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An extension of existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces

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Abstract

In this paper we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore we prove a mean convergence theorem of Baillon's type for widely more generalized hybrid non-self mappings in Hilbert spaces.

1 Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H. In 2010, Kocourek, Takahashi and Yao [14] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2\leq\beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for any $x, y\in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive [19] for $\alpha=1$ and $\beta=0$, that is, $\|Tx-Ty\|\leq\|x-y\|$ for any $x, y\in C$. It is nonspreading [16] for $\alpha=2$ and $\beta=1$, that is, $2\|Tx-Ty\|^2\leq\|Tx-y\|^2+\|Ty-x\|^2$ for any $x, y\in C$. It is also hybrid [20] for $\alpha=\frac{3}{2}$ and $\beta=\frac{1}{2}$, that is, $3\|Tx-Ty\|^2\leq\|x-y\|^2+\|Tx-y\|^2+\|Ty-x\|^2$ for any $x, y\in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [15] and Iemoto and Takahashi [9]. Moreover they proved a nonlinear ergodic theorem. Furthermore they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be super hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}&\alpha\|Tx-Ty\|^2+(1-\alpha+\gamma)\|x-Ty\|^2\\&\quad\leq(\beta+(\beta-\alpha)\gamma)\|Tx-y\|^2+(1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2\\&\qquad+(\alpha-\beta)\gamma\|x-Tx\|^2+\gamma\|y-Ty\|^2\end{aligned}$$

for any $x, y \in C$. A generalized hybrid mapping with a fixed point is quasinonexpansive. However a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the author [12] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0\end{aligned}$$

for any $x, y \in C$. Furthermore the author [13] defined a class of nonlinear mappings in a Hilbert space which covers the class of super hybrid mappings and the class of widely generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Then we prove fixed point theorems for such new mappings in a Hilbert space. Furthermore we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [14] for super hybrid mappings. On the other hand, Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be extended hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}\alpha(1+\gamma)\|Tx - Ty\|^2 + (1 - \alpha(1+\gamma))\|x - Ty\|^2 \\ \leq (\beta + \alpha\gamma)\|Tx - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ -(\alpha - \beta)\gamma\|x - Tx\|^2 - \gamma\|y - Ty\|^2\end{aligned}$$

for any $x, y \in C$. Furthermore they proved a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping.

In this paper we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore we prove a mean convergence theorem of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a non-empty subset of H. We denote by $\overline{co}C$ the closure of the convex hull of C. In a Hilbert space it is known that

$$\|(1-\lambda)x+\lambda y\|^2=(1-\lambda)\|x\|^2+\lambda\|y\|^2-(1-\lambda)\lambda\|x-y\|^2$$

for any $x,y\in H$ and for any $\lambda\in\mathbb{R}$; see [19]. Furthermore in a Hilbert space we obtain that

$$2\langle x-y,z-w\rangle=\|x-w\|^2+\|y-z\|^2-\|x-z\|^2-\|y-w\|^2$$

for any $x,y,z,w\in H$. Let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T)\neq\emptyset$ is said to be quasi-nonexpansive if $\|x-Ty\|\le\|x-y\|$ for any $x\in F(T)$ and for any $y\in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space; see, for instace, [22]. Let C be a non-empty closed convex subset of H and $x\in H$. Then, we know that there exists a unique nearest point $z\in C$ such that $\|x-z\|=\inf_{y\in C}\|x-y\|$. We denote such a correspondence by $z=P_Cx$. The mapping P_C is said to be the metric projection from H onto C. It is known that P_C is nonexpansive and

$$\langle x-P_Cx,P_Cx-u\rangle\ge 0$$

for any $x\in H$ and for any $u\in C$; see [19] for more details. For proving a mean convergence theorem, we also need the following lemma proved by Takahashi and Toyoda [21].

Lemma 2.1. *Let C be a non-empty closed convex subset of H. Let P be the metric projection from H onto C. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1}-u\|\le\|u_n-u\|$ for any $u\in C$ and for any $n\in\mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0\in C$.*

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then we denote by $\mu(f)$ the value of μ at $f=(x_1,x_2,x_3,\dots)\in\ell^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is said to be a mean if $\mu(e)=\|\mu\|=1$, where $e=(1,1,1,\dots)$. A mean μ is said to be a Banach limit on ℓ^∞ if $\mu_n(x_{n+1})=\mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f=(x_1,x_2,x_3,\dots)\in\ell^\infty$,

$$\liminf_{n\to\infty}x_n\le\mu_n(x_n)\le\limsup_{n\to\infty}x_n.$$

In particular, if $f=(x_1,x_2,x_3,\dots)\in\ell^\infty$ and $x_n\to a\in\mathbb{R}$, then we obtain $\mu(f)=\mu_n(x_n)=a$. See [18] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we have the following result; see [17] and [18].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on ℓ^∞. Then there exists a unique point $z_0\in\overline{co}\{x_n\mid n\in\mathbb{N}\}$ such that*

$$\mu_n\langle x_n,y\rangle=\langle z_0,y\rangle$$

for any $y\in H$.

The author [13] proved by Lemma 2.2 the following fixed point theorem.

Theorem 2.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $\zeta+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

As a direct consequence of Theorem 2.1, we obtain the following.

Theorem 2.2. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $\zeta+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

Using Theorem 2.2, we prove a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space; see [11].

Theorem 2.3. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, *and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and* $(\alpha+\beta)\lambda+\zeta+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, *and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and* $(\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0$.

Suppose that for any $x \in C$ there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x + m(y-x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

Moreover, for proving a mean convergence theorem of Baillon's type in a Hilbert space, we need the following lemmas and theorems; see [11].

Lemma 2.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition:*

$$\alpha+\gamma+\varepsilon+\eta>0, \text{ or } \alpha+\beta+\zeta+\eta>0.$$

Then $F(T)$ is closed.

Lemma 2.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

(1) $\alpha+\beta+\gamma+\delta\geq 0$ *and* $\alpha+\gamma+\varepsilon+\eta>0$;

(2) $\alpha+\beta+\gamma+\delta\geq 0$ *and* $\alpha+\beta+\zeta+\eta>0$.

Then $F(T)$ is convex.

Lemma 2.5. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

(1) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\gamma>0$ *and* $\varepsilon+\eta\geq 0$;

(2) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\beta>0$ *and* $\zeta+\eta\geq 0$.

Then T is quasi-nonexpansive.

Moreover we obtain the following.

Lemma 2.6. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

(1) $\alpha+\beta+\gamma+\delta\geq 0$, *and there exists* $\lambda\in\mathbb{R}$ *such that* $0\leq(\alpha+\gamma)\lambda+\varepsilon+\eta<\alpha+\gamma+\varepsilon+\eta$;

(2) $\alpha+\beta+\gamma+\delta\geq 0$, *and there exists* $\lambda\in\mathbb{R}$ *such that* $0\leq(\alpha+\beta)\lambda+\zeta+\eta<\alpha+\beta+\zeta+\eta$.

Then $(1-\lambda)T+\lambda I$ is quasi-nonexpansive.

Now, using the technique developed by Takahashi [17], by Lemmas 2.3, 2.4, 2.5 and 2.6 we obtain the following mean convergence theorems for widely more generalized hybrid non-self mappings in a Hilbert space.

Theorem 2.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

(1) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\gamma>0$ *and* $\varepsilon+\eta\geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \geq 0$.

Then for any $x \in C(T;0) = \{z \mid T^n z \in C,\ \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p *of* T, *where* P *is the metric projection from* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} PT^n x$.

Theorem 2.5. *Let* H *be a real Hilbert space, let* C *be a non-empty closed convex subset of* H *and let* T *be an* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$*-widely more generalized hybrid mapping from* C *into* H *which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Then for any $x \in C(T;\lambda) = \{z \mid ((1-\lambda)T+\lambda I)^n z \in C,\ \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} ((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to a fixed point p *of* T, *where* P *is the metric projection from* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

Moreover we obtain the following.

Theorem 2.6. *Let* H *be a real Hilbert space, let* C *be a non-empty bounded closed convex subset of* H *and let* T *be an* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$*-widely more generalized hybrid mapping from* C *into* H *which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Suppose that for any $x \in C$, *there exist* $m \in \mathbb{R}$ *and* $y \in C$ *such that* $0 \leq (1-\lambda)m \leq 1$ *and* $Tx = x + m(y-x)$. *Then for any* $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} ((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to a fixed point p *of* T, *where* P *is the metric projection from* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

3 Fixed point theorem

Let H be a real Hilbert space and let C be a non-empty subset of H. A mapping T from C into H was said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ +\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$; see Introduction. Such a mapping T is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid; see [13]. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [14] if $\alpha+\beta=-\gamma-\delta=1$ and $\varepsilon=\zeta=\eta=0$. Moreover it is an extension of widely generalized hybrid mappings in the sence of Kawasaki and Takahashi [12]; see also [11].

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta>0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and* $(\alpha+\beta)\lambda+\zeta+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta>0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and* $(\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0$.

Let

$$M=\begin{cases}[0, \infty), & \text{if } \lambda<1, \\ (-\infty, 0], & \text{if } \lambda>1.\end{cases}$$

Suppose that for any $x \in C$ there exist $m \in M$ and $y \in C$ such that $Tx=x+m(y-x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta>0$ on the conditions (1) *and* (2).

As a direct consequence of Theorem 3.1, we obtain the following fixed point theorem for generalized hybrid mappings as an extension of [8, Theorem 3.4].

Theorem 3.2. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an (α, β)-generalized hybrid mapping from C into H. Suppose that for any $x \in C$ there exist $m \in [0, \infty)$ and $y \in C$ such that $Tx=x+m(y-x)$. Then T has a fixed point.*

Example 3.1. Let $H=\mathbb{R}$, let $C=\left[0, \frac{\pi}{2}\right]$, let $Tx=(1+2x)\cos x-2x^2$ and let $\alpha=1$, $\beta=\gamma=11$, $\delta=-22$, $\varepsilon=\zeta=-12$ and $\eta=1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H. Moreover $\alpha+\beta+\gamma+\delta=1 \geq 0$ and $\alpha+\gamma+\varepsilon+\eta=1>0$ hold. Let $\lambda=\frac{2+3\pi}{3(1+\pi)}$. Then we obtain $(\alpha+\beta)\lambda+\zeta+\eta=\frac{\pi-3}{1+\pi} \geq 0$. Let $m=1+\pi$ and $y=x+\frac{(1+2x)(\cos x-x)}{1+\pi}$ for any $x \in C$. Then we obtain $y \in C$ and $Tx=x+m(y-x)$. Therefore by Theorem 3.1 T has a unique fixed point.

Example 3.2. Let $H = \mathbb{R}^2$, let $C = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$, let

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1 + 2\sqrt{x_1x_2}) \cos\sqrt{x_1x_2} - 2x_1x_2 \\ (1 + x_1 + x_2) \cos\frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} \end{pmatrix}$$

and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H. Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $(\alpha + \beta)\lambda + \zeta + \eta = \frac{\pi-3}{1+\pi} \geq 0$. Let $m = 1 + \pi$ and

$$y = \begin{pmatrix} x_1 + \frac{(1+2\sqrt{x_1x_2}) \cos\sqrt{x_1x_2} - 2x_1x_2 - x_1}{1+\pi} \\ x_2 + \frac{(1+x_1+x_2) \cos\frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} - x_2}{1+\pi} \end{pmatrix}$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in C$. Then we obtain $y \in C$ and $Tx = x + m(y - x)$. Therefore by Theorem 3.1 T has a unique fixed point.

4 Mean convergence theorem

In this section we prove a mean convergence theorem of Baillon's type in a Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which satisfies $F(T) \neq \emptyset$ and the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Let

$$M = \begin{cases} [0, \infty), & \text{if } \lambda < 1, \\ (-\infty, 0], & \text{if } \lambda > 1. \end{cases}$$

Suppose that for any $x \in C$ there exist $m \in M$ and $y \in C$ such that $Tx = x + m(y - x)$. Then for any $x \in C$,

$$S_nx = \frac{1}{n}\sum_{k=0}^{n-1}((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1 - \lambda)T + \lambda I)^n x$.

As a direct consequence of Theorem 4.1, we obtain the following mean convergence theorem for super hybrid mappings [8, Theorem 4.2].

Theorem 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an (α, β, γ)-super hybrid mapping from C into itself with $F(T) \neq \emptyset$. Suppose that $\gamma \geq 0$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{1}{1+\gamma} T + \frac{\gamma}{1+\gamma} I \right)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P \left(\frac{1}{1+\gamma} T + \frac{\gamma}{1+\gamma} I \right)^n x$.

Example 4.1. Let $H = \mathbb{R}$, let $C = \left[0, \frac{\pi}{2}\right]$, let $Tx = (1+2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H. Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $0 \leq (\alpha+\gamma)\lambda + \varepsilon + \eta = \frac{\pi-3}{1+\pi} < 1 = \alpha + \gamma + \varepsilon + \eta$. Let $m = 1+\pi$ and $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then we obtain $y \in C$ and $Tx = x + m(y-x)$. Therefore by Theorem 4.1 for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

Example 4.2. Let $H = \mathbb{R}^2$, let $C = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$, let

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1 + 2\sqrt{x_1 x_2}) \cos \sqrt{x_1 x_2} - 2x_1 x_2 \\ (1 + x_1 + x_2) \cos \frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} \end{pmatrix}$$

and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H. Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $0 \leq (\alpha+\gamma)\lambda + \varepsilon + \eta = \frac{\pi-3}{1+\pi} < 1 = \alpha + \gamma + \varepsilon + \eta$. Let $m = 1 + \pi$ and

$$y = \begin{pmatrix} x_1 + \frac{(1+2\sqrt{x_1x_2})\cos\sqrt{x_1x_2} - 2x_1x_2 - x_1}{1+\pi} \\ x_2 + \frac{(1+x_1+x_2)\cos\frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} - x_2}{1+\pi} \end{pmatrix}$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in C$. Then we obtain $y \in C$ and $Tx = x + m(y - x)$. Therefore by Theorem 4.1 for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

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