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GENERALIZATION OF YOUNG DIAGRAMS AND HOOK FORMULA

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1. Preliminaries

First, we give several notations for root systes. We always fix a root datum $(A; \mathfrak{h}, \mathfrak{h}^*, \Pi, \Pi^\vee)$:

$$A = (a_{i,j})_{i,j\in I} : \text{a generalized Cartan matrix.}$$
$$\mathfrak{h} : \mathbb{R}\text{-vector space,}$$
$$\mathfrak{h}^* : \text{the dual space of } \mathfrak{h},$$
$$\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R} : \text{the canonical bilinear form.}$$
$$\Pi := \left\{ \alpha_i \;\middle|\; i \in I \right\} \subset \mathfrak{h}^* : \text{linearly independent subset}$$
$$\Pi^\vee := \left\{ \alpha_i^\vee \;\middle|\; i \in I \right\} \subset \mathfrak{h} : \text{linearly independent subset}$$
$$\text{such that } \langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}.$$

For each $i \in I$, we define the *simple reflection* $s_i \in GL(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$
$$\text{equivalently,} \qquad s_i : h \mapsto h - \langle \alpha_i, h \rangle \alpha_i^\vee, \quad h \in \mathfrak{h}.$$
$$W := \langle s_i \mid i \in I \rangle : \text{the Weyl group}$$

We define a (real) root system and a (real) coroot system:

$$\Phi := W\Pi \quad \left(\subseteq \bigoplus_{i\in I} \mathbb{Z}\alpha_i \right) : \text{(real) root system}$$
$$\Phi_+ := \Phi \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i : \text{(real) positive root system}$$
$$\Phi_- := \Phi \cap \bigoplus_{i\in I} \mathbb{Z}_{\leq 0}\alpha_i : \text{(real) negative root system}$$
$$\Phi = \Phi_+ \amalg \Phi_- \quad \text{(disjoint union)}$$
$$\Phi^\vee := W\Pi^\vee \quad \left(\subseteq \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee \right) : \text{(real) coroot system}$$
$$\Phi_+^\vee := \Phi^\vee \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee : \text{(real) positive coroot system}$$
$$\Phi_-^\vee := \Phi^\vee \cap \bigoplus_{i\in I} \mathbb{Z}_{\leq 0}\alpha_i^\vee : \text{(real) negative coroot system}$$
$$\Phi^\vee = \Phi_+^\vee \amalg \Phi_-^\vee \quad \text{(disjoint union)}$$

For a real root $\beta = w(\alpha_i) \in \Phi$, we define the dual coroot $\beta^\vee \in \Phi^\vee$ of β by:

$$\beta^\vee = w(\alpha_i^\vee).$$

Remark 1. This is independent from the choice of $w \in W$ and $\alpha_i \in \Pi$.

The map $\Phi \ni \beta \mapsto \beta^\vee \in \Phi^\vee$ is a bijection.

For each $\beta \in \Phi$, we define the reflection $s_\beta \in W$ by:

$$s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*,$$
$$s_\beta(h) = h - \langle \beta, h \rangle \beta^\vee, \quad h \in \mathfrak{h}.$$

Definition 1. Let $w \in W$. We define the inversion set $\Phi(w)$ of w by:

$$\Phi(w) := \left\{ \gamma \in \Phi_+ \mid w^{-1}(\gamma) < 0 \right\}.$$

Definition 2. Let $w \in W$. We denote by Red(w) the set of reduced decompositions of w:

$$\mathrm{Red}(w) := \left\{ s_{i_1} s_{i_2} \cdots s_{i_d} \mid \text{reduced decompositions of } w \right\}.$$

Definition 3. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if:

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

The set of integral weights is denoted by P.

Definition 4. An integral weight $\lambda \in P$ is said to be *dominant* if:

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} = \mathbb{N}, \quad i \in I.$$

The set of dominant integral weights is denoted by $P_{\geq 0}$.

2. Minuscule elements and Peterson-Proctor hook formula

Definition 5 (Peterson (see [1])). Let $\Lambda \in P_{\geq 0}$. An element $w \in W$ is said to be Λ-minuscule if there exists a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_d} \in \mathrm{Red}(w)$ of w such that

$$\langle s_{i_{k+1}} \cdots s_{i_d}(\Lambda), \alpha_{i_k}^\vee \rangle = 1, \quad k = 1, 2, \cdots, d.$$

Remark 2. This definition is independent from the choice of reduced decompositions of w.

Example 1. A Grassmannian permutation is a Λ-minuscule element in the Weyl group of type A (symmetric group).

Theorem 2.1 (Proctor (see e.g. [7])). *Suppose that the underlying generalized Cartan matrix is simply-laced. Then there exists a one-to-one correspondence between* $\{(\Lambda, w)\}$ *and d-complete posets.*

Theorem 2.2 (Peterson-Proctor (see [1])). *Let* $\Lambda \in P_{\geq 0}$ *and* $w \in W$ *a* Λ*-minuscule element. Then we have:*

$$\#\mathrm{Red}(w) = \frac{\ell(w)!}{\prod_{\beta \in \Phi(w)} \mathrm{ht}(\beta)}.$$

This hook formula is, of course, a generalization of hook length formula for a Young diagram due to Frame-Robinson-Thrall [2], and a shifted Younf diagram due to Thrall [9].

In terms of d-complete posets, this counts the number of linear extensions of the d-complete posets.

Now, we have three approaches to prove Peterson-Proctor hook formula.

multivariate hook formula Proctor (1997) N. (preprint)	colored hook formula N. (2008)	probabilistic algorithm Okamura (2003) N.-Okamura (preprint)
$\searrow$	$\Downarrow$	$\swarrow$

Peterson-Proctor hook formula

3. Finite predominant integral weights

Definition 6. An integral weight $\lambda \in P$ is said to be *pre-dominant* if:

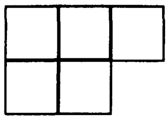
$$\langle \lambda, \beta^\vee \rangle \geq -1\,, \quad \beta \in \Phi_+.$$

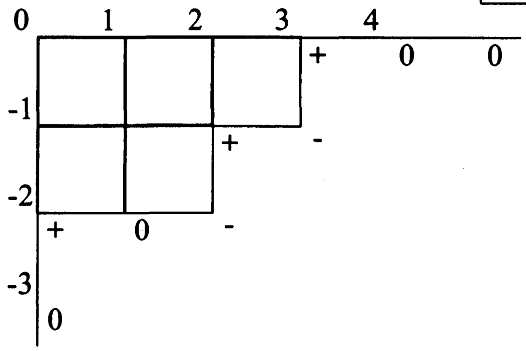
The set of pre-dominant integral weights is denoted by $P_{\geq -1}$.

Definition 7. Let $\lambda \in P_{\geq -1}$. We define a set $\mathrm{D}(\lambda)$ by:

$$\mathrm{D}(\lambda) := \left\{ \beta \in \Phi_+ \mid \langle \lambda, \beta^\vee \rangle = -1 \right\}.$$

The set $\mathrm{D}(\lambda)$ is called a *diagram of* λ. A pre-dominant integral weight λ is said to be *finite* if $\#\mathrm{D}(\lambda) < \infty$. The set of finite pre-dominant integral weights is denoted by $P^{\mathrm{fin}}_{\geq -1}$.

Example 2. As an example, we consider how Young diagram  is realized as $\mathrm{D}(\lambda)$.



According to the above picture, we put $\lambda := 1\Lambda_{-2} + (-1)\Lambda_0 + 1\Lambda_1 + (-1)\Lambda_2 + 1\Lambda_3$, in the root system of type A_6 with index $I = \{-2, -1, 0, 1, 2, 3\}$, where Λ_i denotes i-th fundamental weight. Then we have $\lambda \in P^{\mathrm{fin}}_{\geq -1}$ such that $(\mathrm{D}(\lambda); <)$ is order-isomorphic to the original Young diagram.

$\mathrm{D}(\lambda) =$

$\alpha_{-1}+\alpha_0+\alpha_1+\alpha_2$	$\alpha_0+\alpha_1+\alpha_2$	α_2
$\alpha_{-1}+\alpha_0$	α_0	

Thus, we recover the original Young diagram.

Theorem 3.1. *Let $\Lambda \in P_{\geq 0}$ and $w \in W$ a Λ-minuscule element. Then we have $w(\Lambda) \in P^{\mathrm{fin}}_{\geq -1}$. Furthermore, this correspondence is bijective between $P^{\mathrm{fin}}_{\geq -1}$ and the set of such pairs* (Λ, w).

$$\begin{array}{ccc} \{(\Lambda, w)\} & \to & P^{\mathrm{fin}}_{\geq -1} \\ \uplus & & \uplus \\ (\Lambda, w) & \mapsto & w(\Lambda) \end{array}.$$

Put $\lambda := w(\Lambda)$. *Then we have*

$$\Phi(w) = \mathrm{D}(\lambda).$$

Definition 8. Let $\lambda \in P^{\mathrm{fin}}_{\geq -1}$ and $\beta \in \mathrm{D}(\lambda)$. We define a set $\mathrm{H}_\lambda(\beta)$ by:

$$\mathrm{H}_\lambda(\beta) := \left\{\gamma \in \mathrm{D}(\lambda) \mid s_\beta(\gamma) < 0\right\} = \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right).$$

We call the set $\mathrm{H}_\lambda(\beta)$ the *hook at* β.

Proposition 3.2. *Let* $\lambda \in P^{\mathrm{fin}}_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Then we have:*

(1) $\#\mathrm{H}_\lambda(\beta) = \mathrm{ht}(\beta)$.
(2) $s_\beta(\lambda) \in P^{\mathrm{fin}}_{\geq -1}$.
(3) $\mathrm{D}(s_\beta(\lambda)) = s_\beta(\mathrm{D}(\lambda) \setminus \mathrm{H}_\lambda(\beta))$.

Definition 9. Let $\lambda \in P^{\mathrm{fin}}_{\geq -1}$. A sequence $(\beta_1, \beta_2, \cdots, \beta_l)$ $(l \geq 0)$ of positive real roots is said to be a *λ-path* if:

$$\beta_k \in \mathrm{D}(s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda)), \qquad (k = 1, 2, \cdots, l).$$

The set of λ-paths is denoted by $\mathrm{Path}(\lambda)$.

Definition 10. Let $\lambda \in P^{\mathrm{fin}}_{\geq -1}$. A λ-path of maximal length is called a *maximal λ-path*. The set of maximal λ-paths is denoted by $\mathrm{MPath}(\lambda)$.

Note that if $\#\mathrm{D}(\lambda) = d$ then length of maximal λ-path is d, and hence that maximal λ-path is of a form $(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_d})$.

Example 3. Back to Example 2, put $\lambda := \Lambda_{-1} - \Lambda_0 + \Lambda_1 - \Lambda_2 + \Lambda_3$. Then we have 5 maximal λ-paths below:

$(\alpha_0, \alpha_{-1}, \alpha_2, \alpha_1, \alpha_0) \cdots$

5	4	3
2	1	

$(\alpha_0, \alpha_2, \alpha_{-1}, \alpha_1, \alpha_0) \cdots$

5	4	2
3	1	

$(\alpha_2, \alpha_0, \alpha_{-1}, \alpha_1, \alpha_0) \cdots$

5	4	1
3	2	

$(\alpha_0, \alpha_2, \alpha_1, \alpha_{-1}, \alpha_0) \cdots$

5	3	2
4	1	

$(\alpha_2, \alpha_0, \alpha_1, \alpha_{-1}, \alpha_0) \cdots$

5	3	1
4	2	

Now we restate the Peterson-Proctor hook formula:

Theorem 3.3. *Let* $\lambda \in P^{\mathrm{fin}}_{\geq -1}$. *Put* $d := \#\mathrm{D}(\lambda)$. *Then we have:*

$$\#\mathrm{MPath}(\lambda) = \frac{d!}{\prod_{\beta \in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}.$$

We give two of three approaches to prove the above theorem in section 4 and 5.

4. Colored Hook Formula

Let $\lambda \in P^{\mathrm{fin}}_{\geq -1}$, and put $d = \mathrm{D}(\lambda)$. Then we have:

Theorem 4.1 ([4]).

$$\sum_{(\beta_1,\beta_2,\cdots,\beta_l)\in\mathrm{Path}(\lambda),l\geq 0} \frac{1}{\beta_1}\frac{1}{\beta_1+\beta_2}\cdots\frac{1}{\beta_1+\cdots+\beta_l} = \prod_{\beta\in\mathrm{D}(\lambda)}\left(1+\frac{1}{\beta}\right).$$

Taking the lowest degree, we get:

Corollary 4.2.

$$\sum_{(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_d})\in\mathrm{MPath}(\lambda)} \frac{1}{\alpha_{i_1}}\frac{1}{\alpha_{i_1}+\alpha_{i_2}}\cdots\frac{1}{\alpha_{i_1}+\cdots+\alpha_{i_d}} = \prod_{\beta\in\mathrm{D}(\lambda)}\frac{1}{\beta}.$$

Taking the specialization $\alpha_i \mapsto 1$, we get:

Corollary 4.3 (Peterson-Proctor hook formula).

$$\#\mathrm{MPath}(\lambda) = \frac{d!}{\prod_{\beta\in\mathrm{D}(\lambda)}\mathrm{ht}(\beta)}.$$

5. Probabilistic Algorithm

For simplicity of description, we assume that the underlying root datum is simply-laced. We call the following algorithm the *algorithm A for* Γ:

GNW1.: Set $k := 0$ and set $\lambda_0 := \lambda$.
GNW2.: (Now $\mathrm{D}(\lambda_k)$ has $d-k$ roots.) Pick a root $\beta \in \mathrm{D}(\lambda_k)$ with the probability $1/(d-k)$.
GNW3.: If $\#\mathrm{H}_{\lambda_k}(\beta) - \{\beta\} \neq 0$, then pick a $\gamma \in \mathrm{H}_{\lambda_k}(\beta) - \{\beta\}$ with the probability $1/\#(\mathrm{H}_{\lambda_k}(\beta) - \{\beta\})$, put $\beta := \gamma$ and repeat GNW3.
GNW4.: (Now $\#(\mathrm{H}_{\lambda_k}(\beta) - \{\beta\}) = 0$.) ($\beta = \alpha_i$.) Set $\alpha_{i_{k+1}} := \alpha_i$ and set $\lambda_{k+1} := s_i(\lambda_k)$.
GNW5.: Set $k := k+1$. If $k < d$, return to GNW2; if $k = d$, terminate.

Then, by the definition of the algorithm A for λ, the sequence ($\mathcal{B}$ =)$(\alpha_{i_1},\cdots,\alpha_{i_d})$ generated above is a maximal λ-path. We denote by $\mathrm{Prob}_\lambda(\mathcal{B})$ the probability we get $\mathcal{B} \in \mathrm{MPath}(\lambda)$ by the algorithm A. The algorithm A for λ gives a probability measure $\mathrm{Prob}_\lambda()$ over (a finite set) $\mathrm{MPath}(\lambda)$.

Theorem 5.1 (S. Okamura [6],N-S. Okamura [5]). *Let* $\mathcal{B} \in \mathrm{MPath}(\lambda)$. *Then we have:*

$$\mathrm{Prob}_\lambda(\mathcal{B}) = \frac{\prod_{\beta\in\mathrm{D}(\lambda)}\mathrm{ht}(\beta)}{d!}. \tag{5.1}$$

Since the right-hand side of (5.1) is independent from the choice of $\mathcal{B} \in \mathrm{MPath}(\lambda)$, the probability measure is uniform. Hence, taking the inverse, we get:

Corollary 5.2 (Peterson-Proctor hook formula).

$$\#\mathrm{MPath}(\lambda) = \frac{d!}{\prod_{\beta\in\mathrm{D}(\lambda)}\mathrm{ht}(\beta)}.$$

See [3] for Young diagram case due to Greene-Nijenhuis-Wilf, and [8] for shifted Young diagram case due to Sagan.

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