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精度保証付き数値計算による楕円型作用素の逆作用素ノルム評価

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Abstract

本稿では，2 階楕円型線形作用素に対する可逆性と逆作用素ノルムの上界値を数学的に厳密な意味で保証する数値計算法をいくつか紹介する．

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain ($d = 1, 2, 3$), and for some integer m, let $H^m(\Omega)$ denote the complex L^2-Sobolev space of order m on Ω. We define the Hilbert space

$$H_0^1(\Omega) := \{u(x) \in H^1(\Omega) \mid u(x) = 0,\ x \in \partial\Omega\}$$

with the inner product $(\nabla u, \nabla v)_{L^2(\Omega)}$ and the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$, where $(u, v)_{L^2(\Omega)}$ implies L^2-inner product on Ω. Let

$$H(\Delta; L^2(\Omega)) := \{ u(x) \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega) \}$$

be a Banach space with respect to the graph norm $\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$. Since Ω is in a class of the bounded domain with a Lipschitz continuous boundary, the embedding $H(\Delta; L^2(\Omega)) \hookrightarrow H_0^1(\Omega)$ is compact by the Rellich compactness theorem.

Consider the linear elliptic operator

$$\mathscr{L}u := -\Delta u + b \cdot \nabla u + cu \tag{1}$$

for $b \in L^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$ with norms

$$\|b\|_{L^\infty(\Omega)^d} = \operatorname*{ess\,sup}_{x\in\Omega} \sqrt{|b_1(x)|^2 + \cdots + |b_d(x)|^2}, \qquad \|c\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |c(x)|,$$

respectively.

The aim of this paper is to propose some procedures for verifying the invertibility of an $\mathscr{L}$ with a computable upper bound $M > 0$ satisfying

$$\|u\|_{H_0^1(\Omega)} \le M \|\mathscr{L}u\|_{H^{-1}(\Omega)}, \qquad \forall u \in H_0^1(\Omega) \tag{2}$$

or

$$\|u\|_{H_0^1(\Omega)} \le M \|\mathscr{L}u\|_{L^2(\Omega)}, \qquad \forall u \in H(\Delta; L^2(\Omega)) \tag{3}$$

or

$$\|\Delta u\|_{L^2(\Omega)} \le M \|\mathscr{L}u\|_{L^2(\Omega)}, \qquad \forall u \in H(\Delta; L^2(\Omega)). \tag{4}$$

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For example, when one try to find $u \in H_0^1(\Omega)$ (weak sense) satisfying nonlinear problems

$$-\Delta u(x) = f(x, u, \nabla u), \quad x \in \Omega \tag{5}$$

with certain properties for f and apply infinite-dimensional verification approach for u, the norm estimations (2), (3), (4) are required [13, 16, 18, 19, 20]. We note that the upper bound M can also be applied to verified computations of eigenvalue exclosures in Hilbert spaces [25].

2 Approximation subspace and notations

Let S_h be a finite dimensional approximation subspace of $H_0^1(\Omega)$ dependent on the parameter $h > 0$. For example, S_h is taken to be a finite element subspace with mesh size h. Let $P_h : H_0^1(\Omega) \to S_h$ denote the H_0^1-projection defined by

$$(\nabla(\phi - P_h\phi), \nabla v)_{L^2(\Omega)} = 0, \quad \forall v \in S_h, \tag{6}$$

and suppose that P_h has the following approximation properties.

$$\|v - P_h v\|_{H_0^1(\Omega)} \le C(h)\|\Delta v\|_{L^2(\Omega)}, \qquad \forall v \in H(\Delta; L^2(\Omega)), \tag{7}$$

$$\|v - P_h v\|_{L^2(\Omega)} \le C(h)\|v - P_h v\|_{H_0^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega), \tag{8}$$

where $C(h) > 0$ is a positive constant which is *numerically* determined with the property that $C(h) \to 0$ as $h \to 0$. We emphasize that especially the estimate (7) is indispensable in our argument and the compactness of the embedding $H(\Delta; L^2(\Omega)) \hookrightarrow H^1(\Omega)$ is essential in getting the constant $C(h)$ with desired property. Usually the second estimation (8) for P_h is derived by using a technique so called Aubin-Nitsche's trick [1].

These assumptions (7) and (8) hold for many finite element subspaces of $H_0^1(\Omega)$ [1, 9, 10, 11, 12, 15] or function spaces of Fourier series with finite truncation [23]. For example it can be taken as $C(h) = h/\pi$ and $h/(2\pi)$ for bilinear and biquadratic element, respectively, for the rectangular mesh on the square domain [9], and $C(h) = 0.493h$ for the linear and uniform triangular mesh of the convex polygonal domain [3, 6]. Furthermore, a constructive a priori L^∞ error estimate for the projection P_h can also be obtained [7, 8]. In case of nonconvex polygonal domain, there are some useful techniques and consideration to obtain mathematically rigorous upper bounds for the constant $C(h)$ satisfying (7) with adequate order for such nonconvex domains [2, 5, 14, 26, 27, 28].

Define basis function of S_h by $\{\phi_i\}_{i=1}^N$ for $N := \dim S_h$ and $N \times N$ matrices G, D, L, and Hermitian matrix E by

$$[G]_{ij} = (\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)} + (b \cdot \nabla\phi_j + c\,\phi_j, \phi_i)_{L^2(\Omega)}, \tag{9}$$

$$[D]_{ij} = (\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)}, \tag{10}$$

$$[L]_{ij} = (\phi_j, \phi_i)_{L^2(\Omega)}, \tag{11}$$

$$[E]_{ij} = (b \cdot \nabla\phi_j + c\,\phi_j, b \cdot \nabla\phi_i + c\,\phi_i)_{L^2(\Omega)}, \tag{12}$$

respectively. Since D and L are positive definite, they can be decomposed as $D = D^{1/2}D^{H/2}$ and $L = L^{1/2}L^{H/2}$ where H indicates the conjugate transposition. Usually $D^{1/2}$ and $L^{1/2}$ are the lower triangular matrices. We assume that G has the inverse and let $C_p > 0$ denote the Poincaré or Rayleigh-Ritz constants which satisfies

$$\|u\|_{L^2(\Omega)} \le C_p\|\nabla u\|_{L^2(\Omega)}, \quad u \in H_0^1(\Omega). \tag{13}$$

3 Estimation (2)

This section is devoted to an upper bound M safisfying

$$\|u\|_{H_0^1(\Omega)} \le M\|\mathscr{L}u\|_{H^{-1}(\Omega)}, \qquad \forall u \in H_0^1(\Omega)$$

with the invertibility of $\mathscr{L}$.

It is well-known that for each $\xi \in H^{-1}(\Omega)$ there exists a unique $\psi \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta\psi &= \xi \quad \text{in} \quad \Omega, \\ \psi &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By define this mapping $\xi \mapsto \psi$ by $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, a map $(-\Delta)^{-1}|_{L^2(\Omega)} : L^2(\Omega) \to H_0^1(\Omega)$ becomes compact because ψ belongs to $H(\Delta; L^2(\Omega))$ and the embedding $H(\Delta; L^2(\Omega)) \hookrightarrow H^1(\Omega)$ is compact. We define a linear compact operator $F : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$Fu := (-\Delta)^{-1}|_{L^2(\Omega)}(-b \cdot \nabla u - cu). \tag{14}$$

Then since the term $-b \cdot \nabla u - cu$ maps each bounded set of $H_0^1(\Omega)$ to a bounded set of $L^2(\Omega)$, the operator F becomes compact on $H_0^1(\Omega)$, and the following is true.

Lemma 1. [13, Theorem 2.3]
If $I - F$ on $H_0^1(\Omega)$ is invertible then so is $\mathscr{L}$, and $M > 0$ of (2) can be taken as satisfying

$$\|(I - F)^{-1}u\|_{H_0^1(\Omega)} \leq M\|u\|_{H_0^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega). \tag{15}$$

3.1 1st estimation of (2)

Our first result for (2) is as follows.

Theorem 1. [17, Theorem 1] For

$$C_1 := \|b\|_{L^\infty(\Omega)^d} + C_p\|c\|_{L^\infty(\Omega)}, \tag{16}$$

if $C_pC_1 < 1$ then $I - F$ is invertible and M of (2) can be taken as

$$M = \frac{1}{1 - C_pC_1}. \tag{17}$$

3.2 2nd estimation of (2)

We define

$$C_2 := \|b\|_{L^\infty(\Omega)^d} + C(h)\|c\|_{L^\infty(\Omega)}, \tag{18}$$

$$K := \begin{cases} C(h)\left(C_p\|\nabla \cdot b\|_{L^\infty(\Omega)} + C_1\right), & \text{if} \quad b \in W^{1,\infty}(\Omega)^d, \\ C_pC_2, & \text{if} \quad b \in L^\infty(\Omega)^d, \end{cases} \tag{19}$$

$$\rho := \|D^{T/2}G^{-1}D^{1/2}\|_2, \tag{20}$$

where $\|\cdot\|_2$ stands for matrix 2-norm. Note that ρ can be represented by

$$\rho^{-1} = \min\{|\lambda| \mid Gx = \lambda Dx, \ \mathbf{0} \neq x \in \mathbb{C}^n\},$$

and its verified upper bound can be computed [22]. The below is our second estimation of (2).

Theorem 2. [17, Theorem 2] If

$$\kappa := C(h)(\rho C_1K + C_2) < 1 \tag{21}$$

then $I - F$ is invertible and $M > 0$ of (2) is obtained by

$$M = \frac{1}{1 - \kappa}\left\| \begin{bmatrix} \rho(1 - C_2C(h)) & \rho K \\ \rho C_1C(h) & 1 \end{bmatrix} \right\|_2.$$

3.3 3rd estimation of (2)

Defining

$$\widetilde{K} := C(h)\left(\|b\|_{L^\infty(\Omega)^d} C_1 + \|c\|_{L^\infty(\Omega)}\right),$$
$$C_3 := C(h)\|b\|_{L^\infty(\Omega)^d},$$

we have the following result.

Theorem 3. [17, Theorem 3] If $\widetilde{\kappa} := \widetilde{K}\left(\rho C_p K + C(h)\right) < 1$, $I - F$ is invertible and $M > 0$ of (2) is obtained by

$$M = \frac{1}{1-\widetilde{\kappa}} \left\| \begin{bmatrix} \rho(1 - \widetilde{K}C(h) + KC_3) & \rho K(1 + C_3) \\ \rho\widetilde{K}C_p + C_3 & 1 + C_3 \end{bmatrix} \right\|_2 .$$

If $b \in W^{1,\infty}(\Omega)$, $K = O(C(h))$ and then $\widetilde{\kappa} = O(C(h))^2$.

3.4 Numerical examples

3.4.1 One-dimensional operators

We use interval arithmetic toolbox INTLAB Version 7 [21] with MATLAB 8.0.0.783 (R2012b) on Intel Core i7 3.4GHz. Divide the interval $(0,1)$ by equal partition size $h > 0$ and take S_h as the set of piecewise linear functions on each subinterval. We can take $C(h) = h/\pi$ and $C_p = 1/\pi$.

Table 1 and 2 show verification results. The bold letters indicate the smallest M in the theorems.

Table 1: Verification results for $b = \sin(\pi x)$, $c = 1$, $\rho = 1.0035$ $(1/h = 32)$

	Theorem 1		Theorem 2		Theorem 3	
$1/h$	C_1C_p	M	κ	M	$\tilde{\kappa}$	M
4	0.4197	1.7231	0.1057	1.2507	0.0258	**1.2186**
8	0.4197	1.7231	0.0464	1.1106	0.0065	**1.0976**
16	0.4197	1.7231	0.0216	1.0521	0.0016	**1.0461**
32	0.4197	1.7231	0.0104	1.0258	0.0004	**1.0229**

Table 2: Verification results for $b = -\sin(\pi x)$, $c = -5$, $\rho = 2.0001$ $(1/h = 32)$

	Theorem 1		Theorem 2		Theorem 3	
$1/h$	C_1C_p	M	κ	M	$\tilde{\kappa}$	M
4	0.8250	5.7116	0.2248	2.5155	0.1539	**2.4918**
8	0.8250	5.7116	0.0770	2.1125	0.0393	**2.1122**
16	0.8250	5.7116	0.0293	**2.0280**	0.0099	2.0285
32	0.8250	5.7116	0.0123	**2.0082**	0.0025	2.0084

3.4.2 Two-dimensional non-self-adjoint operators

Consider the case for

$$b = R\begin{bmatrix} -y + 1/2 \\ x - 1/2 \end{bmatrix}, \qquad c \in \mathbb{C}, \qquad \Omega = (0,1) \times (0,1) \tag{22}$$

We take linear and uniform triangular meshes on Ω with the element side length $h > 0$ for a given finite element mesh. We can take $C(h) = 0.493h$ and $C_p = 1/(\pi\sqrt{2})$. Table 3, 4, and 5 show verification results.

Table 3: Verification results for $R = 4,\ c = 0,\ \rho = 1.0001\ (1/h = 10)$

	Theorem 1		Theorem 2		Theorem 3	
$1/h$	C_1C_p	M	κ	M	$\tilde{\kappa}$	M
2	0.6367	**2.7521**	1.1835	—	0.7956	12.5322
5	0.6367	2.7521	0.3567	1.8230	0.1273	**1.7994**
10	0.6367	2.7521	0.1589	**1.2914**	0.0319	1.3180

Table 4: Verification results for $R = 6.75,\ c = -1 - 1.5i,\ \rho = 1.0487\ (1/h = 10)$

	Theorem 1		Theorem 2		Theorem 3	
$1/h$	C_1C_p	M	κ	M	$\tilde{\kappa}$	M
4	1.1658	—	1.0408	—	0.8928	**23.7783**
5	1.1658	—	0.7608	5.6411	0.5721	**5.1856**
10	1.1658	—	0.3081	**1.7124**	0.1433	1.8585

3.5 Report for estimation (2)

We consider three computer-assisted procedures to verify the invertibility of second order linear elliptic operators with a bound for the norm of its inverse. Although it has the limitation, the method of Theorem 1 does not need the computation of ρ (2-norm). The method based on Theorem 3 has the second order for $C(h)$ when $b \in W^{1,\infty}(\Omega)$ and some verification results show that it *could* be an alternative of Theorem 2, especially, some confirmation of the only invertibility for $\mathscr{L}$ are quite essential. We still conclude our second approach of Theorem 2 is robust and reliable than other two approaches.

4 Estimation (3)

Now we consider an upper bound M safisfying

$$\|u\|_{H_0^1(\Omega)} \le M\|\mathscr{L}u\|_{L^2(\Omega)} \quad \forall u \in H(\Delta; L^2(\Omega)).$$

We have three approaches.

4.1 1st estimation of (3)

Our first result is a direct application of Theorem 2.

Theorem 4. [13, Theorem 2.3] If $\kappa = C(h)(\rho C_1 K + C_2) < 1$ then $\mathscr{L}$ is invertible and $M > 0$ of (3) is obtained by

$$M = \frac{C_p}{1-\kappa}\left\| \begin{bmatrix} \rho(1 - C_2C(h)) & \rho K \\ \rho C_1 C(h) & 1 \end{bmatrix} \right\|_2.$$

In Theorem 4, it is expected that $M \to C_p \max\{\rho, 1\}$.

4.2 2nd estimation of (3)

For

$$\hat{\rho} := \|D^{H/2}G^{-1}L^{1/2}\|_2, \tag{23}$$

we obtained the second estimation.

Table 5: Verification results for $R = 5,\ c = -15,\ \rho = 4.0804\ (1/h = 20)$

	Theorem 1		Theorem 2		Theorem 3	
$1/h$	C_1C_p	M	κ	M	$\tilde{\kappa}$	M
5	1.5558	—	1.9949	—	2.3104	—
10	1.5558	—	0.6596	**11.0853**	0.6723	13.9871
20	1.5558	—	0.2148	**4.9111**	0.1761	5.1964

Theorem 5. [24, Theorem 4.2] If

$$\hat{\kappa} := C(h)C_2\,(\hat{\rho}C_1 + 1) < 1 \qquad (24)$$

then $\mathscr{L}$ is invertible and $M > 0$ of (3) is obtained by

$$M = \frac{\sqrt{\hat{\rho}^2 + C(h)^2(1+\hat{\rho}C_1)^2}}{1-\hat{\kappa}}.$$

In Theorem 5, it is expected that $M \to \hat{\rho}$.

4.3 3rd estimation of (3)

We also present the following estimate based on a fixed-point formulation.

Theorem 6. [4, Theorem 3] If $\kappa = C(h)(\rho C_1 K + C_2) < 1$ then $\mathscr{L}$ is invertible and $M > 0$ of (3) is obtained by

$$M = \frac{\sqrt{\rho^2(C_p + C(h)(K - C_pC_2))^2 + C(h)^2(1+\rho C_pC_1)^2}}{1-\kappa}.$$

In Theorem 6, it is expected that $M \to C_p\,\rho$.

Comparing three theorems for (3), Theorem 5 could converge to the exact operator norm for $\mathscr{L}^{-1}$. Because of it holds that $\hat{\rho} \le C_p\,\rho$, when $\hat{\rho} \sim C_p\rho$, Theorem 6 would apply suffient "good" M with low computational cost. From the *actual computational* point of view, since the criterion $\hat{\kappa} < 1$ is sometimes harder than $\kappa < 1$ for fixed h experimentally, Theorem 4 and 6 have a room to be effective.

4.4 Numerical examples

Our numerical environment and S_h for one- or two-dimensional operators are same as the previous section.

4.4.1 One-dimensional operators

Table 6, 7, 8, and 9 show verification results for some $b(x) = r\sin(\pi x)$ and $c \in \mathbb{R}$ in $\Omega = (0,1)$.

4.4.2 Two-dimensional non-self-adjoint operators

Consider the case for (22). Table 10 and 11 show verification results.

4.4.3 Two-dimensional operators

We now report on a case for $b = 0$. Consider an operator: $\mathscr{L} = -\Delta - 1 - 2u_h + 3au_h^2$ which is the linearized the equation

$$\begin{cases} -\Delta u &= 1 + u + u^2 - au^3 \quad \text{in } (0,1)\times(0,1), \\ \quad u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

Table 6: Verification results for $b = 2.5\sin(\pi x)$, $c = -10$

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
10	12.6637	3.6970	0.6865	12.4285	1.9761	—	0.6865	**12.2786**
30	12.9669	3.8003	0.0956	4.4655	0.6249	10.1500	0.0956	**4.4598**
50	12.9916	3.8084	0.0409	4.2504	0.3696	6.0452	0.0409	**4.2485**
100	13.0020	3.8119	0.0142	4.1667	0.1827	4.6645	0.0142	**4.1663**
200	13.0047	3.8128	0.0056	4.1465	0.0908	4.1936	0.0056	**4.1464**
500	13.0054	3.8130	0.0019	4.1409	0.0362	**3.9561**	0.0019	4.1409
1000	13.0055	3.8131	0.0009	4.1401	0.0181	**3.8832**	0.0009	4.1401

Table 7: Verification results for $b = -20\sin(\pi x)$, $c = -20$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
10	2.6420	0.3552	3.9293	—	6.8074	—	3.9293	—
30	2.5044	0.3542	0.5592	1.8684	2.2167	—	0.5592	**1.5439**
50	2.4950	0.3542	0.2518	1.0293	1.3246	—	0.2518	**0.9502**
100	2.4911	0.3542	0.0948	0.8417	0.6603	1.0469	0.0948	**0.8249**
200	2.4911	0.3542	0.0396	0.8040	0.3296	**0.5289**	0.0396	0.8002
500	2.4899	0.3542	0.0140	0.7943	0.1318	**0.4080**	0.0140	0.7938
1000	2.4899	0.3542	0.0067	0.7930	0.0659	**0.3792**	0.0067	0.7929

at two finite element approximate solutions u_h whose named "lower" and "upper."
Table 12 and 13 show verification results.

4.5 Report for estimation (3)

The computer-assisted procedure (Theorem 6) is our latest approach to compute a verified bound of the norm for second order linear elliptic operators $\mathscr{L}$. The criterion for the invertibility of $\mathscr{L}$ is the same as Theorem 4, however, it has no limitation such that the lower bound of M is not less than 1. Although the proposed procedure would not converge to its exact operator norm, some verification examples show that it has a better bound than the approach in Theorem 5. We conclude that our proposed method should be a bridge the gap between the two previous approaches, and one may choice an appropriate procedure taking into consideration given problem or computational cost, and so on.

5 Estimation (4)

Finally we consider an upper bound M safisfying

$$\|\Delta u\|_{L^2(\Omega)} \le M\|\mathscr{L}u\|_{L^2(\Omega)}, \qquad \forall u \in H(\Delta; L^2(\Omega)).$$

We have two approaches.

5.1 1st estimation of (4)

Redefining $\rho_{10} := \|D^{H/2}G^{-1}L^{1/2}\|_2$ and defining $\rho_{00} := \|L^{H/2}G^{-1}L^{1/2}\|_2$, we have the first estimation.

Table 8: Verification results for $b = \sin(\pi x)$, $c = 100$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
10	0.9183	0.0500	1.1665	—	0.3516	**0.1508**	1.1665	—
30	0.9911	0.0499	0.1458	0.4977	0.0577	**0.0608**	0.1458	0.3920
50	0.9969	0.0499	0.0553	0.4060	0.0275	**0.0542**	0.0553	0.3426
100	0.9992	0.0499	0.0155	0.3568	0.0111	**0.0512**	0.0155	0.3242
200	0.9998	0.0499	0.0047	0.3365	0.0049	**0.0504**	0.0047	0.3198
500	1.0000	0.0499	0.0012	0.3254	0.0018	**0.0501**	0.0012	0.3186
1000	1.0000	0.0499	0.0005	0.3218	0.0009	**0.0500**	0.0005	0.3184

Table 9: Verification results for $b = \sin(\pi x)$, $c = -10$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
10	94.9621	29.6261	2.1281	—	5.2424	—	2.1281	—
30	231.4257	72.4346	0.5767	**172.3900**	3.5678	—	0.5767	172.4427
50	261.5470	81.8835	0.2366	**108.4156**	2.3262	—	0.2366	108.4277
100	276.7469	86.6517	0.0641	**93.8348**	1.1938	—	0.0641	93.8375
200	280.8268	87.9316	0.0171	**90.7977**	0.5964	217.8445	0.0171	90.7983
500	281.9909	88.2967	0.0032	**89.9844**	0.2373	115.7653	0.0032	89.9846
1000	282.1580	88.3491	0.0010	**89.8696**	0.1184	100.2071	0.0010	89.8697

Theorem 7. If $\kappa_7 := C(h)C_2(\rho_{10}C_1 + 1) < 1$ then $\mathscr{L}$ is invertible and $M > 0$ of (4) is obtained by

$$M = 1 + \|b\|_{L^\infty(\Omega)^d} A_1 + \|c\|_{L^\infty(\Omega)} A_0,$$

where

$$A_0 = \frac{\rho_{00} + C(h)^2(1 + \rho_{10}C_1)}{1 - \kappa_7}, \qquad A_1 = \frac{\sqrt{\rho_{10}^2 + C(h)^2(1 + \rho_{10}C_1)^2}}{1 - \kappa_7}.$$

5.2 2nd estimation of (4)

Our second result is somewhat constructive than the previous approach.

Theorem 8. For

$$M_h := \sqrt{\left\|(G^{-1}L^{1/2})^H E (G^{-1}L^{1/2})\right\|_2}$$

if it holds that

$$\kappa_8 := C(h)C_2(1 + M_h) < 1$$

then $\mathscr{L}$ is invertible and a bound $M > 0$ of (4) can be taken as

$$M = \frac{1 + M_h}{1 - \kappa_8}.$$

Note that if E is positive definite, by using $E = E^{1/2}E^{H/2}$, it is true that

$$M_h = \left\|E^{H/2}G^{-1}L^{1/2}\right\|_2.$$

Table 10: Verification results for $R = 10$, $c = -10 - 5i$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
5	1.7039	0.3656	2.3287	—	3.6305	—	2.3287	—
10	1.7751	0.3946	0.7724	1.8734	1.7974	—	0.7724	**1.6510**
20	1.7941	0.4025	0.2814	0.5384	0.8798	3.4926	0.2814	**0.5033**
50	1.7995	0.4047	0.0869	0.4222	0.3456	0.6227	0.0869	**0.4174**
100	1.8001	0.4050	0.0392	0.4092	0.1716	0.4897	0.0392	**0.4082**
130	1.8004	0.4051	0.0294	0.4076	0.1318	0.4670	0.0294	**0.4070**

Table 11: Verification results for $R = 10$, $c = 15$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
5	0.9732	0.1270	1.8758	—	1.9610	—	1.8758	—
8	0.9903	0.1276	0.9032	3.3368	1.1493	—	0.9032	**2.6387**
10	0.9939	0.1277	0.6488	0.8671	0.8987	1.6951	0.6488	**0.6589**
20	0.9986	0.1279	0.2497	0.3543	0.4284	**0.2453**	0.2497	0.2760
50	0.9999	0.1279	0.0818	0.2632	0.1663	**0.1559**	0.0818	0.2316
100	1.0001	0.1279	0.0379	0.2426	0.0823	**0.1400**	0.0379	0.2267

5.3 Numerical examples

Consider the case for two-dimensional non-self-adjoint operators (22). Our numerical environment and S_h is same as the previous section. Table 14 and 15 show verification results.

5.4 Report for estimation (4)

We propose two computer-assisted procedures to compute a verified bound $M > 0$ satisfying (4). Some verification examples show that Theorem 8 has a better bound than the approach in Theorem 7. If we are indifferent to computational costs, instead of an estimation

$$\|b \cdot \nabla u_h + cu_h\|_{L^2(\Omega)} \leq M_h(C(h)C_2\|\Delta u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})$$

in the proof of the Theorem 8, it can be possible to use a bound such that

$$\|b \cdot \nabla u_h + cu_h + f\|_{L^2(\Omega)} \leq \hat{M}_h\|f\|_{L^2(\Omega)}$$

with numerically determined $\hat{M}_h > 0$ directly (more constructive).

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Table 12: Verification results for "lower" u_h at $a = 0.001$. $\hat{\rho}/(C_p\,\rho) \sim 0.9995$ $(1/h = 50)$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
10	1.0586	0.2356	0.0030	0.2391	0.0030	0.2421	0.0030	0.2447
20	1.0599	0.2379	0.0008	0.2388	0.0008	0.2395	0.0008	0.2402
30	1.0601	0.2383	0.0004	0.2387	0.0004	0.2391	0.0004	0.2394
40	1.0602	0.2385	0.0002	0.2387	0.0002	0.2389	0.0002	0.2391
50	1.0603	0.2386	0.0002	0.2387	0.0002	0.2388	0.0002	0.2389

Table 13: Verification results for "upper" u_h at $a = 0.001$. $\hat{\rho}/(C_p\,\rho) \sim 0.6040$ $(1/h = 50)$.

			Theorem 4		Theorem 5		Theorem 6	
$1/h$	ρ	$\hat{\rho}$	κ	M	$\hat{\kappa}$	M	κ	M
10	2.5948	0.3545	1.1823	—	0.7722	**1.9668**	1.1823	—
20	2.6622	0.3624	0.2856	0.9204	0.1861	**0.4756**	0.2856	0.8883
30	2.6758	0.3640	0.1262	0.7216	0.0822	**0.4087**	0.1262	0.7074
40	2.6807	0.3645	0.0709	0.6671	0.0461	**0.3887**	0.0709	0.6590
50	2.6830	0.3648	0.0453	0.6438	0.0295	**0.3800**	0.0453	0.6386

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Table 14: Verification results for $R = 20$, $c = 0$.

$1/h$	Theorem 7	Theorem 8	M_h	ρ_{10}	ρ_{00}	A_0	A_1
20	—	**3.5386**	0.5843	0.2238	0.0501	—	—
30	108.0393	**2.5102**	0.5854	0.2242	0.0503	1.6592	7.5689
40	12.9217	**2.1922**	0.5861	0.2243	0.0504	0.1867	0.8430
50	8.7114	**2.0372**	0.5865	0.2244	0.0504	0.1214	0.5453
100	5.4943	**1.7847**	0.5872	0.2244	0.0504	0.0712	0.3178
130	5.0987	**1.7350**	0.5873	0.2244	0.0504	0.0650	0.2899

Table 15: Verification results for $R = 10$, $c = -10 - 10i$.

$1/h$	Theorem 7	Theorem 8	M_h	ρ_{10}	ρ_{00}	A_0	A_1
20	16.3072	**3.2671**	1.0451	0.3172	0.0712	0.3314	1.5020
30	7.7991	**2.7135**	1.0469	0.3177	0.0714	0.1483	0.6651
40	6.3207	**2.5057**	1.0475	0.3179	0.0715	0.1164	0.5199
50	5.7107	**2.3968**	1.0478	0.3180	0.0715	0.1032	0.4600
100	4.8430	**2.2077**	1.0482	0.3181	0.0716	0.0843	0.3750
130	4.6888	**2.1683**	1.0483	0.3181	0.0716	0.0809	0.3599

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