$$

It is not difficult to check that $\mathcal{E}$ is measurable and finite almost surely.

2.2.3. *A martingale version of the energy inequality.* An alternative formulation of the energy inequality that, on the one hand is compatible with conditional probabilities, and on the other hand does not involve additional quantities (such as the processes $z_{\mathbb{P}}$ and $v_{\mathbb{P}}$) can be given in terms of super–martingales. The additional advantage is that this definition is keen to generalization to state–dependent noise.

Define, for every $n \geqslant 1$, the process

$$\mathscr{E}_t^1 = \|\xi_t\|_H^2 + 2\nu\int_0^t \|\xi_s\|_V^2\, ds - 2\,\mathrm{Tr}(\mathcal{S}^\star\mathcal{S}),$$

and, more generally, for every $n \geqslant 1$,

$$\mathscr{E}_t^n = \|\xi_t\|_H^{2n} + 2n\nu\int_0^t \|\xi_s\|_H^{2n-2}\|\xi_s\|_V^2\, ds - n(2n-1)\,\mathrm{Tr}(\mathcal{S}^\star\mathcal{S})\int_0^t \|\xi_s\|_H^{2n-2}\, ds,$$

when $\xi \in L^\infty_{loc}([0,\infty);H) \cap L^2_{loc}([0,\infty);V)$, and ∞ elsewhere.

We say that a solution $\mathbb{P}$ of the martingale problem associated to (2.1) (as in Definition 2.1) satisfies the *super–martingale energy inequality* if for each $n \geqslant 1$, the process $\mathscr{E}_t^n$ defined above is $\mathbb{P}$–integrable and for almost every $s \geqslant 0$ (including $s = 0$) and all $t \geqslant s$,

$$\mathbb{E}[\mathscr{E}_t^n | \mathscr{F}_s^{NS}] \leqslant \mathscr{E}_s^n,$$

or, in other words, if each $\mathscr{E}^n$ is an almost sure supermartingale.

[1]Notice that W is measurable with respect to the solution process.

2.3. ***Legit* families of weak solutions.** Following the spirit of [FR08], given $x \in H$, denote by $\mathscr{C}(x)$ any family of non–empty sets of probability measures on $(\Omega_{NS}, \mathscr{F}^{NS})$ that are solutions of (1.1) with initial condition x, as specified by Definition 2.1, and such that the following properties hold,

- the sets $(\mathscr{C}(x))_{x\in H}$ are close under conditioning, namely for every $\mathbb{P} \in \mathscr{C}(x)$ and every $t > 0$, if $(\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_t})_{\omega\in\Omega_{NS}}$ is the regular conditional probability distribution of $\mathbb{P}$ given $\mathscr{F}^{NS}_t$, then $\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_t} \in \mathscr{C}(\omega(t))$, for $\mathbb{P}$–a.e. $\omega \in \Omega_{NS}$,
- weak–strong uniqueness holds, namely for every $x \in D(A)$ and every $\mathbb{P} \in \mathscr{C}(x)$, $\xi(t) = u_R(t, x)$ for every $t < \tau_R$, $\mathbb{P}$–a.s, where $u_R(\cdot; x)$ is the local smooth solution with initial condition x.

We will call each family $(\mathscr{C}(x))_{x\in H}$ satisfying the two above property a *legit* family.

It is clear that the classes defined in [FR08] (detailed in section 2.2.3) and in [Rom08, Rom10] (detailed in section 2.2.2) are of this kind, as they actually satisfy the more restrictive condition called *reconstruction* in the above–mentioned papers.[2] It is also straightforward that the x–wise set union of two *legit* families is again a *legit* family. A less obvious fact is that the family of sets of solutions obtained as limits of Galerkin approximations is *legit*. This is remarkable as limits of Galerkin approximations do not satisfy the reconstruction property. To see this fact, we first observe that limit of Galerkin approximation satisfy the energy inequality, and hence fall in the same class defined in [Rom08, Rom10]. In particular, due to the energy inequality, weak–strong uniqueness holds. Moreover, once the sub–sequence of Galerkin approximations is identified, the regular conditional probability distributions of the approximations, along the sub–sequence, converge to the corresponding regular conditional probability distributions of the limit (uniquely identified by the sub–sequence).

3. Existence of densities

In this section we give a short review of the results contained in the papers [DR14, Rom14c, Rom14a] (see also [Rom13]). To this end we recall the definition of Besov spaces. The general definition is based on the Littlewood–Paley decomposition, but it is not the best suited for our purposes. We shall use an alternative equivalent definition (see [Tri83, Tri92]) in terms of differences. Define

$$(\Delta^1_h f)(x) = f(x+h) - f(x),$$

$$(\Delta^n_h f)(x) = \Delta^1_h(\Delta^{n-1}_h f)(x) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} f(x+jh),$$

then the following norms, for $s > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$,

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \Big(\int_{\{|h|\leqslant 1\}} \frac{\|\Delta^n_h f\|^q_{L^p}}{|h|^{sq}}\frac{dh}{|h|^d}\Big)^{\frac{1}{q}},$$

and for $q = \infty$,

$$\|f\|_{B^s_{p,\infty}} = \|f\|_{L^p} + \sup_{|h|\leqslant 1}\frac{\|\Delta^n_h f\|_{L^p}}{|h|^{\alpha}},$$

[2] Reconstruction, roughly speaking, requires that if one has a $\mathscr{F}^{NS}_t$ measurable map $x \mapsto \mathbb{Q}_x$, with $\mathbb{Q}_x \in \mathscr{C}(x)$, and $\mathbb{P} \in \mathscr{C}(x_0)$, then the probability measure given by $\mathbb{P}$ on $[0, t]$, and, conditionally on $\omega(t)$, by the values of $\mathbb{Q}_\cdot$, is an element of $\mathscr{C}(x_0)$.

where n is any integer such that $s < n$, are equivalent norms of $B^s_{p,q}(\mathbf{R}^d)$ for the given range of parameters.

The technique introduced in [DR14] is based on two ideas. The first is the following analytic lemma, which provides a quantitative integration by parts. The lemma is implicitly given in [DR14] and explicitly stated and proved in [Rom14c].

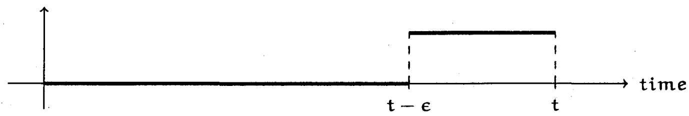
Lemma 3.1 (smoothing lemma). *If μ is a finite measure on $\mathbf{R}^d$ and there are an integer $m \geqslant 1$, two real numbers $s > 0$, $\alpha \in (0,1)$, with $\alpha < s < m$, and a constant $c_1 > 0$ such that for every $\phi \in C^\alpha_b(\mathbf{R}^d)$ and $h \in \mathbf{R}^d$,*

$$\Big|\int_{\mathbf{R}^d} \Delta^m_h \phi(x)\, \mu(dx)\Big| \leqslant c_1 |h|^s \|\phi\|_{C^\alpha_b},$$

then μ has a density f_μ with respect to the Lebesgue measure on $\mathbf{R}^d$ and $f_\mu \in B^r_{1,\infty}$ for every $r < s - \alpha$. Moreover, there is $c_2 = c_2(r)$ such that

$$\|f_\mu\|_{B^r_{1,\infty}} \leqslant c_2 c_1.$$

The second idea is to use the random perturbation to perform the "fractional" integration by parts along the noise to be used in the above lemma. The bulk of this idea can be found in [FP10]. Our method is based on the one hand on the idea that the Navier–Stokes dynamics is "good" for short times, and on the other hand that Gaussian processes have smooth densities. When trying to estimate the Besov norm of the density, we approximate the solution by splitting the time interval in two parts,



On the first part the approximate solution u_ϵ is the same as the original solution, on the second part the non–linearity is killed. By Gaussianity this is enough to estimate the increments of the density of u_ϵ. Since u_ϵ is the one-step explicit Euler approximation of u, the error in replacing u by u_ϵ can be estimated in terms of ϵ. By optimizing the increment versus ϵ we have an estimate on the derivatives of the density.

The final result is given in the proposition below. In comparison with Theorem 5.1 of [DR14], here we give an explicit dependence of the Besov norm of the density with respect to time. The estimate looks not optimal though.

The regularity of the density can be slightly improved from $B^{1-}_{1,\infty}$ to $B^{2-}_{1,\infty}$ if u is the stationary solution, namely the solution whose statistics are independent from time.

Proposition 3.2. *Given $x \in H$ and a finite dimensional subspace F of $D(A)$ generated by the eigenvectors of A, namely $F = \operatorname{span}[e_{n_1}, \dots, e_{n_F}]$ for some arbitrary indices $n_1, \dots, n_F$, assume that $\pi_F \mathcal{S}$ is invertible on F. Then for every $t > 0$ the projection $\pi_F u(t)$ has an almost everywhere positive density $f_{F,t}$ with respect to the Lebesgue measure on F, where u is any solution of (2.1) which is limit point of the spectral Galerkin approximations.*

Moreover, for every $\alpha \in (0,1)$, $f_{F,t} \in B^\alpha_{1,\infty}(\mathbf{R}^d)$ and for every small $\epsilon > 0$, there exists $c_3 = c_3(\epsilon) > 0$ such that

$$\|f_{F,t}\|_{B^\alpha_{1,\infty}} \leqslant \frac{c_3}{(1 \wedge t)^{\alpha+\epsilon}} (1 + \|x\|^2_H).$$

Proof. Given a finite dimensional space F as in the statement of the proposition, fix $t > 0$, and let $\alpha \in (0,1)$, $\phi \in C^\alpha_b$, and $h \in F$, with $|h| \leqslant 1$. For $m \geqslant 1$, consider two cases. If $|h|^{2n/(2\alpha+n)} < t$, then we use the same estimate in [DR14] to get

$$\big|\mathbb{E}[\Delta_h^m \phi(\pi_F u(t))]\big| \leqslant c_4(1 + \|x\|_H^{2\alpha})\|\phi\|_{C_b^\alpha}|h|^{\frac{2n\alpha}{2\alpha+n}}.$$

If on the other hand $t \leqslant |h|^{2n/(2\alpha+n)}$, we introduce the process u_ϵ as above, but with $\epsilon = t$. As in [DR14],

$$\mathbb{E}[\Delta_h^m \phi(\pi_F u(t))] = \mathbb{E}[\Delta_h^m \phi(\pi_F u_\epsilon(t))] + \mathbb{E}[\Delta_h^m \phi(\pi_F u(t)) - \Delta_h^m \phi(\pi_F u_\epsilon(t))]$$

and

$$\big|\mathbb{E}[\Delta_h^m \phi(\pi_F u(t)) - \Delta_h^m \phi(\pi_F u_\epsilon(t))]\big| \leqslant c_5(1 + \|x\|_H^{2\alpha})\|\phi\|_{C_b^\alpha} t^\alpha.$$

For the probabilistic error we use the fact that $u_\epsilon(t)$ is Gaussian, hence

$$\big|\mathbb{E}[\Delta_h^m \phi(\pi_F u_\epsilon(t))]\big| \leqslant c_6\|\phi\|_\infty \Big(\frac{|h|}{\sqrt{t}}\Big)^{\frac{2n\alpha}{2\alpha+n}}$$

In conclusion, from both cases we finally have

$$\big|\mathbb{E}[\Delta_h^m \phi(\pi_F u(t))]\big| \leqslant c_7(1 + \|x\|_H^2)\|\phi\|_{C_b^\alpha}|h|^{\frac{2n\alpha}{2\alpha+n}}(1 \wedge t)^{-\frac{n\alpha}{2\alpha+n}}.$$

The choice of n and α yields the final result. □

Remark 3.3. In [DR14] we introduced three different methods to prove existence of densities. The first method is based on the Markov machinery developed in [FR08] (see also [DPD03]), while the third one is the one on Besov bounds detailed above. A second possibility is to use an appropriate version of the Girsanov change of measure. It turns out that, together, the Girsanov change of measure and the Besov bounds yield time regularity of the densities of finite dimensional projections [Rom14c].

As it may be expected, the time regularity obtained is "half" the space regularity, and the density is at most $\frac{1}{2}$ Hölder in time with values in $B_{1,\infty}^\alpha$, for $\alpha < 1$.

Remark 3.4. An apparent drawback of the method is that it can only handle finite dimensional projections. There are interesting functions of the solution, the energy for instance, that cannot be seen in any way as finite dimensional projections. On the other hand, one can use the same ideas (fractional integration by parts and smoothing effect of the noise) directly on such quantities.

Following this idea, in [Rom14b] it is shown that the two quantities $\|u(t)\|_{H^{-s}}^2$ and $\int_0^t \|u(t)\|_{H^{1-s}}^2$, with $s < \frac{3}{4}$, have a density. Unfortunately, there is a regularity issue that prevents getting densities when $s \geqslant \frac{3}{4}$, unless $s = 0$. The special quantity

$$\|u(t)\|_H^2 + 2\nu\int_0^t \|u(s)\|_V^2\,ds,$$

which represents the energy balance and is quite relevant in the theory, admits a density. This is possible due to the fundamental cancellation property of the Navier–Stokes non–linearity.

Remark 3.5. An interesting question, that has been completely answered for the two–dimensional case in [MP06], concerns the existence of densities when the covariance of the driving noise is essentially non–invertible. The typical perturbation in (1.1) we consider here is

$$\dot\eta(t,y) = \sum_{k\in\mathcal{K}} \sigma_k\dot\beta_k(t)e_k(y),$$

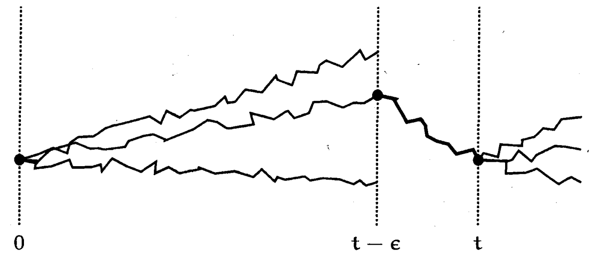


FIGURE 1. The strategy for the transfer principle: we only look at the smooth solution immediately before the evaluation time.

where $\mathcal{Z} \neq \mathbf{Z}^3_\star$ and is usually much smaller (finite, for instance). The idea is that the noise influence is spread, by the non–linearity, to all Fourier components. The condition that should ensure this has been already well understood [Rom04], and corresponds to the fundamental algebraic property that $\mathcal{K}$ should generate the whole group $\mathbf{Z}^3$.

It is clear that the method we have used to obtain Besov bounds cannot work in this case, because the non–linearity plays a major role. On the other hand in [Rom14a] we prove, using ideas similar to those leading to the transfer principle (Theorem 4.1), the existence of a density. No regularity properties are possible, though.

4. THE TRANSFER PRINCIPLE

In this final section we present two results in the direction of extending results proved only for a special class of solutions (limits of spectral Galerkin approximations in [DR14]) to every *legit* solution of (1.1). As already mentioned, the transfer principle allows the extension of instantaneous properties, namely properties that depend on a single time.

Given $t_0 > 0$, consider the following event in Ω_{NS},

$$L(t_0) = \big\{\omega : \text{ there is } \epsilon > 0 \text{ such that } \sup_{t\in[t_0-\epsilon,t_0]} \|A\omega(t)\|_H < \infty\big\}.$$

From [Rom14b] we know that, if $(\mathscr{C}(x))_{x\in H}$ is a *legit* family, if $x \in H$ and $\mathbb{P} \in \mathscr{C}(x)$, then $\mathbb{P}[L(t_0)] = 1$ for a.e. $t_0 > 0$. To be more precise, the proof is given in [Rom14b] only for those *legit* families introduced in [FR08] and [Rom10], but the two crucial properties used in the proof of the probability one statement are exactly those defining a *legit* family.

Our main theorem is given below. The intuitive idea is that if we are able to prove existence of a density (with respect to a suitable Lebesgue measure) for a finite dimensional functional of a solution, then the same holds for any other solution, regardless of the way we were able to produce it.

In other words, we can prove existence of a density for solutions obtained from Galerkin approximation, and this result will extend straight away to any other solutions, for instance those produced by the Leray regularization (see for instance [Lio96]). Or we can use the special properties of Markov solutions given in [FR08, Rom10] to prove existence of densities of a large class of finite dimensional functionals, as done in the first part of [DR14], and again this extends immediately to any (*legit*) solution.

Theorem 4.1 (Transfer principle). *Let $d \geqslant 1$ and let $F : D(A) \to \mathbf{R}^d$ be a measurable function. Assume that we are given a legit class $(\mathscr{C}(x))_{x\in H}$ and a family $(\mathbb{Q}_x)_{x\in H}$ of solutions of* (1.1) *satisfying (only) weak–strong uniqueness.*

If for every $x \in D(A)$ and almost every $t_0 > 0$ the random variable $\omega \mapsto F(\omega(t_0))$ on $(\Omega_{NS}, \mathscr{F}^{NS}, \mathbb{Q}_x)$ has a density with respect to the Lebesgue measure on $\mathbf{R}^d$, then for every $x \in H$, every $\mathbb{P} \in \mathscr{C}(x)$ and almost every $t_0 > 0$, the random variable $\omega \mapsto F(\omega(t_0))$ on $(\Omega_{NS}, \mathscr{F}^{NS}, \mathbb{P})$ has a density with respect to the Lebesgue measure on $\mathbf{R}^d$.

Proof. Following [Rom14b], consider for every $\epsilon \leqslant 1$ and every $R \geqslant 1$ the event $L_{\epsilon,R}(t_0)$ defined as

$$L_{\epsilon,R}(t_0) = \{\sup_{t\in[t_0-\epsilon,t_0]} \|A\omega(t)\|_H \leqslant R\}.$$

Clearly $L(t_0) = \bigcup L_{\epsilon,R}(t_0)$. Given a measurable function F as in the standing assumptions, a Lebesgue null set $E \subset \mathbf{R}^d$, a state $x \in H$ and a solution $\mathbb{P} \in \mathscr{C}(x)$,

$$\mathbb{P}[F(\omega(t_0)) \in E] = \sup_{\epsilon\leqslant 1, R\geqslant 1} \mathbb{P}[\{F(\omega(t_0)) \in E\} \cap L_{\epsilon,R}(t_0)].$$

Given $\epsilon \leqslant 1$ and $R \geqslant 1$, we condition $\mathbb{P}$ at time $t_0 - \epsilon$ and we know that $\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t_0-\epsilon}} \in \mathscr{C}(\omega(t_0-\epsilon))$, where $\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t_0-\epsilon}}$ is the regular conditional probability distribution of $\mathbb{P}$ given $\mathscr{F}^{NS}_{t_0-\epsilon}$. Hence, using weak–strong uniqueness,

$$\begin{aligned}
\mathbb{P}[\{F(\omega(t_0)) \in E\} \cap L_{\epsilon,R}(t_0)] &= \mathbb{E}^{\mathbb{P}}\big[\mathbb{P}[\{F(\omega(t_0)) \in E\} \cap L_{\epsilon,R}(t_0) | \mathscr{F}^{NS}_{t_0-\epsilon}]\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t_0-\epsilon}}[F(\omega'(\epsilon)) \in E, \tau_R \geqslant \epsilon]\mathbb{1}_{A_{\epsilon,R}}\big] \\
&\leqslant \mathbb{E}^{\mathbb{P}}\big[\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t_0-\epsilon}}[F(\omega'(\epsilon)) \in E, \tau_{2R} > \epsilon]\mathbb{1}_{A_{\epsilon,R}}\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[\mathbb{P}^{2R}_{\omega(t_0-\epsilon)}[F(u_{2R}(\epsilon)) \in E, \tau_{2R} > \epsilon]\mathbb{1}_{A_{\epsilon,R}}\big],
\end{aligned}$$

where $A_{\epsilon,R} = \{\|A\omega(t_0 - \epsilon)\|_H \leqslant R\}$. Again by weak–strong uniqueness, $\mathbb{P}^{2R}_y$ and $\mathbb{Q}_y$ agree on the event $\{\tau_{2R} > \epsilon\}$ of positive probability for every y with $\|Ay\|_H \leqslant R$, hence for all such y,

$$\mathbb{P}^{2R}_y[F(u_{2R}(\epsilon)) \in E, \tau_{2R} > \epsilon] = 0.$$

Therefore

$$\mathbb{P}[\{F(\omega(t_0)) \in E\} \cap L_{\epsilon,R}(t_0)] = 0$$

for every $\epsilon \leqslant 1$ and every $R \geqslant 1$. In conclusion $\mathbb{P}[F(\omega(t_0)) \in E] = 0$. □

The previous theorem has two crucial drawbacks. The first is that it deals only with instantaneous properties, namely properties depending only on one single time, and it looks hardly possible, by the nature of the proof, that the principle might ever be extended, at this level of generality, to multi–time statements, such as the existence of a joint density for multiple times (see Remark 4.3 in [DR14]).

The second drawback is that the result is qualitative in nature. Whenever one can find quantitative bounds on the density, such as the Besov bounds in [DR14], it is again a non–trivial task, one that the present author is not able to figure out in general, to prove that the bounds are "universal", hence true for any solution.

If we try to repeat the proof of our main theorem above, with the purpose of extending the Besov bound in a quantitative way, in general we are doomed to failure. Proposition 3.2 above shows that the control of the Besov norm of the density becomes singular for short times. This is clearly expected when the initial condition is deterministic.

Let us try to understand what is preventing us from getting a quantitative estimate, for instance in the case the finite dimensional map of the Theorem above is a finite–dimensional projection as in Proposition 3.2. We proceed in a slightly different way, following loosely the idea of Lemma 3.7 in [Rom08]. Let $\mathbb{P}$ be a weak solution of (1.1) from a *legit* class, and fix $\phi \in L^\infty(F)$ with $\|\phi\|_\infty \leqslant 1$, $h \in F$ with $|h| \leqslant 1$, and $m \geqslant 1$ large. For $\epsilon \in (0,1)$ and $R \geqslant 1$ set

$$A_{\epsilon,R} = \{\|A\omega(t-\epsilon)\|_H \leqslant R\}, \qquad B_{\epsilon,R} = \Big\{\sup_{[t-\epsilon,t]} \|A\omega(s)\|_H \leqslant 2R\Big\}.$$

We have

$$\mathbb{E}^{\mathbb{P}}[\Delta_h^m \phi(\pi_F\omega(t))] = \mathbb{E}[\Delta_h^m \phi(\pi_F\omega(t))\mathbb{1}_{B_{\epsilon,R}\cap A_{\epsilon,R}}] + \boxed{\text{error}},$$

where the error can be simply estimated as

$$\boxed{\text{error}} \leqslant \mathbb{P}[B_{\epsilon,R}^c \cup A_{\epsilon,R}^c] \leqslant \mathbb{P}[B_{\epsilon,R}^c \cap A_{\epsilon,R}] + \mathbb{P}[A_{\epsilon,R}^c].$$

For the second term of the error, there is not much we can do, so we keep it unchanged. As it regards the first term, we exploit the *legit* class assumption on $\mathbb{P}$ and use Proposition 3.5 in [FR07] (or [Rom11, Proposition 5.7]) to get,

$$\mathbb{P}[B_{\epsilon,R}^c \cap A_{\epsilon,R}] = \mathbb{E}^{\mathbb{P}}[\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t-\epsilon}}[\tau_{2R} \leqslant \epsilon]\mathbb{1}_{A_{\epsilon,R}}] \leqslant c_8\, e^{-c_9\frac{R^2}{\epsilon}}, \tag{4.1}$$

if $R^2\epsilon \leqslant c_{10}$, for some constants $c_8, c_9, c_{10} > 0$. Finally, again by the *legit* class condition,

$$\mathbb{E}^{\mathbb{P}}[\Delta_h^m \phi(\pi_F\omega(t))\mathbb{1}_{B_{\epsilon,R}\cap A_{\epsilon,R}}] = \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t-\epsilon}}}[\Delta_h^m \phi(\pi_F\omega(\epsilon))\mathbb{1}_{\{\tau_{2R}\geqslant\epsilon\}}]\mathbb{1}_{A_{\epsilon,R}}\big].$$

On the event $\{\tau_{2R} \geqslant \epsilon\}$, by weak–strong uniqueness, the inner expectation does not depend on $\mathbb{P}$, but only on the smooth solution starting from $\omega(t-\epsilon)$, in particular the Besov estimate holds and for $\alpha \in (0,1)$, by Proposition 3.2,

$$\mathbb{E}^{\mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t-\epsilon}}}[\Delta_h^m \phi(\pi_F\omega(\epsilon))\mathbb{1}_{\{\tau_{2R}\geqslant\epsilon\}}] \leqslant \frac{c_3}{\epsilon^{\alpha+\delta}}(1 + \|\omega(t-\epsilon)\|_H^2)|h|^\alpha + \mathbb{P}|^{\omega}_{\mathscr{F}^{NS}_{t-\epsilon}}[\tau_{2R} \leqslant \epsilon].$$

Using again [FR07, Proposition 3.5] as in (4.1),

$$\mathbb{E}^{\mathbb{P}}[\Delta_h^m \phi(\pi_F\omega(t))\mathbb{1}_{B_{\epsilon,R}\cap A_{\epsilon,R}}] \leqslant \frac{c_3}{\epsilon^{\alpha+\delta}}(1 + \mathbb{E}[\|\omega(t-\epsilon)\|_H^2])|h|^\alpha + c_8\, e^{-c_9\frac{R^2}{\epsilon}},$$

with δ small (so that $\alpha + \delta < 1$). In conclusion,

$$\begin{aligned}|\mathbb{E}^{\mathbb{P}}[\Delta_h^m \phi(\pi_F\omega(t))]| \leqslant{}& \frac{c_3}{\epsilon^{\alpha+\delta}}(1 + \mathbb{E}[\|\omega(t-\epsilon)\|_H^2])|h|^\alpha + 2c_8\, e^{-c_9\frac{R^2}{\epsilon}} + \\ &+ \mathbb{P}[A_{\epsilon,R}^c]\end{aligned}$$

Choose $R \approx \epsilon^{-1/2}$, so that the constraint $\epsilon R^2 \leqslant c_{10}$ is satisfied. Integrate the above inequality over $\epsilon \in (0,\epsilon_0)$, $\epsilon_0 \leqslant 1$, and use the moment in $D(A)$ proved in [Rom14a] to get

$$|\mathbb{E}^{\mathbb{P}}[\Delta_h^m \phi(\pi_F\omega(t))]| \leqslant c_{11}(1 + \|x\|_H^2)\Big(\frac{|h|^\alpha}{\epsilon_0^{\alpha+\delta}} + e^{-\frac{c_9}{\epsilon}}\Big) + \frac{1}{\epsilon_0}\int_0^{\epsilon_0} \mathbb{P}[A_{\epsilon,R}^c]\, d\epsilon. \tag{4.2}$$

Neglect, only for a moment, the last term. The choice $\epsilon = -c_{12}/\log|h|$ would finally give the same result as Proposition 3.2 (up to a logarithmic correction). Unfortunately the last term prevents this computation. There is no much we can do here, our best option seems Chebychev's inequality and the $\frac{2}{3}$–moment in $D(A)$ proved in [Rom14a],

$$\begin{aligned}\frac{1}{\epsilon_0}\int_0^{\epsilon_0}\mathbb{P}[A^c_{\epsilon,R}]\,d\epsilon &\leqslant \frac{1}{\epsilon_0}\mathbb{E}^{\mathbb{P}}\Big[\int_0^{\epsilon_0}\frac{1}{R^{\frac{2}{3}}}\|A\omega(t-\epsilon)\|_H^{\frac{2}{3}}\,d\epsilon\Big]\\ &\leqslant \epsilon_0^{-\frac{2}{3}}\mathbb{E}^{\mathbb{P}}\Big[\int_0^t\|A\omega\|_H^{\frac{2}{3}}\,ds\Big],\end{aligned}\tag{4.3}$$

and no quantitative counterpart of the transfer principle can be proved in this way.

Notice that the same technique is successful in [Rom08]. The reason is that in the mentioned paper (a different form of) the transfer principle was used for moments of the solution. Here we are estimating the size of an increment. It makes a non–trivial difference, since here the crucial mechanism is the smoothing effect of the random perturbation, as it can be seen in a simple example with a one dimensional Brownian motion $(B_t)_{t\geqslant 0}$. Indeed, it is elementary to compute that $|\mathbb{E}[\phi(B_{t+h})-\phi(B_t)]|$ is bounded by $\|\phi\|_\infty\frac{h}{t}$, where $\frac{h}{t}$ is the total variation distance between the laws of B_t and B_{t+h}. On the other hand, the seemingly similar quantity $\mathbb{E}[|\phi(B_{t+h})-\phi(B_t)|]$ is much worse.

There is one case though where our computations above can be carried on. If we assume that u is a stationary solution, with time marginal μ, the quantity in (4.3) can be estimated as (recall that $R\approx\epsilon^{-\gamma}$),

$$\frac{1}{\epsilon_0}\int_0^{\epsilon_0}\mathbb{P}[A^c_{\epsilon,R}]\,d\epsilon \leqslant \frac{1}{\epsilon_0}\mathbb{E}^{\mathbb{P}}\Big[\int_0^{\epsilon_0}\frac{1}{R^{\frac{2}{3}}}\|A\omega(t-\epsilon)\|_H^{\frac{2}{3}}\,d\epsilon\Big] \leqslant \mathbb{E}^{\mu}[\|Ax\|_H^{\frac{2}{3}}]\epsilon_0^{\frac{1}{3}},$$

and (4.2) this time reads,

$$|\mathbb{E}^{\mathbb{P}}[\Delta_h^m\phi(\pi_F\omega(t))]| \leqslant c_{13}\Big(\epsilon_0^{\frac{1}{3}}+\frac{|h|^\alpha}{\epsilon_0^{\alpha+\delta}}\Big).$$

A suitable choice of ϵ_0 by optimization and Lemma 3.1 show that the density is Besov. Since α can run over all values in $(0,2)$ by Theorem 5.2 in [DR14], we obtain the following result.

Theorem 4.2 (Quantitative transfer principle). *Let $d\geqslant 1$ and consider a d dimensional sub–space F of $D(A)$ spanned by a finite number of eigenvectors of the Stokes operator.*

Assume that we are given a legit class $(\mathscr{C}(x))_{x\in H}$, and let $\mathbb{P}_\star$ a stationary solution whose conditional probabilities at time 0 are elements of the legit class $(\mathscr{C}(x))_{x\in H}$.

Denote by $u_\star$ a process with law $\mathbb{P}_\star$, then $\pi_F u_\star(t)$ has a density with respect the Lebesgue measure on F. Moreover, the density belongs to the Besov space $B^\alpha_{1,\infty}(F)$ for every $\alpha<\frac{2}{7}$.

One can slightly improve the exponent $\frac{2}{7}$ by using moments in a different topology than $D(A)$, see Remark 2.2.

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