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Time-periodic problem for the compressible Navier-Stokes equation on the whole space

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1 Introduction

We consider time periodic problem of the following compressible Navier-Stokes equation for barotropic flow in $\mathbb{R}^n$ $(n\geq 3)$:

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho v)=0,\\ \rho(\partial_t v+(v\cdot\nabla)v)-\mu\Delta v-(\mu+\mu')\nabla(\nabla\cdot v)+\nabla p(\rho)=\rho g.\end{cases} \tag{1.1}$$

Here $\rho=\rho(x,t)$ and $v=(v_1(x,t),\cdots,v_n(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t\in\mathbb{R}$ and position $x\in\mathbb{R}^n$; $p=p(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying $p'(\rho_*)>0$ for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu>0$, $\frac{2}{n}\mu+\mu'\geq 0$; and $g=g(x,t)$ is a given external force periodic in t. We assume that $g=g(x,t)$ satisfies the condition

$$g(x,t+T) \;=\; g(x,t) \quad (x\in\mathbb{R}^n,\, t\in\mathbb{R}) \tag{1.2}$$

for some constant $T>0$.

Time periodic flow is one of basic phenomena in fluid mechanics, and thus, time periodic problems for fluid dynamical equations have been extensively studied. We refer, e.g., to [8, 9, 12, 18] for the incompressible Navier-Stokes case, and to [1, 2, 3, 6, 16, 17] for the compressible case. In this paper we are interested in time periodic problem for the compressible Navier-Stokes equation on unbounded domains. Ma, Ukai, and Yang [16] proved the existence and stability of time periodic solutions on the whole space $\mathbb{R}^n$. They showed that if $n\geq 5$, there exists a time periodic solution (ρ_{per},v_{per}) around $(\rho_*,0)$ for a sufficiently small $g\in C^0(\mathbb{R};H^{N-1}\cap L^1)$ with $g(x,t+T)=g(x,t)$, where $N\in\mathbb{Z}$ satisfying $N\geq n+2$. Furthermore, we put $u(t):=(\rho(t),v(t))-(\rho_{per}(t),v_{per}(t))$, then if $\|u(0)\|_{H^{N-1}\cap L^1}<<1$, the time periodic solution is stable and there holds the estimate

$$\|u(t)\|_{H^{N-1}}\leq C(1+t)^{-\frac{n}{4}}\|u(0)\|_{H^{N-1}\cap L^1}.$$

Here H^k denotes the L^2-Sobolev space on $\mathbb{R}^n$ of order k.

On the other hand, it was shown in [6] that, for $n \geq 3$, if the external force g satisfies the oddness condition

$$g(-x,t) = -g(x,t) \quad (x \in \mathbb{R}^n,\ t \in \mathbb{R}) \tag{1.3}$$

and if g is small enough in some weighted Sobolev space, then there exists a time periodic solution (ρ_{per}, v_{per}) for (1.1) around $(\rho_*, 0)$ and $u_{per}(t) = (\rho_{per}(t) - \rho_*, v_{per}(t))$ satisfies

$$\begin{aligned}&\sup_{t\in[0,T]} (\|u_{per}(t)\|_{L^2} + \|x\nabla u_{per}(t)\|_{L^2}) \\ &\quad \leq C\{\|(1+|x|)g\|_{C([0,T];L^1\cap L^2)} + \|(1+|x|)g\|_{L^2(0,T;H^{m-1})}\}.\end{aligned} \tag{1.4}$$

Furthermore, if $\|u(0)\|_{H^s\cap L^1} << 1$, the time periodic solution (ρ_{per}, v_{per}) is asymptotically stable, and the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^2} = O(t^{-\frac{n}{4}}) \text{ as } t \to \infty. \tag{1.5}$$

In this paper we will show the existence of a time periodic solution for (1.1) without assuming the oddness condition (1.3) for $n \geq 3$ under sufficiently small g. Furthermore, we show that the time periodic solution (ρ_{per}, v_{per}) is asymptotically stable under sufficiently small g and initial perturbations, and the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^\infty} \to 0$$

as $t \to \infty$.

We will prove the existence of a time periodic solution around $(\rho_*, 0)$ by an iteration argument by using the time-T-map associated with the linearized problem at $(\rho_*, 0)$. As in [6] we formulate the time periodic problem as a system of equations for low frequency part and high frequency part of the solution. (Cf., [7, 11].) In the proof of the existence of a time periodic solution without assuming the oddness condition (1.3), there are two key observations. One is concerned with the spectrum of the time-T-map for the low frequency part. Another one is concerned with the convection term $v\cdot\nabla v$. As for the former matter, we need to investigate $(I - S_1(T))^{-1}$, where $S_1(T) = e^{-TA}$ with A being the linearized operator around $(\rho_*, 0)$ which acts on functions whose Fourier transforms have their supports in $\{\xi \in \mathbb{R}^n; |\xi| \leq r_\infty\}$ for some $r_\infty > 0$. (See (3.16) and (3.17) bellow.) We will show that the leading part of $(I - S_1(T))^{-1}$ coincides with the solution operator for the linearized stationary problem used by Shibata-Tanaka in [14]. In fact, the Fourier transform of $(I - S_1(T))^{-1}F$ takes the form $(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}$, where $\hat{F}$ is the Fourier transform of F and

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top\xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top\xi. \end{pmatrix}$$

By using the spectral resolution, we see that

$$(I - e^{-T\hat{A}_\xi})^{-1} \sim -\frac{1}{T}\begin{pmatrix} \frac{\nu+\tilde{\nu}}{\gamma^2} & -\frac{i^\top\xi}{\gamma|\xi|^2} \\ -\frac{i\xi}{\gamma|\xi|^2} & \frac{1}{\nu|\xi|^2}\left(I_n - \frac{\xi^\top\xi}{|\xi|^2}\right) \end{pmatrix} \quad \text{as } \xi \to 0.$$

The right-hand side is the solution operator for the linearized stationary problem in the Fourier space. This motivates us to introduce a weighted L^∞ space for the low frequency part employed in the study of the stationary problem in [14].

As for the high frequency part, we will employ the weighted energy estimates established in [6].

Another point in our analysis is concerned with the convection term $v \cdot \nabla v$. Due to the slow decay of $v(x,t)$ as $|x| \to \infty$, there appears some difficulty in estimating $v \cdot \nabla v$. To overcome this, we will use the momentum formulation for the low frequency part, which takes a form of a conservation lows, and the velocity formulation for the high frequency part, for which the energy method works well. We also note that, in estimating the high frequency part of $v \cdot \nabla v$, we will use the fact that a Poincaré type inequality $\|f\|_{L^2} \le C\|\nabla f\|_{L^2}$ holds for the high frequency part.

The asymptotic stability of the time periodic solution (ρ_{per}, v_{per}) can be proved as in the argument in Kagei and Kawashima [4] by using the Hardy inequality.

2 Main results

To state our results, we define function spaces with spatial weight.

For a nonnegative integer ℓ and $1 \le p \le \infty$, we denote by L^p_ℓ the weighted L^p space defined by

$$L^p_\ell = \{u \in L^p; \|u\|_{L^p_\ell} := \|(1+|x|)^\ell u\|_{L^p} < \infty\}.$$

Let k and ℓ be nonnegative integers. We define the weighted L^2-Sobolev space H^k_ℓ by

$$H^k_\ell = \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$H^k_\ell = \left\{ u \in H^k; \|u\|_{H^k_\ell} := \left(\sum_{|\alpha| \le k} \|\partial_x^\alpha u\|^2_{L^2_\ell} \right)^{\frac{1}{2}} < +\infty \right\}$$

We also introduce function spaces of T-periodic functions in t. We denote by $C_{per}(\mathbb{R}; X)$ the set of all T-periodic continuous functions with values in X equipped with the norm $\|\cdot\|_{C([0,T];X)}$; and we denote by $L^2_{per}(\mathbb{R};X)$ the set of all T-periodic locally square integrable functions with values in X equipped with the norm $\|\cdot\|_{L^2(0,T;X)}$.

Our result on the existence of a time periodic solution is stated as follows.

Theorem 2.1. *Let $n \geq 3$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right]+1$. Assume that $g(x,t)$ satisfies (1.2) and $g \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_n) \cap L^2_{per}(\mathbb{R}; H^{s-1}_{n-1})$. We set*

$$[g]_s := \|g\|_{C([0,T];L^1\cap L^\infty_n)\cap L^2(0,T;H^{s-1}_{n-1})}$$

Then there exists a constant $\delta > 0$ such that if $[g]_s \leq \delta$, then the system (1.1) has a time-periodic solution $u_{per} = (\rho_{per} - \rho_*, v_{per}) \in C_{per}(\mathbb{R}; H^s)$ satisfying

$$\sup_{t\in[0,T]} \left(\||x|^{n-1}\rho_{per}(t)\|_{L^\infty} + \||x|^{n-2}v_{per}(t)\|_{L^\infty} + \||x|^{n-1}\nabla v_{per}(t)\|_{L^\infty}\right) \leq C[g]_s.$$

We next consider the stability of the time-periodic solution obtained in Theorem 2.1.

Let ${}^\top(\rho_{per}, v_{per})$ be the periodic solution given in Theorem 2.1. We denote the perturbation by $u = {}^\top(\phi, w)$, where $\phi = \rho - \rho_{per}, w = v - v_{per}$. Substituting $\rho = \phi + \rho_{per}$ and $v = w + v_{per}$ into (1.1), we see that the perturbation $u = {}^\top(\phi, w)$ is governed by

$$\begin{cases} \partial_t\phi + v_{per}\cdot\nabla\phi + \phi\mathrm{div}v_{per} + \rho_{per}\mathrm{div}w + w\cdot\nabla\rho_{per} = f^0, \\ \partial_t w + v_{per}\cdot\nabla w + w\cdot\nabla v_{per} - \frac{\mu}{\rho_{per}}\Delta w - \frac{\mu+\mu'}{\rho_{per}}\nabla\mathrm{div}w \\ \quad +\frac{\phi}{\rho_{per}^2}(\mu\Delta v_{per} + (\mu+\mu')\nabla\mathrm{div}v_{per}) + \nabla(\frac{p'(\rho_{per})}{\rho_{per}}\phi) = \tilde{f}, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} f^0 &= -\mathrm{div}(\phi w), \\ \tilde{f} &= -w\cdot\nabla w - \frac{\phi}{\rho_{per}(\rho_{per}+\phi)}(\mu\Delta w + (\mu+\mu')\nabla\mathrm{div}w) \\ &\quad + \frac{\phi}{\rho_{per}(\rho_{per}+\phi)}(\frac{\phi}{\rho_{per}}\mu\Delta v_{per} + \frac{\phi}{\rho_{per}}(\mu+\mu')\nabla\mathrm{div}v_{per}) \\ &\quad + \frac{\phi}{\rho_{per}^2}\nabla(p^{(2)}(\rho_{per},\phi)\phi) + \frac{\phi^2}{\rho_{per}^2(\rho_{per}+\phi)}\nabla(p(\rho_{per}+\phi)) \\ &\quad + \frac{1}{\rho_{per}}\nabla(p^{(3)}(\rho_{per},\phi)\phi^2), \\ p^{(2)}(\rho_{per},\phi) &= \int_0^1 p'(\rho_{per}+\theta\phi)d\theta, \end{aligned}$$

$$p^{(3)}(\rho_{per},\phi)=\int_0^1(1-\theta)p''(\rho_{per}+\theta\phi)d\theta.$$

We consider the initial value problem for (2.1) under the initial condition

$$u|_{t=0}=u_0={}^\top(\phi_0,w_0). \tag{2.2}$$

Our result on the stability of the time-periodic solution is stated as follows.

Theorem 2.2. *Let $n\geq 3$ and let s be an integer satisfying $s\geq\left[\frac{n}{2}\right]+1$. Assume that $g(x,t)$ satisfies (1.2) and $g\in C_{per}(\mathbb{R};L^1\cap L^\infty_n)\cap L^2_{per}(\mathbb{R};H^s_{n-1})$. Then there exists constants $\delta_1>$ and $\epsilon>0$ such that if*

$$[g]_{s+1}\leq\delta_1,\quad \|(\rho(0)-\rho_{per}(0),v(0)-v_{per}(0))\|_{H^s}\leq\epsilon,$$

then there exists a unique global solution $u={}^\top(\phi,w)$ of (2.1)-(2.2) satisfies

$$u(t)\in C([0,\infty);H^s),$$
$$\|u(t)\|^2_{H^s}+\int_0^t\|\nabla u(\tau)\|^2_{H^{s-1}\times H^s}d\tau\leq C\|u(0)\|^2_{H^s},$$
$$\|u(t)\|_{L^\infty}\to 0\quad(t\to\infty).$$

It is not difficult to see that Theorem 3.2 can be proved by the energy method ([4], [10]), since the Hardy inequality works well to deal with the linear terms including (ρ_{per},v_{per}) due to the estimate for (ρ_{per},v_{per}) in Theorem 3.1; and so the proof is omitted here.

3 Outline of the proof of the main result

3.1 Formulation

We formulate (1.1) as follows. Substituting $\phi=\frac{\rho-\rho_*}{\rho_*}$ and $w=\frac{v}{\gamma}$ with $\gamma=\sqrt{p'(\rho_*)}$ into (1.1), we see that (1.1) is rewritten as

$$\partial_t u+Au=-B[u]u+G(u,g), \tag{3.1}$$

where

$$A=\begin{pmatrix}0 & \gamma\mathrm{div}\\ \gamma\nabla & -\nu\Delta-\tilde\nu\nabla\mathrm{div}\end{pmatrix},\quad \nu=\frac{\mu}{\rho_*},\quad \tilde\nu=\frac{\mu+\mu'}{\rho_*}, \tag{3.2}$$

$$B[\tilde u]u=\gamma\begin{pmatrix}\tilde w\cdot\nabla\phi\\ 0\end{pmatrix}\ \text{for } u={}^\top(\phi,w),\ \tilde u={}^\top(\tilde\phi,\tilde w) \tag{3.3}$$

and

$$G(u,g) = \begin{pmatrix} F^0(u) \\ \tilde{F}(u,g) \end{pmatrix}, \quad (3.4)$$

$$F^0(u) = -\gamma\phi \mathrm{div} w, \quad (3.5)$$

$$\tilde{F}(u,g) = -\gamma(1+\phi)(w\cdot\nabla w) - \phi\partial_t w - \nabla(p^{(1)}(\phi)\phi^2) + \frac{1+\phi}{\gamma} g, \quad (3.6)$$

$$p^{(1)}(\phi) = \frac{\rho_*}{\gamma}\int_0^1 (1-\theta)p''((1+\theta\phi))d\theta.$$

As in [6], to solve the time periodic problem for (3.1), we decompose u into a low frequency part u_1 and a high frequency part u_∞, and then, we rewrite the problem into a system of equations for u_1 and u_∞.

To decompose u, We introduce operators which decompose a function into its low and high frequency parts. Operators P_1 and P_∞ on L^2 are defined by

$$P_j f = \mathcal{F}^{-1}\hat{\chi}_j \mathcal{F}[f] \quad (f \in L^2, j = 1, \infty),$$

where

$$\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j = 1,\infty), \quad 0 \le \hat{\chi}_j \le 1 \quad (j = 1, \infty),$$

$$\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \le r_1), \\ 0 & (|\xi| \ge r_\infty), \end{cases}$$

$$\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi), \ 0 < r_1 < r_\infty.$$

We fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ in such a way that the estimate (3.19) in Lemma 3.11 below holds for $|\xi| \le r_\infty$.

As in [6], we set

$$u_1 = P_1 u, \quad u_\infty = P_\infty u.$$

Applying the operators P_1 and P_∞ to (3.1), we obtain,

$$\partial_t u_1 + A u_1 = F_1(u_1 + u_\infty, g), \quad (3.7)$$

$$\partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) = F_\infty(u_1 + u_\infty, g). \quad (3.8)$$

Here

$$F_1(u_1 + u_\infty, g) = P_1[-B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g)],$$

$$F_\infty(u_1 + u_\infty, g) = P_\infty[-B[u_1 + u_\infty]u_1 + G(u_1 + u_\infty, g)].$$

Suppose that (3.7) and (3.8) are satisfied by some functions u_1 and u_∞. Then by adding (3.7) to (3.8), we obtain

$$\partial_t(u_1 + u_\infty) + A(u_1 + u_\infty) = -P_\infty(B[u_1 + u_\infty]u_\infty) + (P_1 + P_\infty)F(u_1 + u_\infty, g)$$

$$= \quad -B[u_1+u_\infty](u_1+u_\infty)+G(u_1+u_\infty,g).$$

Set $u=u_1+u_\infty$, then we have

$$\partial_t u + Au + B[u]u = G(u,g).$$

Consequently, if we show the existence of a pair of functions $\{u_1,u_\infty\}$ satisfying (3.7)-(3.8), then we can obtain a solution u of (3.1).

We next introduce function spaces for the low frequency part and the high frequency part.

We set $\mathscr{Z}_{(1)}(a,b) := C([a,b];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})$ for the low frequency part, where

$$\begin{aligned}
&\mathscr{X}_{(1)} = \{\phi;\text{supp }\hat{\phi}\subset\{|\xi|\le r_\infty\}, \|\phi\|_{\mathscr{X}_{(1)}}<+\infty\},\\
&\|\phi\|_{\mathscr{X}_{(1)}} := \|\nabla\phi\|_{L^2_1}+\|\phi\|_{L^\infty_{n-1}},\\
&\mathscr{Y}_{(1)} = \{w;\text{supp }\hat{w}\subset\{|\xi|\le r_\infty\}; \|w\|_{\mathscr{Y}_{(1)}}<+\infty\},\\
&\|w\|_{\mathscr{Y}_{(1)}} := \sum_{j=1}^{2}\|\nabla^j w\|_{L^2_{j-1}}+\sum_{j=0}^{1}\|\nabla^j w\|_{L^\infty_{n-2+j}}.
\end{aligned}$$

These spaces are similar to the ones introduced in the stationary problem by Shibata-Tanaka [14].

On the other hand, we define the weighted Sobolev space for the high frequency part by

$$H^k_{(\infty),n-1} = \{u\in H^k;\text{ supp }\hat{u}\subset\{|\xi|\ge r_1\},\ \|u\|_{H^k_{n-1}}<+\infty\}$$

for $k=s,s-1$. Then we introduce a function space for the high frequency part by

$$\mathscr{Z}^k_{(\infty),n-1}(a,b) = C([a,b];H^k_{(\infty),n-1})\times[C([a,b];H^k_{(\infty),n-1})\cap L^2([a,b];H^{k+1}_{(\infty),n-1})]$$

Finally, We set

$$\begin{aligned}
&X^k(a,b) := \{\{u_1,u_\infty\};u_1\in\mathscr{Z}_{(1)}(a,b),u_\infty\in\mathscr{Z}^k_{(\infty)}(a,b)\},\\
&\|\{u_1,u_\infty\}\|_{X^k(a,b)} = \|u_1\|_{\mathscr{Z}_{(1)}(a,b)}+\|u_\infty\|_{\mathscr{Z}^k_{(\infty)}(a,b)}.
\end{aligned}$$

In this paper, we consider the low frequency part u_1 in a weighted L^∞ space. To do so, the velocity formulation is not suitable, and, instead, we use the momentum formulation for the low frequency part.

Let us now reformulate the system (3.7)-(3.8) by using the momentum. We set m_1 and $u_{1,m}$ by

$$m_1 = w_1 + P_1(\phi w),\quad u_{1,m} = {}^{\top}(\phi_1,m_1), \tag{3.9}$$

where $\phi = \phi_1 + \phi_\infty$, and $w = w_1 + w_\infty$. Then, we see that $\{u_{1,m}, u_\infty\}$ defined by (3.9) satisfies the following system of equations.

Lemma 3.1. ([15, Lemma 4.5]) *Assume that* $\{u_1, u_\infty\}$ *satisfies the system* (3.7)-(3.8). *Then,* $\{u_{1,m}, u_\infty\}$ *satisfies the following system:*

$$\begin{aligned} \partial_t u_{1,m} + A u_{1,m} &= F_{1,m}(u_1 + u_\infty, g), \\ \partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \end{aligned} \tag{3.10}$$

Here

$$\begin{aligned} F_{1,m}(u_1 + u_\infty, g) &= {}^\top(0, \tilde{F}_{1,m}(u_1 + u_\infty, g)), \\ \tilde{F}_{1,m}(u_1 + u_\infty, g) &= -P_1\{\mu\Delta(\phi w) + \tilde{\mu}\nabla \mathrm{div}\,(\phi w) + \frac{\rho_*}{\gamma}\nabla(p^{(1)}(\phi)\phi^2) \\ &\quad + \gamma \mathrm{div}\,((1+\phi) w \otimes w) - \frac{1}{\gamma}((1+\phi)g)\}. \end{aligned} \tag{3.11}$$

Conversely, one can see that the momentum formulation (3.8), (3.9) and (3.10) gives the solution $\{u_1, u_\infty\}$ of (3.7)-(3.8) if $\phi = \phi_1 + \phi_\infty$ is sufficiently small. In fact, we have the following Lemma.

Lemma 3.2. ([15, Lemma 4.6]) (i) *Let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. Then there exists a positive constant δ_0 such that if $\phi = \phi_1 + \phi_\infty$ satisfies $\sup_{t\in[a,b]} \|\phi\|_{L^\infty_{n-1}} \leq \delta_0$, then there uniquely exists $w_1 \in C([a,b]; \mathscr{Y}_{(1)})$ that satisfies*

$$w_1 = m_1 - P_1(\phi(w_1 + w_\infty)) \tag{3.12}$$

where $\phi = \phi_1 + \phi_\infty$. Furthermore, there hold the estimates

$$\|w_1\|_{C([a,b];\mathscr{Y}_{(1)})} \leq C(\|m_1\|_{C([a,b];\mathscr{Y}_{(1)})} + \|w_\infty\|_{C([a,b];L^2)}). \tag{3.13}$$

(ii) *Let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. Assume that $\phi = \phi_1 + \phi_\infty$ satisfies*

$$\sup_{t\in[a,b]} \|\phi\|_{L^\infty_{n-1}} \leq \delta_0$$

and $\{u_{1,m}, u_\infty\}$ satisfies

$$\begin{aligned} \partial_t u_{1,m} + A u_{1,m} &= F_{1,m}(u_1 + u_\infty, g), \\ w_1 &= m_1 - P_1(\phi w), \\ \partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \end{aligned}$$

Here $w = w_1 + w_\infty$ with w_1 defined by (3.12). *Then $\{u_1, u_\infty\}$ with $u_1 = {}^\top(\phi_1, w_1)$ satisfies* (3.7)-(3.8).

By Lemma 3.2, if we show the existence of a pair of functions $\{u_{1,m}, u_\infty\} \in X^s(a,b)$ satisfying (3.8), (3.10) and (3.12), then we can obtain a solution $\{u_1, u_\infty\} \in X^s(a,b)$ satisfying (3.7)-(3.8). Therefore, we will consider (3.8), (3.10) and (3.12) instead of (3.7)-(3.8).

We look for a time periodic solution u for the system (3.8), (3.10) and (3.12). To solve the time periodic problem for (3.8), (3.10) and (3.12), we introduce solution operators for the following linear problems:

$$\begin{cases} \partial_t u_{1,m} + Au_{1,m} = F_{1,m}, \\ u_{1,m}|_{t=0} = u_{01,m}, \end{cases} \tag{3.14}$$

and

$$\begin{cases} \partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \\ u_\infty|_{t=0} = u_{0\infty}, \end{cases} \tag{3.15}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01,m}, u_{0\infty}, F_{1,m}$ and F_∞ are given functions.

To formulate the time periodic problem, we denote by $S_1(t)$ the solution operator for (3.14) with $F_{1,m} = 0$, and by $\mathscr{S}_1(t)$ the solution operator for (3.14) with $u_{01,m} = 0$. We also denote by $S_{\infty,\tilde{u}}(t)$ the solution operator for (3.15) with $F_\infty = 0$ and by $\mathscr{S}_{\infty,\tilde{u}}(t)$ the solution operator for (3.15) with $u_{0\infty} = 0$. (The precise definition of these operators will be given later.)

As in [6], we will look for a $\{u_{1,m}, u_\infty\}$ satisfying

$$\begin{cases} u_{1,m}(t) = S_1(t)u_{01,m} + \mathscr{S}_1(t)[F_{1,m}(u,g)], \\ u_\infty(t) = S_{\infty,u}(t)u_{0\infty} + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)], \end{cases} \tag{3.16}$$

where

$$\begin{cases} u_{01,m} = (I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)], \\ u_{0\infty} = (I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)], \end{cases} \tag{3.17}$$

$u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

Let us explain the relation between (3.16)-(3.17) and the time periodic problem (3.8), (3.10) and (3.12) for the reader's convenience.

If $\{u_{1,m}, u_\infty\}$ satisfies (3.8), (3.10) and (3.12), then $u_{1,m}(t)$ and $u_\infty(t)$ satisfy (3.16). Suppose that $\{u_{1,m}, u_\infty\}$ is a T-time periodic solution of (3.16). Then, since $u_{1,m}(T) = u_{1,m}(0)$ and $u_\infty(T) = u_\infty(0)$, we see that

$$\begin{cases} (I - S_1(T))u_{1,m}(0) = \mathscr{S}_1(T)[F_{1,m}(u,g)], \\ (I - S_{\infty,u}(T)u_\infty(0) = \mathscr{S}_{\infty,u}(T)[F_\infty(u,g)], \end{cases}$$

where $u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

Therefore if $(I - S_1(T))$ and $(I - S_{\infty,u}(T))$ are invertible in a suitable sense, then one obtains (3.16)-(3.17).

Hereafter we abbreviate $u_{1,m}$ to u_1. We set

$$\Gamma_{(1)}[\{u_1, u_\infty\}] := S_1(t)(I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)] + \mathscr{S}_1(t)[F_{1,m}(u,g)],$$

$$\Gamma_{(\infty)}[\{u_1, u_\infty\}] := S_{\infty,u}(t)u_{0\infty}(I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)] + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)],$$

where $u = {}^\top(\phi, w)$ is a function given by u_1 and u_∞ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

To obtain a T-time periodic solution of (3.8), (3.10) and (3.12), we look for a pair of functions $\{u_1, u_\infty\}$ satisfying

$$\begin{cases} u_1 = \Gamma_{(1)}[\{u_1, u_\infty\}], \\ u_\infty = \Gamma_{(\infty)}[\{u_1, u_\infty\}]. \end{cases}$$

Hence, We estimate $\Gamma_{(1)}[\{u_1, u_\infty\}]$ in subsection 3.2; and we estimate $\Gamma_{(\infty)}[\{u_1, u_\infty\}]$ in subsection 3.3.

In the remaining of this subsection we introduce some lemmas which will be used in the proof of Theorem 2.1.

We first derive some inequalities for the low frequency part.

Lemma 3.3. ([6, Lemma 4.3]) (i) *Let k be a nonnegative integer. Then P_1 is a bounded linear operator from L^2 to H^k. In fact, it holds that*

$$\|\nabla^k P_1 f\|_{L^2} \le C\|f\|_{L^2} \qquad (f \in L^2).$$

As a result, for any $2 \le p \le \infty$, P_1 is bounded from L^2 to L^p.

(ii) *Let k be a nonnegative integer. Then there hold the estimates*

$$\|\nabla^k f_1\|_{L^2} + \|f_1\|_{L^p} \le C\|f_1\|_{L^2} \quad (f \in L^2_{(1)}),$$

where $2 \le p \le \infty$.

The following inequality is concerned with the estimates of the weighted L^p norm for the low frequency part.

Lemma 3.4. ([15, Lemma 4.3]) *Let k and ℓ be nonnegative integers and let $1 \le p \le \infty$. Then there holds the estimate*

$$\| |x|^\ell \nabla^k f_1 \|_{L^p} \le C \| |x|^\ell f_1 \|_{L^p} \quad (f \in L^2_{(1)} \cap L^p_\ell).$$

The following lemma is related to the estimates for the integral kernels which will appear in the analysis of the low frequency part.

Lemma 3.5. ([15, Lemma 4.8]) *Let ℓ be a nonnegative integer and let $E(x) = \mathscr{F}^{-1}\hat{\Phi}_\ell$ $(x \in \mathbb{R}^n)$, where $\hat{\Phi}_\ell \in C^\infty(\mathbb{R}^n - \{0\})$ is a function satisfying*

$$\partial_\xi^\alpha \hat{\Phi}_\ell \in L^1 \quad (|\alpha| \le n - 3 + \ell),$$
$$|\partial_\xi^\beta \hat{\Phi}_\ell| \le C|\xi|^{-2-|\beta|+\ell} \quad (\xi \ne 0,\ |\beta| \ge 0).$$

Then the following estimate holds for $x \ne 0$.

$$|E(x)| \le C|x|^{-(n-2+\ell)}.$$

We will also use the following lemma for the analysis of the low frequency part.

Lemma 3.6. ([15, Lemma 4.9]) (i) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C}{(1+|x|)^{|\alpha|+n-2}} \quad (|\alpha| = 0, 1, 2). \tag{3.18}$$

Assume that f is a scalar function satisfying $\|f\|_{L^\infty_n \cap L^1} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.

$$|[\partial_x^\alpha E * f](x)| \le \frac{C}{(1+|x|)^{|\alpha|+n-2}} \|f\|_{L^\infty_n \cap L^1}.$$

(ii) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying* (3.18). *Assume that f is a scalar function of the form: $f = \partial_{x_j} f_1$ for some $1 \le j \le n$ satisfying $\|\partial_{x_j} f_1\|_{L^\infty_n} + \|f_1\|_{L^\infty_{n-1}} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.*

$$|[\partial_x^\alpha E * f](x)| \le \frac{C}{(1+|x|)^{|\alpha|+n-2}} (\|\partial_{x_j} f_1\|_{L^\infty_n} + \|f_1\|_{L^\infty_{n-1}}).$$

(iii) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C}{(1+|x|)^{|\alpha|+n-1}} \quad (|\alpha| = 0, 1).$$

Assume that f is a scalar function satisfying $\|f\|_{L^\infty_n} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.

$$|[\partial_x^\alpha E * f](x)| \leq \frac{C \log|x|}{(1+|x|)^{|\alpha|+n-1}} \|f\|_{L^\infty_n}.$$

Remark 3.7. When $n = 3$, Lemma 3.6 (i) and (ii) are given in the stationary problem [14, Lemma 2.5].

As for the high frequency part, we have the following Poincaré type inequalities.

Lemma 3.8. ([6, Lemma 4.4]) (i) *Let k be a nonnegative integer. Then P_∞ is a bounded linear operator on H^k.*

(ii) *There hold the inequalities*

$$\begin{aligned} \|P_\infty f\|_{L^2} &\leq C\|\nabla f\|_{L^2} \quad (f \in H^1), \\ \|f_\infty\|_{L^2} &\leq C\|\nabla f_\infty\|_{L^2} \quad (f_\infty \in H^1_{(\infty)}). \end{aligned}$$

Lemma 3.9. ([15, Lemma 4.13]) *Let $\ell \in \mathbb{N}$. Then there exists a positive constant C depending only on ℓ such that*

$$\|P_\infty f\|_{L^2_\ell} \leq C\|\nabla f\|_{L^2_\ell}.$$

3.2 The estimates for $\Gamma_{(1)}$

In this section we investigate $S_1(t)$ and $\mathscr{S}_1(t)$ and establish estimates for $\Gamma_{(1)}$.

We denote by A_1 the restriction of A on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. Using Lemma 3.4, we have the following properties of $S_1(t)$ and $\mathscr{S}_1(t)$.

Proposition 3.10. ([15, Proposition 5.1]) (i) *A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and $S_1(t) = e^{-tA_1}$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. Furthermore, $S_1(t)$ satisfies*

$$S_1(t)u_1 \in C([0,T']; \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}), \quad \partial_t S_1(\cdot)u_1 \in C([0,T']; L^2)$$

for each $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and all $T' > 0$,

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \,(= -AS_1(t)u_1), \ S_1(0)u_1 = u_1 \ \ \textit{for} \ \ u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \le C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}},$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $k = 0, 1$, *where* $T' > 0$ *is any given positive number and* C *is a positive constant depending on* T'.

(ii) *Let the operator* $\mathscr{S}_1(t)$ *be defined by*

$$\mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)\,d\tau$$

for $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$. *Then*

$$\mathscr{S}_1(\cdot)F_1 \in C^1([0,T];\mathscr{X}_{(1)}) \times [C([0,T];\mathscr{Y}_{(1)}) \times H^1(0,T;\mathscr{Y}_{(1)})]$$

for each $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$ *and*

$$\partial_t \mathscr{S}_1(t)F_1 + A_1\mathscr{S}_1(t)F_1 = F_1(t), \;\; \mathscr{S}_1(0)F_1 = 0,$$

$$\|\partial_t^k \mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})},$$

for $k = 0, 1$, *where* C *is a positive constant depending on* T.

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')F_1 = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ *and* $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$.

To estimate $\Gamma_{(1)}$, we prepare the following lemmas. The first Lemma is related to the asymptotic expansion of the linearlized semigroup around $|\xi| = 0$.

Lemma 3.11. ([10]) (i) *The set of all eigenvalues of* $-\hat{A}_\xi$ *consists of* $\lambda_j(\xi)$ $(j = 1, \pm)$, *where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$, *then*

$$\mathrm{Re}\,\lambda_\pm = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2, \quad \mathrm{Im}\,\lambda_\pm = \pm\gamma|\xi|\sqrt{1 - \frac{(\nu+\tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *For* $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$, $e^{-t\hat{A}_\xi}$ *has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *is eigenprojections for* $\lambda_j(\xi)\,(j=1,\pm)$, *and* $\Pi_j(\xi)\,(j=1,\pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix},$$

$$\Pi_\pm(\xi) = \pm\frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} -\lambda_\mp & -i\gamma^\top\xi \\ -i\gamma\xi & \lambda_\pm\frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$, *then there exist a constant* $C > 0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \le C\,(j=1,\pm) \tag{3.19}$$

hold for $|\xi| \le r_\infty$.

Hereafter we fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (3.19) in Lemma 3.11 holds for $|\xi| \le r_\infty$.

Lemma 3.12. ([15, Lemma 5.4]) *Let* α *be a multi-index. Then the following estimates hold true uniformly for* ξ *with* $|\xi| \le r_\infty$ *and* $t \in [0,T]$.

(i) $|\partial_\xi^\alpha\lambda_1| \le C|\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha\lambda_\pm| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 0)$.

(ii) $|(\partial_\xi^\alpha\Pi_1)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|$, $|(\partial_\xi^\alpha\Pi_\pm)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{F}_1|$ $(|\alpha| \ge 0)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(iii) $|\partial_\xi^\alpha(e^{\lambda_1 t})| \le C|\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1)$.

(iv) $|\partial_\xi^\alpha(e^{\lambda_\pm t})| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}_1| \le C(|\xi|^{1-|\alpha|}|\hat{F}_1^0| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|)$ $(|\alpha| \ge 1)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(vi) $|\partial_\xi^\alpha(I - e^{\lambda_1 t})^{-1}| \le C|\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0)$.

(vii) $|\partial_\xi^\alpha(I - e^{\lambda_\pm t})^{-1}| \le C|\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0)$.

We are now in a position to give estimates for $\Gamma_{(1)}$.

Proposition 3.13. *Let* $n \ge 3$ *and let* s *be a nonnegative integer satisfying* $s \ge [\frac{n}{2}] + 1$.

(i) *Assume that* $u_1 = {}^\top(\phi_1, m_1)$ *and* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *satisfy*

$$\|\{u_1, u_\infty\}\|_{X^s(0,T)} << 1.$$

Then it holds that

$$\|\Gamma_{(1)}[\{u_1,u_\infty\}]\|_{\mathscr{Z}^s_{(\infty)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)} + C(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)})[g]_s$$

uniformly for u_1 *and* u_∞.

(ii) *Assume that* $u_1^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ *and* $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)})$ *satisfy*

$$\|\{u_1^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} << 1 \quad (k=1,2).$$

Then it holds that

$$\begin{aligned}
&\|\Gamma_{(1)}[\{u_1^{(1)}, u_\infty^{(1)}\}] - \Gamma_{(1)}[\{u_1^{(2)}, u_\infty^{(2)}\}]\|_{\mathscr{Z}^s_{(\infty)}(0,T)} \\
&\le C\sum_{k=1}^{2} \|\{u_1^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \|\{u_1^{(1)} - u_1^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\
&\quad + C[g]_s \|\{u_1^{(1)} - u_1^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}
\end{aligned}$$

uniformly for $u_1^{(k)}$ *and* $u_\infty^{(k)}$ $(k=1,2)$.

Proof. As for (i), we set

$$\begin{aligned}
\Gamma_{(1),1}[\{u_1,u_\infty\}] &:= S_1(t)(I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)], \\
\Gamma_{(1),2}[\{u_1,u_\infty\}] &:= \mathscr{S}_1(t)[F_{1,m}(u,g)],
\end{aligned}$$

where $F_{1,m}(u,g)$ is the same one defined in (3.11). As for $\Gamma_{(1),1}[\{u_1,u_\infty\}]$, by Proposition 3.10 and well-known properties of the Fourier transform, we have

$$\begin{aligned}
\Gamma_{(1),1}[\{u_1,u_\infty\}] &= S_1(t)(I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)] \\
&= \mathscr{F}^{-1}\Big\{e^{-t\hat{A}_\xi}(I - e^{-T\hat{A}_\xi})^{-1}\int_0^T e^{-(T-\tau)\hat{A}_\xi}\hat{F}_{1,m}(\tau,u,g)d\tau\Big\} \\
&=: \int_0^T E_1(t,\tau) * F_{1,m}(\tau,u,g)d\tau, \qquad (3.20)
\end{aligned}$$

where

$$E_1(t,\tau) = \mathcal{F}^{-1}\{\hat{\chi}_0 e^{-t\hat{A}_\xi}(I - e^{-T\hat{A}_\xi})^{-1}e^{-(T-\tau)\hat{A}_\xi}\},$$

χ_0 is a cut-off function defined by $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$ with $\hat{\chi}_0$ satisfying

$$\hat{\chi}_0 \in C^\infty(\mathbb{R}^n), \quad 0 \le \hat{\chi}_0 \le 1, \quad \hat{\chi}_0 = 1 \text{ on } \{|\xi| \le r_\infty\} \text{ and } \operatorname{supp}\hat{\chi}_0 \subset \{|\xi| \le 2r_\infty\}.$$

By Lemma 3.11, $e^{-t\hat{A}_\xi}$ has the spectral resolution

$$e^{-t\hat{A}_\xi} = \sum_j e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where λ_j and Π_j $(j = 1, \pm)$ are the same ones in Lemma 3.11. Therefore, we see that

$$(I - e^{-T\hat{A}_\xi})^{-1} = (I - e^{T\lambda_1})^{-1}\Pi_1 + (I - e^{T\lambda_+})^{-1}\Pi_+ + (I - e^{T\lambda_-})^{-1}\Pi_-. \quad (3.21)$$

Let α be a multi-index satisfying $|\alpha| \geq 0$. It follows from Lemma 3.12 that

$$\sum_j |\partial_x^\alpha E_1(x)| \leq C \int_{|\xi| \leq 2r_\infty} |\xi|^{-2} d\xi \quad (x \in \mathbb{R}^n).$$

Since $\int_{|\xi| \leq r_\infty} |\xi|^{-2} d\xi < \infty$ for $n \geq 3$, we see that

$$\sum_j |\partial_x^\alpha E_1(x)| \leq C \quad (x \in \mathbb{R}^n), \quad (3.22)$$

where $C > 0$ is a constant depending on α, T and n. By Lemma 3.12, we have

$$\begin{aligned} |\partial_\xi^\beta((i\xi)^\alpha \hat{\chi}_0 (I - e^{\lambda_1 T})^{-1}\Pi_1)| &\leq C|\xi|^{-2+|\alpha|-|\beta|} \text{ for } |\beta| \geq 0, \\ |\partial_\xi^\beta((i\xi)^\alpha \hat{\chi}_0 (I - e^{\lambda_\pm T})^{-1}\Pi_\pm)| &\leq C|\xi|^{-1+|\alpha|-|\beta|} \text{ for } |\beta| \geq 0. \end{aligned}$$

It then follows from Lemma 3.5 and (3.21) that

$$|\partial_x^\alpha E_1(x)| \leq C|x|^{-(n-2+|\alpha|)}. \quad (3.23)$$

From (3.22) and (3.23), we obtain that

$$|\partial_x^\alpha E_1(x)| \leq C(1 + |x|)^{-(n-2+|\alpha|)} \quad (3.24)$$

uniformly for $x \in \mathbb{R}^n$.

We here estimate nonlinear and inhomogeneous terms. Concerning the estimate for $P_1(\gamma \mathrm{div}\, w \otimes w)$, by Lemma 3.2 (i), Lemma 3.6, Lemma 3.12, (3.20) and (3.24), we see that

$$\|S_1(t)(I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m,1}(u)]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|\{u_{1,m}, u_\infty\}\|^2_{X^s(0,T)}, \quad (3.25)$$

where $F_{1,m,1}(u) = {}^\top(0, P_1(\gamma \mathrm{div}\, w \otimes w))$. Similarly to (3.25), the remaining terms can be estimated by applying Lemma 3.2 (i), Lemma 3.5, Lemma 3.6, Lemma 3.11 and Lemma 3.12. Hence, we obtain the desired estimate for $\Gamma_{(1),1}$ The estimate for $\Gamma_{(1),2}$ can be proved in a similar manner to the proof of the estimate for $\Gamma_{(1),1}$.

The desired estimate in (ii) can be similarly obtained by applying Lemma 3.2 (i), Lemma 3.5, Lemma 3.6, Lemma 3.11 and Lemma 3.12. This completes the proof. □

3.3 The estimates for $\Gamma_{(\infty)}$

In this section we first state some properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in weighted Sobolev spaces which were obtained in [15]. Using the properties, we derive the estimates of $\Gamma_{(\infty)}$.

Let us consider the following initial value problem (3.15). Concerning the solvability of (3.15), we have the following

Proposition 3.14. ([15, Proposition 6.1]) *Let $n \geq 3$ and let s be an integer satisfying $s \geq [\frac{n}{2}] + 1$. Set $k = s - 1$ or s. Assume that*

$$\begin{aligned} &\nabla \tilde{w} \in C([0,T']; H^{s-1}) \cap L^2(0,T'; H^s), \\ &u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)}, \\ &F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T'; H^k_{(\infty)} \times H^{k-1}_{(\infty)}). \end{aligned}$$

Here T' is a given positive number. Then there exists a unique solution $u_\infty = {}^\top(\phi_\infty, w_\infty)$ of (3.15) satisfying

$$\phi_\infty \in C([0,T']; H^k_{(\infty)}),\ w_\infty \in C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)}) \cap H^1(0,T'; H^{k-1}_{(\infty)}).$$

In view of Proposition 3.14, $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ are defined as follows.

We fix an integer s satisfying $s \geq [\frac{n}{2}] + 1$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfying

$$\tilde{\phi} \in C_{per}(\mathbb{R}; H^s),\ \nabla \tilde{w} \in C_{per}(\mathbb{R}; H^{s-1}) \cap L^2_{per}(\mathbb{R}; H^s) \tag{3.26}$$

Let $k = s - 1$ or s. The operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ is defined by

$$u_\infty(t) = S_{\infty,\tilde{u}}(t) u_{0\infty} \ \text{ for } \ u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_\infty(t)$ is the solution of (3.15) with $F_\infty = 0$; and the operator $\mathscr{S}_{\infty,\tilde{u}}(t)$: $L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ is defined by

$$u_\infty(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \ \text{ for } \ F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_\infty(t)$ is the solution of (3.15) with $u_{0\infty} = 0$.

The operators $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ have the following properties.

Proposition 3.15. ([15, Proposition 6.3]) *Let $n \geq 3$ and let s be a nonnegative integer satisfying $s \geq [\frac{n}{2}] + 1$. Let $k = s - 1$ or s and let ℓ be a nonnegative integer. Assume that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (3.26). Then there exists a constant $\delta > 0$ such that the following assertions hold true if $\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta$.*

(i) *It holds that $S_{\infty,\tilde{u}}(\cdot)u_{0\infty} \in C([0,\infty); H^k_{(\infty),\ell})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell}$ and there exist constants $a > 0$ and $C > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|S_{\infty,\tilde{u}}(t)u_{0\infty}\|_{H^k_{(\infty),\ell}} \leq Ce^{-at}\|u_{0\infty}\|_{H^k_{(\infty),\ell}}$$

for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),\ell}$.

(ii) *It holds that $\mathscr{S}_{\infty,\tilde{u}}(\cdot)F_\infty \in C([0,T]; H^k_{(\infty),\ell})$ for each $F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty]\|_{H^k_{(\infty),\ell}} \leq C\left\{\int_0^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^k_{(\infty),\ell}\times H^{k-1}_{(\infty),\ell}}\, d\tau\right\}^{\frac{1}{2}}$$

for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ with a positive constant C depending on T.

(iii) *It holds that $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T)) < 1$, where $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T))$ is the spectral radius of $S_{\infty,\tilde{u}}(T)$ on $H^k_{(\infty),\ell}$.*

(iv) *$I - S_{\infty,\tilde{u}}(T)$ has a bounded inverse $(I - S_{\infty,\tilde{u}}(T))^{-1}$ on $H^k_{(\infty),\ell}$ and $(I - S_{\infty,\tilde{u}}(T))^{-1}$ satisfies*

$$\|(I - S_{\infty,\tilde{u}}(T))^{-1}u\|_{H^k_{(\infty),\ell}} \leq C\|u\|_{H^k_{(\infty),\ell}} \quad \text{for} \quad u \in H^k_{(\infty),\ell}.$$

Applying Proposition 3.15, we have the following estimate for a solution u_∞ of (3.15) satisfying $u_\infty(0) = u_\infty(T)$.

Proposition 3.16. ([15, Proposition 6.5]) *Let $n \geq 3$ and let s be a nonnegative integer satisfying $s \geq [\frac{n}{2}] + 1$. Assume that*

$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),n-1} \times H^{k-1}_{(\infty),n-1})$$

with $k = s - 1$ or s. Assume also that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (3.26). Then there exists a positive constant δ such that the following assertion holds true if

$$\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta.$$

The function

$$u_\infty(t) := S_{\infty,\tilde{u}}(t)(I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)[F_\infty] + \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \qquad (3.27)$$

is a solution of (3.15) *in* $\mathscr{Z}^k_{(\infty),n-1}(0,T)$ *satisfying* $u_\infty(0) = u_\infty(T)$ *and the estimate*

$$\|u_\infty\|_{\mathscr{Z}^k_{(\infty),n-1}(0,T)} \leq C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),n-1}\times H^{k-1}_{(\infty),n-1})}.$$

Applying Lemma 3.8, Lemma 3.9 and Proposition 3.16, we obtain the following estimates for $\Gamma_{(\infty)}$.

Proposition 3.17. *Let* $n \geq 3$ *and let* s *be a nonnegative integer satisfying* $s \geq [\frac{n}{2}] + 1$.

(i) *Assume that* $u_1 = {}^\top(\phi_1, m_1)$ *and* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *satisfy*

$$\|\{u_1, u_\infty\}\|_{X^s(0,T)} << 1.$$

Assume also that u_1 *and* u_∞ *satisfy* $u_1(0) = u_1(T)$ *and* $u_\infty(0) = u_\infty(T)$. *Then it holds that*

$$\|\Gamma_{(\infty)}[\{u_1, u_\infty\}]\|_{\mathscr{Z}^s_{(\infty)}(0,T)} \leq C\|\{u_1, u_\infty\}\|^2_{X^s(0,T)} + C(1 + \|\{u_1, u_\infty\}\|_{X^s(0,T)})[g]_s$$

uniformly for u_1 *and* u_∞.

(ii) *Assume that* $u_1^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ *and* $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)})$ *satisfy*

$$\|\{u_1^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} << 1 \quad (k = 1, 2).$$

Assume also that $u_1^{(k)}$ *and* $u_\infty^{(k)}$ *satisfy* $u_1^{(k)}(0) = u_1^{(k)}(T)$ *and* $u_\infty^{(k)}(0) = u_\infty^{(k)}(T)$ *for* $k = 1, 2$. *Then it holds that*

$$\begin{aligned} &\|\Gamma_{(\infty)}[\{u_1^{(1)}, u_\infty^{(1)}\}] - \Gamma_{(\infty)}[\{u_1^{(2)}, u_\infty^{(2)}\}]\|_{\mathscr{Z}^s_{(\infty)}(0,T)} \\ &\leq C\sum_{k=1}^{2} \|\{u_1^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)}\|\{u_1^{(1)} - u_1^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\ &\quad + C[g]_s\|\{u_1^{(1)} - u_1^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \end{aligned}$$

uniformly for $u_1^{(k)}$ *and* $u_\infty^{(k)}$ $(k = 1, 2)$.

Proof. As for (i), concerning the estimates for nonlinear and inhomogeneous terms, we here estimate only $P_\infty(w \cdot \nabla w)$, since the computation is not straightforward due to the slow decay of w_1 as $|x| \to \infty$. By Lemma 3.9, we see that

$$\|P_\infty(w \cdot \nabla w)\|_{L^2_{n-1}} \leq \|\nabla(w \cdot \nabla w)\|_{L^2_{n-1}}$$

$$
\begin{aligned}
&\leq C\|\nabla w \cdot \nabla w\|_{L^2_{n-1}} + \|w \cdot \nabla^2 w\|_{L^2_{n-1}} \\
&\leq C(\|(1+|x|)^{n-1}\nabla w\|_{L^\infty}\|\nabla w\|_{L^2} \\
&\quad + \|(1+|x|)^{n-2} w\|_{L^\infty}\|(1+|x|)\nabla^2 w\|_{L^2}). \qquad (3.28)
\end{aligned}
$$

For $1 \leq |\alpha| \leq s-1$, by Lemma 3.4 and Lemma 3.8, we see that

$$
\begin{aligned}
&\|P_\infty \partial_x^\alpha (w \cdot \nabla w)\|_{L^2_{n-1}} \\
&\leq \|w \cdot \partial_x^\alpha \nabla w\|_{L^2_{n-1}} + \|[\partial_x^\alpha, w] \cdot \nabla w\|_{L^2_{n-1}} \\
&\leq C\Big\{\sum_{j=0}^{1}(\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty} + \|w_\infty\|_{H^s_{n-1}})\Big\} \\
&\quad \times\Big\{\sum_{j=1}^{2}(\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2} + \|w_\infty\|_{H^s_{n-1}})\Big\}. \qquad (3.29)
\end{aligned}
$$

It follows from (3.28) and (3.29) that

$$
\begin{aligned}
&\|P_\infty (w \cdot \nabla w)\|_{H^{s-1}_{n-1}} \\
&\leq C\Big\{\sum_{j=0}^{1}\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty} + \|w_\infty\|_{H^s_{n-1}}\Big\} \\
&\quad \times\Big\{\sum_{j=1}^{2}\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2} + \|w_\infty\|_{H^s_{n-1}})\Big\}.
\end{aligned}
$$

Similarly to (3.29), the remaining terms can be estimated by applying Lemma 3.4 and Lemma 3.8. Integrating the obtained inequalities on $(0,T)$ and applying Lemma 3.2 (i), we obtain the desired estimate.

The desired estimate in (ii) can be similarly obtained by applying Lemma 3.2 (i), Lemma 3.8, Lemma 3.9 and Proposition 3.16. This completes the proof. □

By Proposition 3.13, Proposition 3.17 and the iteration argument, we obtain Theorem 3.1.

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