Recent development on James constant of 2-dimensional Lorentz sequence spaces

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Abstract. The James constant of a Banach space X was introduced by Gao and Lau [3] and has recently been studied by several authors. In this paper, we present some recent results on James constant of 2-dimensional Lorentz sequence space.

1 Introduction

In the paper, we consider the James constant $J(X)$ of a Banach space X:

$$J(X)=\sup\{\min\{\|x+y\|,\|x-y\|\}:x,y\in S_{X}\},$$

where $S_{X}=\{x\in X:\|x\|=1\}$. This notion was introduced by Gao and Lau [3] and recently it has been studied by several authors (cf. [5, 6, 17]). $\sqrt{2}\leq J(X)\leq 2$ for any Banach space X. In particular, $J(X)=\sqrt{2}$ if X is a Hilbert space. If $1\leq p\leq\infty$ and $\dim L_{p}\geq 2$, then $J(L_{p})=\max\{2^{1/p},2^{1/p'}\}$, where $1/p+1/p'=1$. $J(X)<2$ holds if and only if X is uniformly non-square; that is, there exists a $\delta>0$ such that $x,y\in S_{X}$ and $\|(x-y)/2\|\geq 1-\delta$ imply $\|(x+y)/2\|\leq 1-\delta$. Banach spaces with uniformly normal structure are characterized in terms of James constant (see [4, 5]). The Schäffer constant $g(X)$ of a Banach space X is defined by

$$g(X)=\inf\{\max\{\|x+y\|,\|x-y\|\}:x,y\in S_{X}\}.$$

We know that $1\leq g(X)\leq\sqrt{2}\leq J(X)\leq 2$ and $g(X)J(X)=2$ for all Banach spaces X.

A norm $\|\cdot\|$ on $\mathbb{R}^{2}$ is said to be absolute if $\|(z,w)\|=\|(|z|,|w|)\|$ for all $(z,w)\in\mathbb{R}^{2}$, and normalized if $\|(1,0)\|=\|(0,1)\|=1$. The ℓ_{p}-norms $\|\cdot\|_{p}$ are such examples;

$$\|(z,w)\|_{p}=\begin{cases}(|z|^{p}+|w|^{p})^{1/p} & \text{if } 1\leq p<\infty,\\ \max\{|z|,|w|\} & \text{if } p=\infty.\end{cases}$$

Let AN_2 be the set of all absolute normalized norms on $\mathbb{R}^2$, and Ψ_2 the set of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$. As in Bonsall and Duncan [1], AN_2 and Ψ_2 are in 1-1 correspondence under the equation

$$\psi(t) = \|(1-t,\, t)\| \ (0 \leq t \leq 1). \tag{1}$$

Indeed, for all $\psi \in \Psi_2$ let

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1). From this result, we can consider many non ℓ_p-type norms easily. Now let $\psi_p(t) = \{(1-t)^p + t^p\}^{1/p} \in \Psi_2$. As is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with ψ_p.

In this note we present some recent results about James constants of two dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ and its dual $d^{(2)}(\omega,q)^*$. In general, $J(X) \neq J(X^*)$ for a space X and its dual X^*, whereas we will obtain $J(d^{(2)}(\omega,q)) = J(d^{(2)}(\omega,q)^*)$ for all w,q.

2 James constant

We first discuss James constant of absolute norms. Let X be a Banach space and $x,y \in X$. We say that x is isosceles orthogonal to y, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$. We define a function $\beta(x)$ on X by

$$\beta(x) = \sup\{\min\{\|x+y\|, \|x-y\|\} : y \in S_X\}.$$

To obtain $J((\mathbb{R}^2, \|\cdot\|_\psi))$, we need the following lemma.

Lemma 1 ([3]). *Let $\psi \in \Psi_2$ and $x \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}$. Then there exists uniquely a vector $y_0 \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}$ with $x \perp_I y_0$. Moreover, $\beta(x) = \|x+y_0\|_\psi$.*

From this lemma, we can write

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \sup\left\{\|x+y\|_\psi : x,y \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}, x \perp_I y\right\}.$$

We recall that an absolute normalized norm $\|\cdot\|$ on $\mathbb{R}^2$ is symmetric in the sense that $\|(x,y)\| = \|(y,x)\|$ for all $(x,y) \in \mathbb{R}^2$ if and only if the corresponding function ψ is symmetric with respect to $t = 1/2$, that is, $\psi(1-t) = \psi(t)$ for every $t \in [0,1]$. Using Lemma 1 we can obtain the following formula by using ψ in Ψ_2.

Theorem 2 ([11]). *Let $\psi \in \Psi_2$. If ψ is symmetric with respect to $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0 \le t \le 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right).$$

Corollary 3 ([11]). *Let $\psi \in \Psi_2$. Assume that ψ is symmetric with respect to $t = 1/2$.*

(i) *If $\psi \ge \psi_2$ and $M_1 = \max_{0 \le t \le 1} \psi(t)/\psi_2(t)$ is taken at $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\psi\left(\frac{1}{2}\right).$$

(ii) *If $\psi \le \psi_2$ and $M_2 = \max_{0 \le t \le 1} \psi_2(t)/\psi(t)$ is taken at $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{1}{\psi(1/2)}.$$

Example 1. Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$.

(i) If $1 \le p \le 2$, then $J((\mathbb{R}^2, \|\cdot\|_p)) = 2\psi_p(1/2) = 2^{1/p}$.

(ii) If $2 \le p \le \infty$, then $J((\mathbb{R}^2, \|\cdot\|_p)) = 1/\psi_p(1/2) = 2^{1/p'}$.

For $0 < \omega < 1$ and $1 \le q < \infty$, the 2-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ is $\mathbb{R}^2$ with the norm

$$\|x\|_{\omega,q} = (x_1^{*q} + \omega x_2^{*q})^{1/q}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where (x_1^*, x_2^*) is the nonincreasing rearrangement of $(|x_1|, |x_2|)$; that is, $x_1^* = \max\{|x_1|, |x_2|\}$ and $x_2^* = \min\{|x_1|, |x_2|\}$. Note here that the norm $\|\cdot\|_{\omega,q}$ of $d^{(2)}(\omega, q)$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi_{\omega,q}(t) = \begin{cases} \left((1-t)^q + \omega t^q\right)^{1/q} & \text{if } 0 \le t \le 1/2, \\ \left(t^q + \omega(1-t)^q\right)^{1/q} & \text{if } 1/2 \le t \le 1. \end{cases}$$

Kato and Maligranda in [6] computed the James constant of $d^{(2)}(w, q)$ in the case where $q \ge 2$.

Theorem 4 ([6]). *Let $q \ge 2$ and $0 < \omega < 1$. Then*

$$J\left(d^{(2)}(\omega, q)\right) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

For the case where $q < 2$, we attempt to calculate it and we had a partial answer in Mitani and Saito [11] and Suzuki, Yamano and Kato [16] by using Theorem 2. In Mitani, Saito and Suzuki [12] we have a complete answer.

Theorem 5 ([11]). *Let $q = 1$. If $0 < \omega \leq \sqrt{2} - 1$, then*

$$J\big(d^{(2)}(\omega, q)\big) = \frac{2}{1+\omega}.$$

If $\sqrt{2} - 1 < \omega < 1$, then

$$J\big(d^{(2)}(\omega, q)\big) = 1 + \omega.$$

Theorem 6 ([12, 14]). *Let $1 < q < 2$.* (i) *If $0 < \omega \leq (\sqrt{2} - 1)^{2-q}$, then*

$$J\big(d^{(2)}(\omega, q)\big) = 2\left(\frac{1}{1+\omega}\right)^{1/q}$$

and

$$g(d^{(2)}(\omega, q)) = (1 + \omega)^{1/q}.$$

(ii) *If $(\sqrt{2} - 1)^{2-q} < \omega < 1$, then there exists a unique pair of real numbers s_0, s_1 such that*

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $(1 + s_i)^{q-1}(1 - \omega s_i^{q-1}) = \omega(1 - s_i)^{q-1}(1 + \omega s_i^{q-1})$ for $i = 0, 1$.

(ii-a) *If $(\sqrt{2} - 1)^{2-q} < \omega \leq \sqrt{2}^q - 1$, then*

$$J(d^{(2)}(\omega, q)) = \max\left\{\left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}}\right)^{1/q}, 2\left(\frac{1}{1+\omega}\right)^{1/q}\right\}$$

and

$$g(d^{(2)}(\omega, q)) = \min\left\{\left(\frac{2(1 + s_1)^{q-1}}{1 + \omega s_1^{q-1}}\right)^{1/q}, (1 + \omega)^{1/q}\right\}.$$

(ii-b) *If $\sqrt{2}^q - 1 < \omega < 1$, then*

$$J(d^{(2)}(\omega, q)) = \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}}\right)^{1/q}$$

and

$$g(d^{(2)}(\omega, q)) = \left(\frac{2(1 + s_1)^{q-1}}{1 + \omega s_1^{q-1}}\right)^{1/q}.$$

We next consider the dual space of $d^{(2)}(\omega,q)$. For $q=1$, it is known that $d^{(2)}(\omega,1)^*$ is a 2-dimensional Marcinkiewicz space m_ω given by the norm

$$\|(x,y)\|_{m_\omega} = \max\left\{x^*, \frac{x^*+y^*}{1+\omega}\right\},$$

where $x^* = \max\{|x|,|y|\}$, $y^* = \min\{|x|,|y|\}$. For $\psi \in \Psi_2$ let $\|\cdot\|_\psi^*$ be the dual of the norm $\|\cdot\|_\psi$. Namely, $\|x\|_\psi^* = \sup\{|\langle x,y\rangle| : y \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}\}$ for any $x \in \mathbb{R}^2$. Then $\|\cdot\|_\psi^* \in AN_2$ and the corresponding convex function ψ^* in Ψ_2 is

$$\psi^*(t) = \sup_{0\le s\le 1} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for t with $0 \le t \le 1$.

Theorem 7 ([13]). *Let $0<\omega<1$. (i) If $1<q<\infty$, then*

$$\psi_{\omega,q}^*(t) = \begin{cases} ((1-t)^p + \omega^{1-p}t^p)^{1/p}, & \text{if } \ 0 \le t < \dfrac{\omega}{1+\omega}, \\ (1+\omega)^{1/p-1}, & \text{if } \ \dfrac{\omega}{1+\omega} \le t < \dfrac{1}{1+\omega}, \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p}, & \text{if } \ \dfrac{1}{1+\omega} \le t \le 1, \end{cases}$$

where $1/p+1/q=1$.
(ii) *If* $q=1$, *then*

$$\psi_{\omega,1}^*(t) = \begin{cases} 1-t, & \text{if } \ 0 \le t < \dfrac{\omega}{1+\omega}, \\ \frac{1}{1+\omega}, & \text{if } \ \dfrac{\omega}{1+\omega} \le t < \dfrac{1}{1+\omega}, \\ t, & \text{if } \ \dfrac{1}{1+\omega} \le t \le 1. \end{cases}$$

Hence,

Theorem 8 ([13]). *Let $0<\omega<1$. (i) If $1<q<\infty$, then*

$$\|(x,y)\|_{\omega,q}^* = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{1/p} & \text{if } \omega|x| \ge |y|, \\ (1+\omega)^{1/p-1}(|x|+|y|) & \text{if } \ \omega|x| \le |y| \le \omega^{-1}|x|, \\ (|y|^p + \omega^{1-p}|x|^p)^{1/p} & \text{if } \ \omega^{-1}|x| \le |y|. \end{cases}$$

(ii) *If* $q=1$, *then*

$$\|(x,y)\|_{\omega,1}^{*}=\begin{cases}\max\{|x|,\omega^{-1}|y|\} & \text{if } \ \omega|x|\ge|y|,\\ \frac{1}{1+\omega}(|x|+|y|) & \text{if } \ \omega|x|\le|y|\le\omega^{-1}|x|,\\ \max\{\omega^{-1}|x|,|y|\} & \text{if } \ \omega^{-1}|x|\le|y|.\end{cases}$$

Namely, $\|(x,y)\|_{\omega,1}^{*}=\|(x,y)\|_{m_\omega}$.

We consider the constant $J\big(d^{(2)}(\omega,q)^*\big)$ for ω, q.

Theorem 9 ([13]). (i) *Let either* $q\ge 2$ *and* $0<\omega<1$, *or* $1<q<2$ *and* $0<\omega\le(\sqrt{2}-1)^{2-q}$. *Then*

$$J\big(d^{(2)}(\omega,q)^*\big)=2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

(ii) *Let* $q=1$. *If* $0<\omega\le\sqrt{2}-1$, *then*

$$J\big(d^{(2)}(\omega,q)^*\big)=\frac{2}{1+\omega}.$$

If $\sqrt{2}-1<\omega<1$, *then*

$$J\big(d^{(2)}(\omega,q)^*\big)=1+\omega.$$

Hence $J\big(d^{(2)}(\omega,q)\big)$ and $J\big(d^{(2)}(\omega,q)^*\big)$ coincide for such cases. In the case where $1<q<2$ and $\omega>(\sqrt{2}-1)^{2-q}$, we recently obtained the following result.

Theorem 10 ([13, 14]). *Let* $1<q<2$ *and* $1/p+1/q=1$. *If* $(\sqrt{2}-1)^{2-q}<\omega<1$, *then there exists a unique pair of real numbers* s_0, s_1 *such that*

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1}<s_0<\omega^{1/(2-q)}<s_1<1$$

and $(1+s_i)^{q-1}(1-\omega s_i^{q-1})=\omega(1-s_i)^{q-1}(1+\omega s_i^{q-1})$ *for* $i=0,1$.

(i) *If* $(\sqrt{2}-1)^{2-q}<\omega\le\sqrt{2}^{q}-1$, *then*

$$J(d^{(2)}(\omega,q)^*)=\max\left\{\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q},2\left(\frac{1}{1+\omega}\right)^{1/q}\right\}$$

and

$$g(d^{(2)}(\omega,q)^*)=\min\left\{\left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q},(1+\omega)^{1/q}\right\}.$$

(ii) *If* $\sqrt{2}^q - 1 < \omega < 1$, *then*

$$J(d^{(2)}(\omega,q)^*) = \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}$$

and

$$g(d^{(2)}(\omega,q)^*) = \left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q}.$$

Thus,

Theorem 11 ([14]). *If* $1 \leq q < \infty$ *and* $0 < \omega < 1$, *then*

$$J\big(d^{(2)}(\omega,q)^*\big) = J\big(d^{(2)}(\omega,q)\big)$$

and

$$g\big(d^{(2)}(\omega,q)^*\big) = g\big(d^{(2)}(\omega,q)\big).$$

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