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DECAY CHARACTERIZATION FOR SOLUTIONS TO DISSIPATIVE EQUATIONS IN TERMS OF THE INITIAL DATUM

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ABSTRACT. By examining the Fourier transform of the initial datum near the origin, we define the *decay character* of the datum and provide a method to study the lower and upper algebraic rates of decay of solutions to a wide class of dissipative system of equations.

1. INTRODUCTION

We address the study of decay rates of solutions to nonlinear dissipative evolution equations satisfying the energy inequality

$$(E) \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u(x,t)|^2\,dx \leq -C\int_{\mathbb{R}^3} |\nabla^\alpha u(x,t)|^2\,dx,$$

where $\alpha \in (0,1]$. The characterization of the decay rates is given first for a class of linear systems by introducing the concept of *decay character*, a number associated to the initial datum that describes its behavior near the origin in frequency space. We then study nonlinear systems with the underlying linear systems for which we have already obtained decay rates. The decay character and the Fourier Splitting method are then used to obtain upper and lower bounds for decay of solutions to appropriate nonlinear dissipative equations, both in the incompressible and compressible case. The method derived in this paper can be applied to most of the equations that satisfy (E). It works for systems like Navier-Stokes, MHD, Quasi-Geostrophic equations and certain compressible systems.

We recall the *original question of Leray*: how does the L^2-energy decay for weak solutions of the Navier-Stokes equations?. We would like to use the decay character in order to give a concise answer to this question not only for the solutions to the Navier-Stokes equations, but for the class of all solutions to dissipative systems satisfying (E). Our goal is to, given the decay character of the initial datum, know whether the solution with that initial datum has uniform decay or not and, if there is uniform decay, what are the upper and lower bounds for these rates.

In this note we only present the results and give ideas of the proofs. The details can be found in [6]. The main basis for the proofs are:

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(1) "Behavior of solutions for large time is determined by low frequencies of the solutions";
(2) Use a time depending filter to study the low frequencies, this is the *Fourier Splitting method* [9], [10].

1.1. **Background.** For the *heat equation in* $\mathbb{R}^n$ it is very easy to see that the decay depends on the behavior of the data near the origin in frequency space. For completeness we describe what happens for such solutions. Let $u = u(x,t)$ be a solution to the heat equation

$$u_t - \Delta u = 0, \ u_0(x) = u(x,0).$$

Then

$$u(x,t) = G_t * u_0(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x).$$

We study now the different possible decay rates.

1.1.1. *Exponential decay.* Let $u_0 \in L^2(\mathbb{R}^n)$ and $\widehat{u_0}(\xi) = 0$, when $|\xi| < \delta$. Then

$$\|\widehat{u}(t)\|^2_{L^2} = \int_{|\xi|>\delta} e^{-8\pi|\xi|^2 t} |\widehat{u_0}(\xi)|^2 \, d\xi \le C\, e^{-8\pi\delta^2 t}.$$

1.1.2. *Slow decay.* Let $\mathcal{B} = \{v : \|v\|_{L^2} = 1\}$. Let $u_0^\lambda(x) = \lambda^{\frac{n}{2}} e^{-\pi\frac{|\lambda x|^2}{2}}$, then $u_0^\lambda(x) \in \mathcal{B}$. However, the norm of the gradient scales as $\|\nabla u_0^\lambda\|_{L^2} = \pi\lambda\|\nabla u_0\|_{L^2}$, so when λ gets smaller, the norm of the gradient gets smaller too, so the right hand side of (E) produces slow decay. Namely, for any fixed $t > 0$, decay for solutions with data $u_0^\lambda \in \mathcal{B}$ will not be uniform, as

$$\frac{\|\widehat{u^\lambda}(t)\|^2_{L^2}}{\|\widehat{u_0^\lambda}\|^2_{L^2}} = \frac{1}{1+4\lambda^2 t} \xrightarrow{\lambda\to 0} 1.$$

So, there exist solutions to the heat equation with data in $L^2(\mathbb{R}^n)$ decaying arbitrarily slowly.

Proposition 1.1. *Given $r, T, \epsilon > 0$, there exists u_0 with $\|u_0\|_{L^2} = r$ so that for the solution to the heat equation with that initial datum.*

$$\frac{\|u(T)\|_{L^2}}{\|u_0\|_{L^2}} \ge 1 - \epsilon.$$

1.1.3. *Algebraic decay.* This is easily seen when $|\widehat{u}_0(\xi)| \approx C|\xi|^k$ and when $|\widehat{u}_0(\xi)| \ge C > 0$, see [1].

1.2. **Ideas for characterizing decay decay.** To characterize the L^2 and/or Sobolev decay of solutions to dissipative equations, we will:

(1) characterize the initial data;
(2) understand behavior of solution to the underlying linear equation;
(3) study influence of the non linear part;
(4) study of difference of linear and nonlinear solutions;
(5) use a reverse triangle inequality to obtain lower bounds of rates of decay.

2. Characterization of the initial datum

In this section we introduce the definitions of the *decay indicator*, *decay character*, *s-decay indicator* and *s-decay character*.

2.1. Definitions.

Definition 2.1. ([1], [6]) Let $u_0 \in L^2(\mathbb{R}^n)$, $r \in \left(-\frac{n}{2}, \infty\right)$. The *decay indicator* $P_r(u_0)$ of u_0 is defined by

$$P_r(u_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\widehat{u_0}(\xi)|^2 \, d\xi,$$

where $B(\rho) = \{\xi : |\xi| \le \rho\}$.

Remark 2.1. The decay indicator compares $|\widehat{u_0}(\xi)|$ with $f(\xi) = |\xi|^r$ at $\xi = 0$. □

Definition 2.2. ([1], [6]) Let $u_0 \in L^2(\mathbb{R}^n)$. The *decay character of* u_0 is $r^* = r^*(u_0)$, the unique $r \in \left(-\frac{n}{2}, \infty\right)$ such that $0 < P_r(u_0) < \infty$, if this number exists. If it does not exist then

$$r^*(u_0) = \begin{cases} -\frac{n}{2}, & \text{if } P_r(u_0) = \infty, \text{ for all } r \in \left(-\frac{n}{2}, \infty\right) \\ \infty, & \text{if } P_r(u_0) = 0, \text{ for all } r \in \left(-\frac{n}{2}, \infty\right). \end{cases}$$

Definition 2.3. ([6]) Let $u_0 \in L^2(\mathbb{R}^n)$, $s > 0$, $r \in \left(-\frac{n}{2} + s, \infty\right)$. The *s-decay indicator* $P_r^s(u_0)$ of $\Lambda^s u_0$ is defined as

$$P_r^s(u_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\xi|^{2s} |\widehat{u_0}(\xi)|^2 \, d\xi$$

where $B(\rho) = \{\xi : |\xi| \le \rho\}$.

Remark 2.2. The s-decay indicator compares $|\Lambda^s u_0(\xi)|$ with $f(\xi) = |\xi|^r$ at $\xi = 0$. □

Definition 2.4. ([6]) The *s-decay character of* $\Lambda^s u_0$ is $r_s^* = r_s^*(u_0)$, the unique $r \in \left(-\frac{n}{2} + s, \infty\right)$ such that $0 < P_r^s(u_0) < \infty$, provided this number exists. If it does not exist then

$$r_s^*(u_0) = \begin{cases} \infty, & \text{if } P_r(u_0) = 0, \text{ for all } q \in \left(-\frac{n}{2} + s, \infty\right) \\ -\frac{n}{2} + s, & \text{if } P_r(u_0) = \infty, \text{ for all } q \in \left(-\frac{n}{2} + s, \infty\right). \end{cases}$$

Remark 2.3. If $u_0 \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), 1 \le p \le 2$, then $r^*(u_0) = -n\left(1 - \frac{1}{p}\right)$. So, if $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $r^*(u_0) = 0$. If $u_0 \in L^2(\mathbb{R}^n)$ but is not in any $L^p(\mathbb{R}^n)$, with $1 \le p < 2$, then $r^*(u_0) = -\frac{n}{2}$. □

2.2. **Results.** Here we state a theorem that shows the relation between the decay character and the s-decay character. Heuristically, if $\widehat{u_0}(\xi)$ is like $|\xi|^r$ near $\xi = 0$, then $\widehat{\Lambda^s u_0}(\xi)$ must be like $|\xi|^{r+s}$ near $\xi = 0$. Then the decay character $r^*(u_0)$ and the s-decay character $r_s^*(u_0)$ should be related through $r^* + s = r_s^*$.

Theorem 2.5. *(Theorem 2.11, [6]) Let $u_0 \in H^s(\mathbb{R}^n), s > 0$.*

(1) *If $-\frac{n}{2} < r^*(u_0) < \infty$ then $-\frac{n}{2} + s < r_s^*(u_0) < \infty$ and $r_s^*(u_0) = s + r^*(u_0)$.*
(2) *$r_s^*(u_0) = \infty$ if and only if $r^*(u_0) = \infty$.*
(3) *$r^*(u_0) = -\frac{n}{2}$ if and only if $r_s^*(u_0) = r^*(u_0) + s = -\frac{n}{2} + s$.*

3. Linear Part: examples and decay

3.1. **Linear Part.** Let $\mathcal{L} : X^n \to \left(L^2(\mathbb{R}^n)\right)^n$ be a pseudodifferential operator on a Hilbert space X for which the symbol $\mathcal{M}(\xi)$ of $\mathcal{L}$ is such that

$$\mathcal{M}(\xi) = P^{-1}(\xi)D(\xi)P(\xi), \qquad \xi - a.e.$$

where $P(\xi) \in O(n)$ and $D(\xi) = -c_i|\xi|^{2\alpha}\delta_{ij}$, for $c_i > c > 0$ and $0 < \alpha \leq 1$. The Laplacian and the fractional Laplacian are examples of such operators.

Given the linear equation

$$\partial_t v = \mathcal{L}v$$

multiplying by v, integrating in space and using properties of $\mathcal{L}$ we obtain

$$\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R}^n)} \leq -C\int_{\mathbb{R}^n} |\xi|^{2\alpha}|\widehat{v}(\xi,t)|^2\,d\xi,$$

which is inequality (E).

Example 3.1. Temam [11] introduced the following compressible approximation to Navier-Stokes equations

$$u_t = \mathcal{L}u = \Delta u + \frac{1}{\epsilon}\nabla div\, u, \qquad \epsilon > 0 \tag{3.1}$$

where the relation $\epsilon p = -div\, u$ eliminates the nonlocal relation between the pressure and the velocity. The symbol for this operator is $(\mathcal{M}(\xi))_{ij} = -|\xi|^2\delta_{ij} - \frac{1}{\epsilon}\xi_i\xi_j$, with $D(\xi) = diag(-|\xi|^2, -|\xi|^2, -\left(1+\frac{1}{\epsilon}\right)|\xi|^2)$ and

$$P(\xi) = \begin{pmatrix} \frac{-\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{-\xi_1\xi_3}{\sqrt{1-\xi_3^2}} & \xi_1 \\ \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{-\xi_2\xi_3}{\sqrt{1-\xi_3^2}} & \xi_2 \\ 0 & \frac{1-\xi_3^2}{\sqrt{1-\xi_3^2}} & \xi_3 \end{pmatrix}$$

Then, the kernel is given by

$$\left(e^{t\mathcal{M}(\xi)}\right)_{ij} = e^{-t|\xi|^2}\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\left(e^{-t|\xi|^2} - e^{-\left(1+\frac{1}{\epsilon}\right)t|\xi|^2}\right), \tag{3.2}$$

see Rusin [8].

3.2. **Decay of linear part.** In this subsection we give the main decay theorems for the linear equations and give the sketch of some of the proofs.

Theorem 3.2. *(Theorem 2.10, [6]) Let $v_0 \in L^2(\mathbb{R}^n)$ have decay character $r^*(v_0) = r^*$. Let $v(t)$ be a solution to the linear equation with initial datum v_0. Then:*

(1) *if $-\frac{n}{2} < r^* < \infty$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1(1+t)^{-\frac{1}{\alpha}(\frac{n}{2}+r^*)} \leq \|v(t)\|_{L^2}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}(\frac{n}{2}+r^*)};$$

(2) *if $r^* = -\frac{n}{2}$, there exists $C = C(\epsilon) > 0$ such that*

$$\|v(t)\|_{L^2}^2 \geq C(1+t)^{-\epsilon}, \qquad \forall \epsilon > 0,$$

i.e. the decay of $\|v(t)\|_{L^2}^2$ is slower than any uniform algebraic rate;

(3) *if $r^* = \infty$, there exists $C > 0$ such that*

$$\|v(t)\|_{L^2}^2 \leq C(1+t)^{-m}, \qquad \forall m > 0,$$

i.e. the decay of $\|v(t)\|_{L^2}$ is faster than any algebraic rate.

Proof We only sketch the proof of (1). We first prove lower bounds for decay. Using the properties of the symbol of $\mathcal{L}$, we obtain

$$|e^{\mathcal{M}(\xi)t}\widehat{v_0}(\xi)| \geq Ce^{-ct\rho^{2\alpha}(t)}|\widehat{v_0}(\xi)|.$$

Then

$$\|v(t)\|_{L^2} \geq C\rho^{2r+n}e^{-ct\rho^{2\alpha}(t)}\rho^{-2r-n}\int_{B(\rho(t))}|\widehat{v_0}(\xi)|^2\,d\xi$$

$$\geq C\rho^{2r+n}e^{-t\rho^{2\alpha}(t)}.$$

Choosing $\rho(t) = \rho_0(1+t)^{-\frac{1}{2\alpha}}$ we have

$$\|v(t)\|_{L^2}^2 \geq C(1+t)^{-\frac{1}{\alpha}(\frac{n}{2}+r^*)}.$$

For the upper bounds, from

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \leq -C\rho^{2\alpha}(t)\int_{B^c(\rho(t))}|\widehat{v}(\xi)|^2\,d\xi,$$

using the Fourier Splitting method we obtain

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + \rho^{2\alpha}(t)\|v(t)\|_{L^2}^2 \leq C\rho^{2\alpha}(t)\int_{B(t)}|\widehat{v}(\xi)|^2\,d\xi.$$

As $P_r(u_0) < \infty$, there are $\rho_0 > 0, C > 0$ such that for $0 < \rho < \rho_0$

$$\rho^{-2r-n}\int_{B(\rho(t))}|\widehat{v_0}(\xi)|^2\,d\xi \leq C.$$

Then

$$\frac{d}{dt}\|v(t)\|_{L^2}^2\rho^{2\alpha}(t) \leq C\rho^{2\alpha+2r+n}(t).$$

and choosing $\rho(t) = m^{\frac{1}{2\alpha}}(1+t)^{-\frac{1}{2\alpha}}$, with $m > r + \frac{n}{2}$ and using the integrating factor $h(t) = (1+t)^m$, we obtain

$$\frac{d}{dt}\left((1+t)^m \|v(t)\|_{L^2}^2\right) \leq C(1+t)^{m-1-\frac{r}{\alpha}-\frac{n}{2\alpha}}. \quad \square$$

For the decay of derivatives in the case of solutions to linear equations we have the following result.

Theorem 3.3. *(Theorem 2.12, [6]) Let $v_0 \in H^s(\mathbb{R}^n)$, with $s > 0$, have decay character $r_s^* = r_s^*(v_0)$. Then the solution of the linear equation with datum v_0 satisfies:*

(1) *if $-\frac{n}{2} \leq r^* < \infty$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1(1+t)^{-\frac{1}{\alpha}\left(\frac{n}{2}+r^*+s\right)} \leq \|v(t)\|_{\dot{H}^s}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}\left(\frac{n}{2}+r^*+s\right)};$$

(2) *if $r^* = \infty$, then*

$$\|v(t)\|_{\dot{H}^s}^2 \leq C(1+t)^{-r}, \qquad \forall r > 0,$$

i.e. the decay of $\|v(t)\|_{\dot{H}^s}$ is faster than any algebraic rate.

4. Applications

4.1. **Quasi-Geostrophic equations.** In this section we study the upper and lower decay rates for solutions to the Dissipative Quasi-Geostrophic (DQGE) equation. The DQGE is given by

$$\theta_t + u \cdot \nabla\theta + (-\Delta)^\alpha \theta = 0, \qquad 0 < \alpha \leq 1$$

where θ is the potential temperature of a fluid in $\mathbb{R}^2$, and

$$u = R^\perp \theta = (-R_2\theta, R_1\theta)$$

is its velocity, where R_i is the Riesz transform in x_i. This equation models important geophysical phenomena ([5], [7]) and when $\alpha = \frac{1}{2}$ it provides a good model for $3D$ Navier-Stokes equations ([2], [4]).

4.1.1. *Results for DQGE.* The following Theorem summarizes all our results.

Theorem 4.1. *([6]) Let $\theta_0 \in L^2(\mathbb{R}^2)$, with decay character $r^* = r^*(u_0)$.*

(1) *If $r^* \leq 1 - \alpha$, then there exists constants $C_1, C_2 > 0$ so that*
$$C_1(1+t)^{-\frac{1}{\alpha}(1+r^*)} \leq \|\theta(t)\|_{L^2}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}(1+r^*)};$$

(2) *If $r^* \geq 1 - \alpha$, $r^* \leq \min\{1, 2(1-\alpha)\}$ then there exist $C_1, C_2 > 0$*
$$C_1(1+t)^{-\frac{1}{\alpha}(1+r^*)} \leq \|\theta(t)\|_{L^2}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}(2-\alpha)};$$

(3) *If $r^* > 1$ and $r^* \geq 2(1-\alpha)$ we have that*
$$\|\theta(t)\|_{L^2}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}(2-\alpha)}.$$

We now give partial statements and sketches of proofs.

Theorem 4.2. *(Theorem 3.1, [6]) Let $\theta_0 \in L^2(\mathbb{R}^2)$, let $r^* = r^*(\theta_0), -1 < r^* < \infty$, and $0 < \alpha \leq 1$. Let θ be a weak solution to QGE with data θ_0. Then:*

(1) *If $r^* \leq 1 - \alpha$, then*

$$\|\theta(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{1}{\alpha}(1+r^*)};$$

(2) *if $r^* \geq 1 - \alpha$, then*

$$\|\theta(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{1}{\alpha}(2-\alpha)}.$$

Proof We give formal estimates, which have to be proven rigurously by taking approximations and passing to the limit. Let

$$B(t) = \{\xi \in \mathbb{R}^2 : |\xi|^{2\alpha} \leq \frac{f'(t)}{2f(t)}\}.$$

Through the Fourier Splitting method we obtain

$$\frac{d}{dt}\left(f(t)\|\theta(t)\|_{L^2}^2\right) \leq f'(t)\int_{B(t)} |\widehat{u}(\xi,t)|^2\, d\xi$$

$$\leq Cf'(t)\left(\|\Theta(t)\|_{L^2}^2 + \int_{B(t)}\left(\int_0^t e^{-(t-s)|\xi|^{2\alpha}}|\xi||\widehat{u\theta}(\xi,s)|^2\, ds\right)^2 d\xi\right), \tag{4.3}$$

where Θ = is the solution to the linear part. First, we obtain a preliminary decay by choosing $f(t) = [\ln(e+t)]^{1+\frac{1}{\alpha}}$, $0 < \alpha < 1$ or $f(t) = [\ln(e+t)]^3$, for $\alpha = 1$. In this case

$$\|\theta(t)\|_{L^2}^2 \leq \|\Theta(t)\|_{L^2}^2 + C[\ln(e+t)]^{-(1+\frac{1}{\alpha})} \leq C[\ln(e+t)]^{-(1+\frac{1}{\alpha})}.$$

Now we bootstrap with the new choice $f = (t+1)^\beta$, $\beta \gg 1$. Plugging in the preliminary decay in (4.3) and dividing by $(t+1)^{\beta-\frac{2}{\alpha}+1}$ we obtain

$$\begin{aligned}(t+1)^{\frac{2}{\alpha}-1}\|\theta(t)\|_{L^2}^2 &\leq \|\theta_0\|_{L^2}^2(t+1)^{-(\beta-\frac{2}{\alpha}+1)} + C(t+1)^{\frac{1}{\alpha}-\frac{r^*}{\alpha}-1}\\ &+ C\int_0^t \frac{(s+1)^{1-\frac{2}{\alpha}}}{\ln(e+s)]^{1+\frac{1}{\alpha}}}(s+1)^{\frac{2}{\alpha}-1}\|\theta(s)\|_{L^2}^2\, ds.\end{aligned}$$

Define

$$\psi(t) = (1+t)^{\frac{2}{\alpha}-1}\|\theta(t)\|_{L^2}^2,\ a(t) = C(t+1)^{-\beta} + (1+t)^{\frac{1}{\alpha}-\frac{r^*}{\alpha}-1},$$
$$b(t) = C[\ln(e+t)]^{-(1+\frac{1}{\alpha})}(s+1)^{1-\frac{2}{\alpha}}.$$

and then use Gronwall's inequality to obtain final estimates. □.

For the derivatives the decay result is the following.

Theorem 4.3. *(Theorem 3.5, [6]) Let $\frac{1}{2} < \alpha \leq 1$, $\alpha \leq s$ and $\theta_0 \in H^s(\mathbb{R}^2)$. For $r^* = r^*(\theta_0)$ the solutions to QGE satisfy*

(1) *if $r^* \leq 1 - \alpha$, then*

$$\|\theta(t)\|_{\dot{H}^s}^2 \leq C(t+1)^{-\frac{1}{\alpha}(s+1+r^*)};$$

(2) *if $r^* \geq 1 - \alpha$, then*

$$\|\theta(t)\|_{\dot{H}^s}^2 \leq C(t+1)^{-\frac{1}{\alpha}(s+2-\alpha)}.$$

We now state the Theorem that deals with the decay of the difference between the linear and the nonlinear parts.

Theorem 4.4. *(Theorem 3.2, [6]) Let* $0 < \alpha \leq 1$, $\theta_0 \in L^2(\mathbb{R}^2)$. *Then*

(1) *if* $-1 < r^* \leq \alpha - 1$ *then*

$$\|\theta(t) - \Theta(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}(2-\alpha+r^*)};$$

(2) *if* $\alpha - 1 < r^* \leq 1 - \alpha$ *then*

$$\|\theta(t) - \Theta(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}\min\{2,2-\alpha+r^*\}};$$

(3) *if* $r^* \geq 1 - \alpha$, *then*

$$\|\theta(t) - \Theta(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}\min\{3-2\alpha,2\}}.$$

Proof The main term to estimate is

$$\left|\int_{\mathbb{R}^2} \Theta(u \cdot \nabla\theta)dx\right| \leq \|\nabla\Theta(t)\|_\infty \|\theta(t)\|_2^2 \leq C(1+t)^{-\gamma} = h(t).$$

For the proof of this theorem we use Fourier splitting and follow the ideas in [3]. □

The bounds for the difference between the linear solution and the nonlinear one, combined with the bounds of the linear solution yields the lower bounds of decay.

Theorem 4.5. *(Theorem 3.3, [6]) Let* $0 < \alpha \leq 1$, $\theta_0 \in L^2(\mathbb{R}^2), r^* = r^*(\theta_0)$. *Then, for* $0 < \alpha \leq \frac{1}{2}$ *and* $-1 < r^* \leq 1$ *or* $\frac{1}{2} < \alpha \leq 1$ *and* $-1 < r^* \leq 2(1-\alpha)$ *we have that*

$$\|\theta(t)\|_{L^2}^2 \geq C(1+t)^{-\frac{1}{\alpha}(1+r^*)}.$$

Proof Follows from the reverse triangle inequality

$$\|\theta(t)\|_{L^2}^2 \geq \|\Theta(t)\|_{L^2}^2 - \|\theta(t) - \Theta(t)\|_{L^2}^2,$$

provided that the linear part has slower decay than the difference between the solutions and the linear part. □.

4.2. **Approximation for compressible Navier-Stokes.** In the Navier-Stokes equations, the pressure is a nonlocal function of the velocity. This poses important problems when using numerical methods to study solutions to this system. Temam [11] introduced a compressible approximation to Navier-Stokes by relating the pressure and the velocity through $\nabla \cdot u = -\epsilon p$. In order to have an energy inequality, he stabilized the equation by introducing a term of the form $\frac{1}{2}(div\, u)\, u$. Then, the system obtained is

$$\begin{aligned} u_t^\epsilon + (u^\epsilon \cdot \nabla)\, u^\epsilon &+ \frac{1}{2}(div\, u^\epsilon)\, u^\epsilon = \Delta u^\epsilon + \frac{1}{\epsilon}\nabla div\, u^\epsilon, \\ u_0^\epsilon(x) &= u^\epsilon(x,0). \end{aligned}$$

The linear part of this system, i.e.

$$u_t = \mathcal{L}u = \Delta u + \frac{1}{\epsilon}\nabla \cdot div\, u = 0,$$
$$(\mathcal{M}_\epsilon(\xi,t))_{kl} = e^{-t|\xi|^2}\delta_{kl} - \frac{\xi_k\xi_l}{|\xi|^2}\left(e^{-t|\xi|^2} - e^{-(1+\frac{1}{\epsilon})t|\xi|^2}\right),$$

fits exactly in our setting, see Example 3.1. As the nonlinear part

$$(u\cdot\nabla)\, u + \frac{1}{2}\,(div\, u)\; u = \nabla\,(u\otimes u) - \frac{1}{2}\,(div\, u)\; u.$$

has a structure similar to Navier-Stokes and the compressible part can be easily handled since

$$\int_{\mathbb{R}^3} u(u\cdot\nabla)u\, dx = -\frac{1}{2}\int_{\mathbb{R}^3} |u|^2 div\, u\, dx,$$

we have the energy inequality of the form (E).

Remark 4.4. Rusin [8] proved existence of weak solutions to these system with u_0^ϵ in $L^2(\mathbb{R}^3)$. He also proved that when ϵ goes to zero, the solutions converge to suitable solutions of the Navier-Stokes system. □

4.2.1. *Results for the compressible approximation to Navier-Stokes.* Here we just list the results and for details refer the reader to [6]. The methods and techniques for the proofs are similar to those for the DQGE.

Theorem 4.6. *([6]) Let $u_0^\epsilon \in L^2(\mathbb{R}^3), r^* = r^*(u_0)$. Then for $-\frac{3}{2} < r^* \le 1$, there exist $C_1, C_2 > 0$ such that*

$$C_1(1+t)^{-(\frac{3}{2}+r^*)} \le \|u^\epsilon(t)\|_{L^2}^2 \le C_2(1+t)^{-(\frac{3}{2}+r^*)}.$$

If $r^* > 1$, then

$$\|u^\epsilon(t)\|_{L^2}^2 \le C(1+t)^{-\frac{5}{2}}.$$

Theorem 4.7. *(Theorem 3.14, [6]) Let $u_0 \in H^r(\mathbb{R}^3), r \ge 1,\ r^* = r^*(u_0)$. Then, for $1 \le s \le r$ we have that*

$$\|u(t)\|_{\dot{H}^s} \le C(1+t)^{-\frac{1}{2}\left(s+\min\{\frac{3}{2},r^*+\frac{3}{2}\}\right)}.$$

Theorem 4.8. *(Theorem 3.10, [6]) Let $\epsilon > 0,\ u_0^\epsilon \in L^2(\mathbb{R}^3)$, and $r^* = r^*(u_0)$ with $-\frac{3}{2} < r^* < \infty$. Then*

$$\|u^\epsilon(t) - \bar{u}(t)\|_{L^2}^2 \le C(1+t)^{-\min\{\frac{7}{4},\frac{7}{4}+r^*\}}$$

Theorem 4.9. *(Theorem 3.11, [6]) Let $u_0^\epsilon \in L^2(\mathbb{R}^3), r^* = r^*(u_0)$. Then for $-\frac{3}{2} < r^* \le 1$ we have that*

$$\|u^\epsilon(t)\|_{L^2}^2 \ge C(1+t)^{-(\frac{3}{2}+r^*)}.$$

Remark 4.5. The estimates for this compressible approximation are the same as those obtained by Bjorland and M.E. Schonbek [1] for the Navier-Stokes equations. Hence, the stabilizing nonlinear damping term $\frac{1}{2}(div\, u^\epsilon)u^\epsilon$ provides enough dissipation to have an energy inequality, but does not alter the range of values of r^* for which the linear part has slower decay. □

5. Final comments

(1) The decay character classifies the L^2 data for dissipative equations, at least when the linear part has slower decays. The linear part has to be studied first, then the whole nonlinear system.

(2) We are able to obtain information on both upper and lower decay rates, sometimes sharply characterizing the decay in terms of the initial data.

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