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Hochschild cohomology ring modulo nilpotence of finite dimensional algebras

東京理科大学 理学部数学科 小原　大樹 (Daiki Obara)
Department of Mathematics
Tokyo University of Science

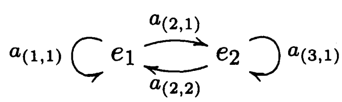
Abstract

This paper is based on my talk given at the Symposium on Cohomology Theory of Finite Groups and Related Topics held at Kyoto University, Japan, 18 February to 20 February 2015. In this paper, we consider finite dimensional quiver algebras over a field k with quantum-like relations. We determine the projective resolution of the algebras and the ring structure of the Hochschild cohomology ring modulo nilpotence.

Introduction

Let A be an indecomposable finite dimensional algebra over a field k and $\mathrm{char}\, k = 0$. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$. In [6], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A. This led us to consider the ring structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [5], Snashal gave the question whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k-algebra. With respect to sufficient condition, Green, Snashall and Solberg have shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for self-injective algebras of finite representation type [1] and for monomial algebras [2].

Let q_{12} and q_{23} be a non-zero element in k. We consider the quiver algebra $A = kQ/I$ where Q is the following quiver:



and I is the ideal of kQ generated by

$$a_{(1,1)}^{n_1}, (a_{(2,1)} + a_{(2,2)})^{2n_2}, a_{(3,1)}^{n_3},$$
$$a_{(1,1)}(a_{2,1}a_{2,2}) - q_{12}(a_{2,1}a_{2,2})a_{(1,1)}, (a_{2,2}a_{2,1})a_{(3,1)} - q_{23}a_{(3,1)}(a_{2,2}a_{2,1}),$$
$$a_{(1,1)}a_{(2,1)}a_{(3,1)}, a_{(3,1)}a_{(2,2)}a_{(1,1)}.$$

Paths are written from right to left.

In this paper, in the case of $n_1 = n_2 = n_3 = 1$ and $q_{12} = q_{23} = 1$, we determine the projective resolution of A and the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

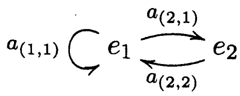
In [3] and [4], we have the minimal projective bimodule resolution and the Hochschild cohomology ring modulo nilpotence of the quiver algebra defined by the two cycles and quantum-like relation. Then, the projective resolution of this algebra was given by the total complex depending the projective resolutions of two Nakayama algebras. Similaly, the projective resolution of A is given by the total complex depending the projective resolutions of the quiver algebra defined by two cycles and the Nakayama algebra. Using this resolution, we have the ring structure of the Hochschild cohomology ring modulo nilpotence.

The content of the paper is organized as follows. In Section 1, we introduce the projective bimodule resolusion and the Hochschild cohomology ring modulo nilpotence of the quiver algebra defined by two cycles and a quantum-like relation given in [3] and [4]. In Section 2, we determine the projective bimodule resolusion and the Hochschild cohomology ring modulo nilpotence of A in the case of $n_1 = n_2 = n_3 = 1$ and $q_{12} = q_{23} = 1$.

1 Quiver algebra defined by two cycles and a quantum-like relation

In [3] and [4], we have the projective resolution and Hochschild cohomology ring modulo nilpotence of the quiver algebra defined by two cycles and a quantum-like relation. In this section, we introduce these of the following algebra as a simple example.

We consider the quiver algebra $A_1 = kQ_1/I_1$ where Q_1 is the following quiver:



and the ideal I_1 of kQ generated by

$$a^2_{(1,1)}, a_{(1,1)}(a_{(2,1)}a_{(2,2)}) - (a_{(2,1)}a_{(2,2)})a_{(1,1)}, (a_{(2,1)} + a_{(2,2)})^2.$$

Then we have the minimal projective resolution of A_1 as the total complex depending on the minimal projective resolutions of two Nakayama algebras.

We define projective left A_1^e-modules, equivalently A_1-bimodules:

$$\begin{aligned}
P_0 &= A_1e_1 \otimes e_1A_1 \oplus A_1e_2 \otimes e_2A_1, \\
Q_{(l_1,0)} &= A_1e_1 \otimes e_1A_1 && \text{for } l_1 \geq 1, \\
Q_{(0,l_2)} &= \begin{cases} A_1e_1 \otimes e_2A_1 \coprod A_1e_2 \otimes e_1A_1 & \text{if } l_2 \text{ is odd,} \\ A_1e_1 \otimes e_1A_1 \coprod A_1e_2 \otimes e_2A_1 & \text{if } l_2 \text{ is even,} \end{cases} && \text{for } l_2 \geq 1, \\
Q_{(l_1,l_2)} &= A_1e_1 \otimes e_1A_1 && \text{for } l_1, l_2 \geq 1,
\end{aligned}$$

Then we have the following complexes.

Proposition 1.1. (1) *If $l_1 \geq 1$ and $l_2 = 0$, we define the left A_1^e-homomorphisms $\partial_{(l_1,0)} : Q_{(l_1,0)} \to Q_{(l_1-1,0)}$ by*

$$\partial_{(l_1,0)} : e_1 \otimes e_1 \mapsto \begin{cases} e_1 \otimes a_{(1,1)} - a_{(1,1)} \otimes e_1 & \textit{if } l_1 \textit{ is odd,} \\ \sum_{i=0}^{1} a^i_{(1,1)} \otimes a^{1-i}_{(1,1)} & \textit{if } l_1 \textit{ is even,} \end{cases}$$

Then we have the complex:

$$P_0 \overset{\partial_{(1,0)}}{\longleftarrow} Q_{(1,0)} \overset{\partial_{(2,0)}}{\longleftarrow} Q_{(2,0)} \overset{\delta_{(3,0)}}{\longleftarrow} \cdots \leftarrow Q_{(n-1,0)} \overset{\partial_{(n,0)}}{\longleftarrow} Q_{(n,0)} \leftarrow \cdots .$$

This complex is the minimal projective resolution of $k[a_{(1,1)}]/\langle a_{(1,1)}^2\rangle$.

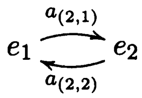
(2) *If* $l_1 = 0$ *and* $l_2 \geq 1$, *we define the left* A_1^e*-homomorphisms* $\delta_{(0,l_2)} : Q_{(0,l_2)} \to Q_{(0,l_2-1)}$ *by*

$$\delta_{(0,l_2)}:$$
$$\begin{cases} \begin{cases} e_1 \otimes e_2 \mapsto e_1 \otimes a_{(2,1)} - a_{(2,1)} \otimes e_2, \\ e_2 \otimes e_1 \mapsto e_2 \otimes a_{(2,2)} - a_{(2,2)} \otimes e_1, \end{cases} & \text{if } l_2 \text{ is odd,} \\ \begin{cases} e_1 \otimes e_1 \mapsto \\ \sum_{i=0}^{1} (a_{(2,1)}a_{(2,2)})^i (e_1 \otimes a_{(2,2)} + a_{(2,1)} \otimes e_1)(a_{(2,1)}a_{(2,2)})^{1-i}, \\ e_2 \otimes e_2 \mapsto \\ \sum_{i=0}^{1} (a_{(2,2)}a_{(2,1)})^i (e_2 \otimes a_{(2,1)} + a_{(2,2)} \otimes e_2)(a_{(2,2)}a_{(2,1)})^{1-i} \end{cases} & \text{if } l_2 \text{ is even,} \end{cases}$$

Then we have the complex:

$$P_0 \overset{\delta_{(0,1)}}{\longleftarrow} Q_{(0,1)} \overset{\delta_{(0,2)}}{\longleftarrow} Q_{(0,2)} \overset{\delta_{(0,3)}}{\longleftarrow} \cdots \leftarrow Q_{(0,n-1)} \overset{\delta_{(0,n)}}{\longleftarrow} Q_{(0,n)} \leftarrow \cdots .$$

This complex is the minimal projective resolution of the Nakayama algebra $kQ'/\langle (a_{(2,1)} + a_{(2,2)})^2 \rangle$ *where* Q' *is the following quiver:*



(3) *If* $l_1, l_2 \geq 1$, *we define the left* A_1^e*-homomorphisms* $\partial_{(l_1,l_2)} : Q_{(l_1,l_2)} \to Q_{(l_1-1,l_2)}$ *and* $\delta_{(l_1,l_2)} : Q_{(l_1,l_2)} \to Q_{(l_1,l_2-1)}$ *as follows:*

$$\partial_{(l_1,l_2)} : e_1 \otimes e_1 \mapsto \begin{cases} (-1)^{l_2}(e_1 \otimes a_{(1,1)} - a_{(1,1)} \otimes e_1) & \text{if } l_1 \text{ is odd,} \\ (-1)^{l_2} \sum_{i=0}^{1} a_{(1,1)}^i e_1 \otimes e_1 a_{(1,1)}^{1-i} & \text{if } l_1 \text{ is even} (\neq 0). \end{cases}$$

$$\delta_{(l_1,l_2)} : e_1 \otimes e_1 \mapsto$$
$$\begin{cases} \begin{cases} e_1 \otimes (a_{(2,1)}a_{(2,2)}) - (a_{(2,1)}a_{(2,2)}) \otimes e_1 & \text{if } l_2 \text{ is odd,} \\ \sum_{i=0}^{1} (a_{(2,1)}a_{(2,2)})^i e_1 \otimes e_1 (a_{(2,1)}a_{(2,2)})^{1-i} & \text{if } l_2 \text{ is even,} \end{cases} & \text{if } l_2 \geq 2, \\ \begin{cases} e_1 \otimes (a_{(2,1)}a_{(2,2)}) - (a_{(2,1)}a_{(2,2)}) \otimes e_1 & \text{if } l_1 \text{ is odd,} \\ (e_1 \otimes a_{(2,2)} + a_{(2,1)} \otimes e_1)(a_{(2,1)}a_{(2,2)}) \\ \quad - (a_{(2,1)}a_{(2,2)})(e_1 \otimes a_{(2,2)} + a_{(2,1)} \otimes e_1) & \text{if } l_1 \text{ is even.} \end{cases} & \text{if } l_2 = 1. \end{cases}$$

Then we have the following complexes:

$$Q_{(0,l_2)} \overset{\partial_{(1,l_2)}}{\longleftarrow} Q_{(1,l_2)} \overset{\partial_{(2,l_2)}}{\longleftarrow} Q_{(2,l_2)} \leftarrow \cdots \leftarrow Q_{(n'-1,l_2)} \overset{\partial_{(n',l_2)}}{\longleftarrow} Q_{(n',l_2)} \leftarrow \cdots,$$

$$Q_{(l_1,0)} \overset{\delta_{(l_1,1)}}{\longleftarrow} Q_{(l_1,1)} \overset{\delta_{(l_1,2)}}{\longleftarrow} Q_{(l_1,2)} \leftarrow \cdots \leftarrow Q_{(l_1,n''-1)} \overset{\delta_{(l_1,n'')}}{\longleftarrow} Q_{(l_1,n'')} \leftarrow \cdots.$$

The total complex $\mathbb{P}_1$ of these complexes hold that

$$\mathbb{P}_1 \otimes A_1/\mathrm{rad}\, A_1.$$

So $\mathbb{P}_1$ is the projective bimodule resolution of A_1.

Theorem 1.2. *We have the minimal projective resolution of* A_1*:*

$$\mathbb{P}_1 : 0 \leftarrow A_1 \overset{\pi}{\longleftarrow} P_0 \overset{d_1}{\longleftarrow} P_1 \overset{d_2}{\longleftarrow} P_2 \leftarrow \cdots P_{n-1} \overset{d_n}{\longleftarrow} P_n \leftarrow \cdots,$$

as the total complex of the above complexes where P_n *is the projective left* A_1^e*-modules:*

$$P_n = \coprod_{l_1+l_2=n} Q_{(l_1,l_2)},$$

and d_n *is the left* A_1^e*-homomorphisms*

$$d_n = \sum_{l_1+l_2=n} \partial_{(l_1,l_2)} + \delta_{(l_1,l_2)},$$

for $l_1, l_2 \geq 0$ *and* $n \geq 1$.

We consider the cohomology of the complex $\mathrm{Hom}_{A_1^e}(\mathbb{P}_1, A_1)$ and Yoneda product. Then we have the generators of the Hochschild cohomology ring of A_1 modulo nilpotence as follows:

$e_{1,(l_1,l_2)} : e_1 \otimes e_1 \to e_1 \in \mathrm{Hom}_{A_1^e}(Q_{(l_1,l_2)}, A_1)$ for l_1 and l_2 are even($\neq 0$), and
$e_{1,(l_1,0)} : e_1 \otimes e_1 \to e_1 \in \mathrm{Hom}_{A_1^e}(Q_{(l_1,0)}), A_1)$ for l_1 is even, and
$e_{1,(0,l_2)} + e_{2,(0,l_2)}$ where
$e_{1,(0,l_2)} : e_1 \otimes e_1 \to e_1 \in \mathrm{Hom}_{A_1^e}(Q_{(0,l_2)}, A_1)$,
$e_{2,(0,l_2)} : e_2 \otimes e_2 \to e_2 \in \mathrm{Hom}_{A_1^e}(Q_{(0,l_2)}, A_1)$ for l_2 is even.

with the following Yoneda product:

$$e_{1,(l_1,l_2)} \circ e_{1,(m_1,m_2)} = e_{1,(l_1+m_1,l_2+m_2)},$$
$$e_{1,(l_1,l_2)} \circ (e_{1,(0,m_2)} + e_{2,(0,m_2)}) = (e_{1,(0,m_2)} + e_{2,(0,m_2)}) \circ e_{1,(l_1,l_2)} = e_{1,(l_1,l_2+m_2)},$$
$$(e_{1,(0,l_2)} + e_{2,(0,l_2)}) \circ (e_{1,(0,m_2)} + e_{2,(0,m_2)}) = (e_{1,(0,l_2+m_2)} + e_{2,(0,l_2+m_2)}).$$

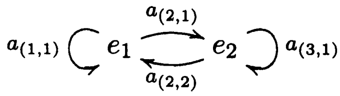
So we have the following result.

Theorem 1.3. *The Hochsxhild cohomology ring of* A_1 *modulo nilpotence is the polynomial ring of two variables.*

$$\mathrm{HH}^*(A_1)/\mathcal{N} = k[e_{1,(2,0)}, e_{1,(0,2)} + e_{2,(0,2)}].$$

2 Quiver algebra with quantum-like relation

In this section, we consider the quiver algebra $A = kQ/I$ defined by the following quiver Q and the ideal I:



I is the ideal of kQ generated by

$$\begin{aligned}
&a_{(1,1)}^2, (a_{(2,1)} + a_{(2,2)})^4, a_{(3,1)}^2,\\
&a_{(1,1)}(a_{2,1}a_{2,2}) - (a_{2,1}a_{2,2})a_{(1,1)}, (a_{2,2}a_{2,1})a_{(3,1)} - a_{(3,1)}(a_{2,2}a_{2,1}),\\
&a_{(1,1)}a_{(2,1)}a_{(3,1)}, a_{(3,1)}a_{(2,2)}a_{(1,1)}.
\end{aligned}$$

The projective resolution of this algebra A is given by the total complex of the following complexes.

(1) Let the complex

$$P_0 \overset{\partial_{(1,0,0)}}{\longleftarrow} R_{(1,0,0)} \overset{\partial_{(2,0,0)}}{\longleftarrow} R_{(2,0,0)} \overset{\partial_{(3,0,0)}}{\longleftarrow} \cdots \leftarrow R_{(n-1,0,0)} \overset{\partial_{(n,0,0)}}{\longleftarrow} R_{(n,0,0)} \leftarrow \cdots,$$

be the minimal projective resolution of A_1 given in Theorem 1. Then we denote $e_1 \otimes e_1 \in Q_{(l_1,l_2)}$ by $\varepsilon_{(i,0,0),(1,1),(l_1,l_2)}$ for $l_1, l_2 \geq 0$ with $l_1 + l_2 = i$. And we denote $e_2 \otimes e_2$, $e_1 \otimes e_2$ and $e_2 \otimes e_1 \in R_{(i,0,0)}$ by $\varepsilon_{(i,0,0),(2,2)}$, $\varepsilon_{(i,0,0),(1,2)}$ and $\varepsilon_{(i,0,0),(2,1)}$.

(2) Let the complex

$$P_0 \overset{\delta_{(0,1,0)}}{\longleftarrow} R_{(0,1,0)} \overset{\delta_{(0,2,0)}}{\longleftarrow} R_{(0,2,0)} \overset{\delta_{(0,3,0)}}{\longleftarrow} \cdots \leftarrow R_{(0,n-1,0)} \overset{\delta_{(0,n,0)}}{\longleftarrow} R_{(0,n,0)} \leftarrow \cdots,$$

be the minimal projective resolution of Nakayama algebra $k[a_{(3,1)}]/a_{(3,1)}^2$. Then we denote $e_2 \otimes e_2 \in R_{(0,j,0)}$ by $\varepsilon_{(0,j,0),(2,2)}$.

(3) We have the complex

$$R_{(i,0,0)} \overset{\delta_{(i,1,0)}}{\longleftarrow} R_{(i,1,0)} \overset{\delta_{(i,2,0)}}{\longleftarrow} R_{(i,2,0)} \overset{\delta_{(i,3,0)}}{\longleftarrow} \cdots \leftarrow R_{(i,n-1,0)} \overset{\delta_{(i,n,0)}}{\longleftarrow} R_{(i,n,0)} \leftarrow \cdots,$$

where $R_{(i,j,0)}$ is the projective module defined by

$$R_{(i,j,0)} = \coprod_{\substack{l_1 + l_2 = i \\ l_1 \geq 1}} A\varepsilon_{(i,j,0),(1,2),(l_1,l_2)}A \coprod A\varepsilon_{(i,j,0),(2,2)}A,$$

and $\delta_{(i,j,0)}$ is A^e-homomorphism defined by the following images:

$$\delta_{(i,j,0)} : R_{(i,j,0)} \to R_{(i,j-1,0)} :$$

$$\varepsilon_{(i,j,0),(1,2),(i,0)} \to$$
$$\begin{cases} \varepsilon_{(i,0,0),(1,1),(i,0)}a_{(2,1)}a_{(3,1)} & \text{if } i \geq 2 \text{ and } j = 1, \\ \varepsilon_{(i,j-1,0),(1,2),(i,0)}a_{(3,1)} & \text{if } j \geq 2, \end{cases}$$

$$\varepsilon_{(i,j,0),(1,2),(1,i-1)} \to$$
$$\begin{cases} \varepsilon_{(1,0,0),(1,1),(1,i-1)}a_{(2,1)}a_{(3,1)} + a_{(1,1)}\varepsilon_{(1,0,0),(1,2)}a_{(3,1)} & \text{if } i \text{ is odd and } j = 1, \\ \varepsilon_{(i,0,0),(1,1),(1,i-1)}a_{(2,1)}a_{(3,1)} & \text{if } i \text{ is even and } j = 1, \\ \varepsilon_{(i,j-1,0),(1,2),(1,i-1)}a_{(3,1)} & \text{if } j \geq 2, \end{cases}$$

$$\varepsilon_{(i,j,0),(1,2),(l_1,l_2)} \to$$
$$\begin{cases} \varepsilon_{(i,0,0),(1,1),(l_1,l_2)}a_{(2,1)}a_{(3,1)} & \text{if } i, l_1 \geq 2 \text{ and } j = 1, \\ \varepsilon_{(i,j-1,0),(1,2),(l_1,l_2)}a_{(3,1)} & \text{if } j \geq 2, \end{cases}$$

$$\varepsilon_{(i,j,0),(2,1),(i,0)} \to$$
$$\begin{cases} a_{(3,1)}a_{(2,2)}\varepsilon_{(i,0,0),(1,1),(i,0)} & \text{if } i \geq 2 \text{ and } j = 1, \\ a_{(3,1)}\varepsilon_{(i,j-1,0),(2,1),(i,0)} & \text{if } j \geq 2, \end{cases}$$

$$\varepsilon_{(i,j,0),(2,1),(1,i-1)} \to$$
$$\begin{cases} a_{(3,1)}a_{(2,2)}\varepsilon_{(1,0,0),(1,1),(1,i-1)} + a_{(3,1)}\varepsilon_{(1,0,0),(2,1)}a_{(1,1)} & \text{if } i \text{ is odd and } j = 1, \\ a_{(3,1)}a_{(2,2)}\varepsilon_{(i,0,0),(1,1),(1,i-1)} & \text{if } i \text{ is even and } j = 1, \\ a_{(3,1)}\varepsilon_{(i,j-1,0),(2,1),(1,i-1)} & \text{if } j \geq 2, \end{cases}$$

$$\varepsilon_{(i,j,0),(2,1),(l_1,l_2)} \to$$
$$\begin{cases} a_{(3,1)}a_{(2,2)}\varepsilon_{(i,0,0),(1,1),(l_1,l_2)} & \text{if } i, l_1 \geq 2 \text{ and } j = 1, \\ a_{(3,1)}\varepsilon_{(i,j-1,0),(1,2),(l_1,l_2)} & \text{if } j \geq 2, \end{cases}$$

$$\varepsilon_{(i,j,0),(2,2)} \to$$
$$\begin{cases} (\varepsilon_{(i,0),(2,1)}a_{(2,1)} + a_{(2,2)}\varepsilon_{(i,0),(1,2)})a_{(3,1)} & \\ \quad - a_{(3,1)}(\varepsilon_{(i,0),(2,1)}a_{(2,1)} + a_{(2,2)}\varepsilon_{(i,0),(1,2)}) & \text{if } i \text{ is odd and } j = 1, \\ \varepsilon_{(i,j-1),(2,2)}a_{(3,1)} - a_{(3,1)}\varepsilon_{(i,j-1),(2,2)} & \text{if } i \text{ is even and } j = 1, \\ \varepsilon_{(i,j-1),(2,2)}a_{(3,1)} - a_{(3,1)}\varepsilon_{(i,j-1),(2,2)} & \text{if } j \text{ is odd and } j \geq 3, \\ \sum_{k=0}^{1} a_{(3,1)}^{k}\varepsilon_{(i,j-1),(2,2)}a_{(3,1)}^{1-k} & \text{if } j \text{ is even.} \end{cases}$$

(4) We have the complex

$$R_{(0,j,0)} \xleftarrow{\partial_{(1,j,0)}} R_{(1,j,0)} \xleftarrow{\partial_{(2,j,0)}} R_{(2,j,0)} \xleftarrow{\partial_{(3,j,0)}} \cdots \leftarrow R_{(n-1,j,0)} \xleftarrow{\partial_{(n,j,0)}} R_{(n,j,0)} \leftarrow \cdots,$$

where $\partial_{(i,j,0)}$ is A^e-homomorphism defined by the following images:

$\partial_{(i,j,0)} : R_{(i,j,0)} \to R_{(i-1,j,0)}$:

$\varepsilon_{(i,j,0),(1,2),(i,0)} \to$

$$\begin{cases} a_{(1,1)}a_{(2,1)}\varepsilon_{(0,j,0),(2,2)} & \text{if } i=1, \\ (-1)^i a_{(1,1)}\varepsilon_{(i-1,j,0),(1,2),(i-1,0)} & \text{if } i \geq 2, \end{cases}$$

$\varepsilon_{(i,j,0),(1,2),(1,i-1)} \to$

$$\begin{cases} \sum_{k=0}^{1}\big(a_{(2,1)}a_{(2,2)}\big)^k \varepsilon_{(i-1,j,0),(1,2),(1,i-2)}\big(a_{(2,2)}a_{(2,1)}\big)^{1-k} & \\ \quad - a_{(1,1)}a_{(2,1)}\varepsilon_{(i-1,j,0),(2,2)} & \text{if } i \text{ is odd and } i \geq 2, \\ \varepsilon_{(i-1,j,0),(1,2),(1,i-2)}\big(a_{(2,2)}a_{(2,1)}\big) & \\ \quad - \big(a_{(2,1)}a_{(2,2)}\big)\varepsilon_{(i-1,j,0),(1,2),(1,i-2)} + a_{(1,1)}a_{(2,1)}\varepsilon_{(i-1,j,0),(2,2)} & \text{if } i \text{ is even and } i \geq 2, \end{cases}$$

$\varepsilon_{(i,j,0),(1,2),(l_1,l_2)} \to$

$$\begin{cases} a_{(1,1)}a_{(2,1)}\varepsilon_{(0,j,0),(2,2)} & \text{if } i=1, \\ \sum_{k=0}^{1}\big(a_{(2,1)}a_{(2,2)}\big)^k \varepsilon_{(i-1,j,0),(1,2),(l_1,l_2-1)}\big(a_{(2,2)}a_{(2,1)}\big)^{1-k} & \\ \quad + (-1)^i a_{(1,1)}\varepsilon_{(i-1,j,0),(1,2),(l_1-1,l_2)} & \text{if } l_2 \text{ is even}(\neq 0) \text{ and } l_1 \geq 2 \\ \varepsilon_{(i-1,j,0),(1,2),(l_1,l_2-1)}\big(a_{(2,2)}a_{(2,1)}\big) & \\ \quad - \big(a_{(2,1)}a_{(2,2)}\big)\varepsilon_{(i-1,j,0),(1,2),(l_1,l_2-1)} & \\ \quad + (-1)^i a_{(1,1)}\varepsilon_{(i-1,j,0),(1,2),(l_1-1,l_2)} & \text{if } l_2 \text{ is odd and } l_1 \geq 2, \end{cases}$$

$\varepsilon_{(i,j,0),(2,1),(i,0)} \to$

$$\begin{cases} \varepsilon_{(0,j,0),(2,2)}a_{(2,2)}a_{(1,1)} & \text{if } i=1, \\ \varepsilon_{(i-1,j,0),(2,1),(i-1,0)}a_{(1,1)} & \text{if } i \geq 2, \end{cases}$$

$\varepsilon_{(i,j,0),(2,1),(1,i-1)} \to$

$$\begin{cases} \sum_{k=0}^{1}\big(a_{(2,2)}a_{(2,1)}\big)^k \varepsilon_{(i-1,j,0),(2,1),(1,i-2)}\big(a_{(2,1)}a_{(2,2)}\big)^{1-k} & \\ \quad + \varepsilon_{(i-1,j,0),(2,2)}a_{(2,2)}a_{(1,1)} & \text{if } i \text{ is odd and } i \geq 2, \\ \varepsilon_{(i-1,j,0),(2,1),(1,i-2)}\big(a_{(2,1)}a_{(2,2)}\big) & \\ \quad - \big(a_{(2,2)}a_{(2,1)}\big)\varepsilon_{(i-1,j,0),(2,1),(1,i-2)} - \varepsilon_{(i-1,j,0),(2,2)}a_{(2,2)}a_{(1,1)} & \text{if } i \text{ is even and } i \geq 2, \end{cases}$$

$\varepsilon_{(i,j,0),(2,1),(l_1,l_2)} \to$

$$\begin{cases} \varepsilon_{(0,j,0),(2,2)}a_{(2,2)}a_{(1,1)} & \text{if } i=1, \\ \sum_{k=0}^{1}\big(a_{(2,2)}a_{(2,1)}\big)^k \varepsilon_{(i-1,j,0),(2,1),(l_1,l_2-1)}\big(a_{(2,1)}a_{(2,2)}\big)^{1-k} & \\ \quad + (-1)^{l_1}\varepsilon_{(i-1,j,0),(2,1),(l_1-1,l_2)}a_{(1,1)} & \text{if } l_2 \text{ is even}(\neq 0) \text{ and } l_1 \geq 2 \\ \varepsilon_{(i-1,j,0),(2,1),(l_1,l_2-1)}\big(a_{(2,1)}a_{(2,2)}\big) & \\ \quad - \big(a_{(2,2)}a_{(2,1)}\big)\varepsilon_{(i-1,j,0),(2,1),(l_1,l_2-1)} & \\ \quad - \varepsilon_{(i-1,j,0),(2,1),(l_1-1,l_2)}a_{(1,1)} & \text{if } l_2 \text{ is odd and } l_1 \geq 2, \end{cases}$$

$\varepsilon_{(i,j,0),(2,2)} \to$

$$\begin{cases} \varepsilon_{(i-1,j),(2,2)}\big(a_{(2,2)}a_{(2,1)}\big) - \big(a_{(2,2)}a_{(2,1)}\big)\varepsilon_{(i-1,j),(2,2)} & \text{if } i \text{ is odd}, \\ \sum_{k=0}^{1}\big(a_{(2,2)}a_{(2,1)}\big)^k \varepsilon_{(i-1,j),(2,2)}\big(a_{(2,2)}a_{(2,1)}\big)^{1-k} & \text{if } i \text{ is even}. \end{cases}$$

(5) Let l_1, l_2, l_3, $l_4 \geq 0$. We have the complex

$$R_{(i,j,0)} \overset{\xi_{(i,j,1)}}{\longleftarrow} R_{(i,j,1)} \overset{\xi_{(i,j,2)}}{\longleftarrow} R_{(i,j,2)} \overset{\xi_{(i,j,3)}}{\longleftarrow} \cdots \leftarrow R_{(i,j,n-1)} \overset{\xi_{(i,j,n)}}{\longleftarrow} R_{(i,j,n)} \leftarrow \cdots,$$

where $R_{(i,j,0)}$ is the projective module defined by

$$R_{(i,j,k)} = \begin{cases} \coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2}} \left(\coprod_{l_3+l_4=k} A\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(l_3,l_4)}A \oplus \coprod_{l_3+l_4=k} A\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(l_3,l_4)}A \right) \\ \oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3,\, l_4 \text{ are odd}}} A\varepsilon_{(i,j,k),(1,1),(1,i-1),(l_3,l_4)}A \oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3,\, l_4 \text{ are even}}} A\varepsilon_{(i,j,k),(2,2),(1,i-1),(l_3,l_4)}A \\ \hfill \text{if } k \text{ is odd}, \\ \coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2}} \coprod_{l_3+l_4=k} \left(A\varepsilon_{(i,j,k),(1,2),(l_1,l_2),(l_3,l_4)}A \oplus \coprod_{l_3+l_4=k} A\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(l_3,l_4)}A \right) \\ \oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3:\text{even},\, l_4:\text{odd}}} A\varepsilon_{(i,j,k),(1,2),(1,i-1),(l_3,l_4)}A \oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3:\text{odd},\, l_4:\text{even}}} A\varepsilon_{(i,j,k),(2,1),(1,i-1),(l_3,l_4)}A \\ \hfill \text{if } k \text{ is even}, \end{cases}$$

for $i, j, k \geq 1$ and $\xi_{(i,j,k)}$ is A^e-homomorphism defined by the following images:

(a) In the case of k is odd,

$\xi_{(i,j,0)} : R_{(i,j,k)} \to R_{(i,j,k-1)}$:

$\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(k,0)} \to \varepsilon_{(i,j,k-1),(1,2),(l_1,l_2),(k-1,0)}a_{(2,2)}a_{(1,1)}$,

$\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(k,0)} \to \varepsilon_{(i,j,k-1),(2,1),(l_1,l_2),(k-1,0)}a_{(2,1)}a_{(3,1)}$,

$\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(l_3,l_4)} \to$

$\varepsilon_{(i,j,k-1),(1,2),(l_1,l_2),(l_3-1,l_4)}a_{(2,2)}a_{(1,1)} + (-1)^{l_1}a_{(1,1)}a_{(2,1)}\varepsilon_{(i,j,k-1),(2,1),(l_1,l_2),(l_3,l_4-1)}$,

$\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(l_3,l_4)} \to$

$\varepsilon_{(i,j,k-1),(2,1),(l_1,l_2),(l_3-1,l_4)}a_{(2,1)}a_{(3,1)} + (-1)^{l_1}a_{(3,1)}a_{(2,2)}\varepsilon_{(i,j,k-1),(1,2),(l_1,l_2),(l_3,l_4-1)}$,

$\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(0,k)} \to a_{(1,1)}a_{(2,1)}\varepsilon_{(i,j,k-1),(2,1),(l_1,l_2),(0,k-1)}$,

$\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(0,k)} \to a_{(3,1)}a_{(2,2)}\varepsilon_{(i,j,k-1),(1,2),(l_1,l_2),(0,k-1)}$,

$\varepsilon_{(i,j,k),(2,2),(1,i-1),(k+1,0)} \to \varepsilon_{(i,j,k-1),(2,1),(1,i-1),(k,0)}a_{(2,1)}a_{(3,1)}$,

$\varepsilon_{(i,j,k),(2,2),(1,i-1),(0,k+1)} \to a_{(3,1)}a_{(2,2)}\varepsilon_{(i,j,k-1),(1,2),(1,i-1),(0,k)}$,

$\varepsilon_{(i,j,k),(1,1),(1,i-1),(l_3,l_4)} \to$

$\varepsilon_{(i,j,k-1),(1,2),(1,i-1),(l_3-1,l_4)}a_{(2,2)}a_{(1,1)} - a_{(1,1)}a_{(2,1)}\varepsilon_{(i,j,k-1),(2,1),(1,i-1),(l_3,l_4-1)}$,

$\varepsilon_{(i,j,k),(2,2),(1,i-1),(l_3,l_4)} \to$

$\varepsilon_{(i,j,k-1),(2,1),(1,i-1),(l_3-1,l_4)}a_{(2,1)}a_{(3,1)} + a_{(3,1)}a_{(2,2)}\varepsilon_{(i,j,k-1),(1,2),(1,i-1),(l_3,l_4-1)}$.

(b) In the case of k is even,

$$\xi_{(i,j,0)} : R_{(i,j,k)} \to R_{(i,j,k-1)} :$$

$$\varepsilon_{(i,j,k),(1,2),(l_1,l_2),(k,0)} \to \varepsilon_{(i,j,k-1),(1,1),(l_1,l_2),(k-1,0)} a_{(2,1)} a_{(3,1)},$$

$$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(k,0)} \to \varepsilon_{(i,j,k-1),(2,2),(l_1,l_2),(k-1,0)} a_{(2,2)} a_{(1,1)},$$

$$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(l_3,l_4)} \to$$

$$\varepsilon_{(i,j,k-1),(2,2),(l_1,l_2),(l_3-1,l_4)} a_{(2,2)} a_{(1,1)} + (-1)^{l_1} a_{(3,1)} a_{(2,2)} \varepsilon_{(i,j,k-1),(1,1),(l_1,l_2),(l_3,l_4-1)},$$

$$\varepsilon_{(i,j,k),(1,2),(l_1,l_2),(l_3,l_4)} \to$$

$$\varepsilon_{(i,j,k-1),(1,1),(l_1,l_2),(l_3-1,l_4)} a_{(2,1)} a_{(3,1)} + (-1)^{l_1} a_{(1,1)} a_{(2,1)} \varepsilon_{(i,j,k-1),(2,2),(l_1,l_2),(l_3,l_4-1)},$$

$$\varepsilon_{(i,j,k),(1,2),(l_1,l_2),(0,k)} \to a_{(1,1)} a_{(2,1)} \varepsilon_{(i,j,k-1),(2,2),(l_1,l_2),(0,k-1)},$$

$$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(0,k)} \to a_{(3,1)} a_{(2,2)} \varepsilon_{(i,j,k-1),(1,1),(l_1,l_2),(0,k-1)},$$

$$\varepsilon_{(i,j,k),(2,1),(1,i-1),(k+1,0)} \to \varepsilon_{(i,j,k-1),(2,2),(1,i-1),(k,0)} a_{(2,2)} a_{(1,1)},$$

$$\varepsilon_{(i,j,k),(1,2),(1,i-1),(0,k+1)} \to a_{(1,1)} a_{(2,1)} \varepsilon_{(i,j,k-1),(2,2),(1,i-1),(0,k)},$$

$$\varepsilon_{(i,j,k),(1,2),(1,i-1),(l_3,l_4)} \to$$

$$\varepsilon_{(i,j,k-1),(1,1),(1,i-1),(l_3-1,l_4)} a_{(2,1)} a_{(3,1)} + a_{(1,1)} a_{(2,1)} \varepsilon_{(i,j,k-1),(2,2),(1,i-1),(l_3,l_4-1)},$$

$$\varepsilon_{(i,j,k),(2,1),(1,i-1),(l_3,l_4)} \to$$

$$\varepsilon_{(i,j,k-1),(2,2),(1,i-1),(l_3-1,l_4)} a_{(2,2)} a_{(1,1)} - a_{(3,1)} a_{(2,2)} \varepsilon_{(i,j,k-1),(1,1),(1,i-1),(l_3,l_4-1)}.$$

(6) We have the complex

$$R_{(i,1,k)} \overset{\delta_{(i,2,k)}}{\longleftarrow} R_{(i,2,k)} \overset{\delta_{(i,3,k)}}{\longleftarrow} R_{(i,3,k)} \overset{\delta_{(i,4,k)}}{\longleftarrow} \cdots \leftarrow R_{(i,n-1,k)} \overset{\delta_{(i,n,k)}}{\longleftarrow} R_{(i,n,k)} \leftarrow \cdots,$$

where $\delta_{(i,j,k)}$ is A^e-homomorphism defined by the following images:

(a) In the case of k is odd,

$$\delta_{(i,j,k)} : R_{(i,j,k)} \to R_{(i,j-1,k)} :$$

$$\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(k,0)} \to a_{(3,1)} \varepsilon_{(i,j-1,k),(2,2),(l_1,l_2),(k,0)},$$

$$\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(0,k)} \to \varepsilon_{(i,j-1,0),(2,2),(l_1,l_2),(0,k)} a_{(3,1)},$$

(b) In the case of k is even,

$$\delta_{(i,j,k)} : R_{(i,j,k)} \to R_{(i,j-1,k)} :$$

$$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(k,0)} \to a_{(3,1)} \varepsilon_{(i,j-1,k),(2,1),(l_1,l_2),(k,0)},$$

$$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(0,k)} \to \varepsilon_{(i,j-1,0),(2,1),(l_1,l_2),(0,k)} a_{(3,1)}.$$

(7) We have the complex

$$R_{(1,j,k)} \overset{\partial_{(2,j,k)}}{\longleftarrow} R_{(2,j,k)} \overset{\partial_{(3,j,k)}}{\longleftarrow} R_{(3,j,k)} \overset{\partial_{(4,j,k)}}{\longleftarrow} \cdots \leftarrow R_{(n-1,j,k)} \overset{\partial_{(n,j,k)}}{\longleftarrow} R_{(n,j,k)} \leftarrow \cdots,$$

where $\partial_{(i,j,k)}$ is A^e-homomorphism defined by the following images:

(a) In the case of k is odd,

$\partial_{(i,j,k)} : R_{(i,j,k)} \to R_{(i-1,j,k)}$:

$\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(k,0)} \to$

$$\begin{cases} (-1)^i a_{(1,1)}\varepsilon_{(i-1,j,k),(1,1),(i-1,0),(k,0)} & \text{if } l_2 = 0, \\ \sum_{k=0}^{1} (a_{(2,1)}a_{(2,2)})^k \varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(k,0)} (a_{(2,1)}a_{(2,2)})^{1-k} & \\ \quad + (-1)^i a_{(1,1)}\varepsilon_{(i-1,j,k),(1,1),(l_1-1,l_2),(k,0)} & \text{if } l_2 \text{ is even}(\neq 0), \\ \varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(k,0)}(a_{(2,1)}a_{(2,2)}) & \\ \quad - (a_{(2,1)}a_{(2,2)})\varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(k,0)} & \\ \quad + (-1)^i a_{(1,1)}\varepsilon_{(i-1,j,k),(1,1),(l_1-1,l_2),(k,0)} & \text{if } l_2 \text{ is odd}, \end{cases}$$

$\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(0,k)} \to$

$$\begin{cases} \varepsilon_{(i-1,j,k),(1,1),(i-1,0),(0,k)}a_{(1,1)} & \text{if } l_2 = 0, \\ \sum_{k=0}^{1} (a_{(2,1)}a_{(2,2)})^k \varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(0,k)} (a_{(2,1)}a_{(2,2)})^{1-k} & \\ \quad + (-1)^{l_1}\varepsilon_{(i-1,j,k),(1,1),(l_1-1,l_2),(0,k)}a_{(1,1)} & \text{if } l_2 \text{ is even}(\neq 0), \\ \varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(0,k)}(a_{(2,1)}a_{(2,2)}) & \\ \quad - (a_{(2,1)}a_{(2,2)})\varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(0,k)} & \\ \quad - \varepsilon_{(i-1,j,k),(1,1),(l_1-1,l_2),(0,k)}a_{(1,1)} & \text{if } l_2 \text{ is odd}, \end{cases}$$

$\varepsilon_{(i,j,k),(1,1),(l_1,l_2),(l_3,l_4)} \to$

$$\begin{cases} \sum_{l=0}^{1} (a_{(2,1)}a_{(2,2)})^l \varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(l_3,l_4)} (a_{(2,1)}a_{(2,2)})^{1-l} & \text{if } i \text{ is odd}, \\ \varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(l_3,l_4)}(a_{(2,1)}a_{(2,2)}) & \\ \quad - (a_{(2,1)}a_{(2,2)})\varepsilon_{(i-1,j,k),(1,1),(l_1,l_2-1),(l_3,l_4)} & \text{if } i \text{ is even}, \end{cases}$$

if $l_2 \geq 1$,

$\varepsilon_{(i,j,k),(2,2),(l_1,l_2),(l_3,l_4)} \to$

$$\begin{cases} \sum_{k=0}^{1} (a_{(2,2)}a_{(2,1)})^k \varepsilon_{(i-1,j,k),(2,2),(l_1,l_2-1),(l_3,l_4)} (a_{(2,2)}a_{(2,1)})^{1-k} & \text{if } i \text{ is odd}, \\ \varepsilon_{(i-1,j,k),(2,2),(l_1,l_2-1),(l_3,l_4)}(a_{(2,2)}a_{(2,1)}) & \\ \quad - (a_{(2,2)}a_{(2,1)})\varepsilon_{(i-1,j,k),(2,2),(l_1,l_2-1),(l_3,l_4)} & \text{if } i \text{ is even}, \end{cases}$$

if $l_2 \geq 1$,

$\varepsilon_{(i,j,k),(2,2),(1,i-1),(l_3,l_4)} \to$

$$\begin{cases} \sum_{k=0}^{1} (a_{(2,2)}a_{(2,1)})^k \varepsilon_{(i-1,j,k),(2,2),(1,i-2),(l_3,l_4)} (a_{(2,2)}a_{(2,1)})^{1-k} & \text{if } i \text{ is odd}, \\ \varepsilon_{(i-1,j,k),(2,2),(1,i-2),(l_3,l_4)}(a_{(2,2)}a_{(2,1)}) & \\ \quad - (a_{(2,2)}a_{(2,1)})\varepsilon_{(i-1,j,k),(2,2),(1,i-2),(l_3,l_4)} & \text{if } i \text{ is even}, \end{cases}$$

$\varepsilon_{(i,j,k),(1,1),(1,i-1),(l_3,l_4)} \to$

$$\begin{cases} \sum_{k=0}^{1} (a_{(2,1)}a_{(2,2)})^k \varepsilon_{(i-1,j,k),(1,1),(1,i-2),(l_3,l_4)} (a_{(2,1)}a_{(2,2)})^{1-k} & \text{if } l_2 \text{ is even}(\neq 0), \\ \varepsilon_{(i-1,j,k),(1,1),(1,i-2),(l_3,l_4)}(a_{(2,1)}a_{(2,2)}) & \\ \quad - (a_{(2,1)}a_{(2,2)})\varepsilon_{(i-1,j,k),(1,1),(1,i-2),(l_3,l_4)} & \text{if } l_2 \text{ is odd}, \end{cases}$$

(b) In the case of k is even,

$\partial_{(i,j,k)} : R_{(i,j,k)} \to R_{(i-1,j,k)}$:

$\varepsilon_{(i,j,k),(1,2),(l_1,l_2),(k,0)} \to$

$$\begin{cases} (-1)^i a_{(1,1)} \varepsilon_{(i-1,j,k),(1,2),(i-1,0),(k,0)} & \text{if } l_2 = 0, \\ \sum_{k=0}^{1} \big(a_{(2,1)}a_{(2,2)}\big)^k \varepsilon_{(i-1,j,k),(1,2),(l_1,l_2-1),(k,0)} \big(a_{(2,2)}a_{(2,1)}\big)^{1-k} & \\ \ + (-1)^i a_{(1,1)} \varepsilon_{(i-1,j,k),(1,2),(l_1-1,l_2),(k,0)} & \text{if } l_2 \text{ is even}(\neq 0), \\ \varepsilon_{(i-1,j,k),(1,2),(l_1,l_2-1),(k,0)} \big(a_{(2,2)}a_{(2,1)}\big) & \\ \ - \big(a_{(2,1)}a_{(2,2)}\big) \varepsilon_{(i-1,j,k),(1,2),(l_1,l_2-1),(k,0)} & \\ \ + (-1)^i a_{(1,1)} \varepsilon_{(i-1,j,k),(1,2),(l_1-1,l_2),(k,0)} & \text{if } l_2 \text{ is odd}, \end{cases}$$

$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(0,k)} \to$

$$\begin{cases} \varepsilon_{(i-1,j,k),(2,1),(i-1,0),(0,k)} a_{(1,1)} & \text{if } l_2 = 0, \\ \sum_{k=0}^{1} \big(a_{(2,2)}a_{(2,1)}\big)^k \varepsilon_{(i-1,j,k),(2,1),(l_1,l_2-1),(0,k)} \big(a_{(2,1)}a_{(2,2)}\big)^{1-k} & \\ \ + (-1)^{l_1} \varepsilon_{(i-1,j,k),(2,1),(l_1-1,l_2),(0,k)} a_{(1,1)} & \text{if } l_2 \text{ is even}(\neq 0), \\ \varepsilon_{(i-1,j,k),(2,1),(l_1,l_2-1),(0,k)} \big(a_{(2,1)}a_{(2,2)}\big) & \\ \ - \big(a_{(2,2)}a_{(2,1)}\big) \varepsilon_{(i-1,j,k),(2,1),(l_1,l_2-1),(0,k)} & \\ \ - \varepsilon_{(i-1,j,k),(2,1),(l_1-1,l_2),(0,k)} a_{(1,1)} & \text{if } l_2 \text{ is odd}, \end{cases}$$

$\varepsilon_{(i,j,k),(1,2),(l_1,l_2),(l_3,l_4)} \to$

$$\begin{cases} \sum_{l=0}^{1} \big(a_{(2,1)}a_{(2,2)}\big)^l \varepsilon_{(i-1,j,k),(1,2),(l_1,l_2-1),(l_3,l_4)} \big(a_{(2,2)}a_{(2,1)}\big)^{1-l} & \text{if } i \text{ is odd}, \\ \varepsilon_{(i-1,j,k),(1,2),(l_1,l_2-1),(l_3,l_4)} \big(a_{(2,2)}a_{(2,1)}\big) & \\ \ - \big(a_{(2,1)}a_{(2,2)}\big) \varepsilon_{(i-1,j,k),(1,2),(l_1,l_2-1),(l_3,l_4)} & \text{if } i \text{ is even}, \end{cases}$$

if $l_2 \geq 1$,

$\varepsilon_{(i,j,k),(2,1),(l_1,l_2),(l_3,l_4)} \to$

$$\begin{cases} \sum_{k=0}^{1} \big(a_{(2,2)}a_{(2,1)}\big)^k \varepsilon_{(i-1,j,k),(2,1),(l_1,l_2-1),(l_3,l_4)} \big(a_{(2,1)}a_{(2,2)}\big)^{1-k} & \text{if } i \text{ is odd}, \\ \varepsilon_{(i-1,j,k),(2,1),(l_1,l_2-1),(l_3,l_4)} \big(a_{(2,1)}a_{(2,2)}\big) & \\ \ - \big(a_{(2,2)}a_{(2,1)}\big) \varepsilon_{(i-1,j,k),(2,1),(l_1,l_2-1),(l_3,l_4)} & \text{if } i \text{ is even}, \end{cases}$$

if $l_2 \geq 1$,

$\varepsilon_{(i,j,k),(2,1),(1,i-1),(l_3,l_4)} \to$

$$\begin{cases} \sum_{k=0}^{1} \big(a_{(2,2)}a_{(2,1)}\big)^k \varepsilon_{(i-1,j,k),(2,1),(1,i-2),(l_3,l_4)} \big(a_{(2,1)}a_{(2,2)}\big)^{1-k} & \text{if } i \text{ is odd}, \\ \varepsilon_{(i-1,j,k),(2,1),(1,i-2),(l_3,l_4)} \big(a_{(2,1)}a_{(2,2)}\big) & \\ \ - \big(a_{(2,2)}a_{(2,1)}\big) \varepsilon_{(i-1,j,k),(2,1),(1,i-2),(l_3,l_4)} & \text{if } i \text{ is even}, \end{cases}$$

$\varepsilon_{(i,j,k),(1,2),(1,i-1),(l_3,l_4)} \to$

$$\begin{cases} \sum_{k=0}^{1} \big(a_{(2,1)}a_{(2,2)}\big)^k \varepsilon_{(i-1,j,k),(1,2),(1,i-2),(l_3,l_4)} \big(a_{(2,2)}a_{(2,1)}\big)^{1-k} & \text{if } l_2 \text{ is even}(\neq 0), \\ \varepsilon_{(i-1,j,k),(1,2),(1,i-2),(l_3,l_4)} \big(a_{(2,2)}a_{(2,1)}\big) & \\ \ - \big(a_{(2,1)}a_{(2,2)}\big) \varepsilon_{(i-1,j,k),(1,2),(1,i-2),(l_3,l_4)} & \text{if } l_2 \text{ is odd}. \end{cases}$$

The total complex $\mathbb{P}$ of these complexes hold that $\mathbb{P} \otimes A/\mathrm{rad}\, A$ is a exaxt sequence. So $\mathbb{P}$ is the projective bimodule resolution of A.

Theorem 2.1. *We have the minimal projective resolution of A:*

$$\mathbb{P} : 0 \leftarrow A \xleftarrow{\pi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \leftarrow \cdots P_{n-1} \xleftarrow{d_n} P_n \leftarrow \cdots$$

as the total complex of the above complexes where P_n is the projective left A^e-modules:

$$P_n = \coprod_{i+j=n} R_{(i,j,0)} \oplus \coprod_{\substack{i+j+k=n \\ i,j,k \geq 1}} R_{(i,j,k)}$$

and d_n is the left A^e-homomorphisms

$$d_n = \sum_{i+j+k=n} \partial_{(i,j,k)} + (-1)^{i+k+1}\delta_{(i,j,k)} + \xi_{(i,j,k)},$$

for $n \geq 1$.

We consider the cohomology of the complex $\mathrm{Hom}_{A^e}(\mathbb{P}, A)$ and Yoneda product. Then we have the generators of the Hochschild cohomology ring of A modulo nilpotence as follows:

$$\begin{aligned} &e_{1,(i,0,0),(0,i)} + e_{2,(i,0,0)} \text{ where} \\ &e_{1,(i,0,0),(0,i)} : \varepsilon_{(i,0,0),(1,1),(0,i)} \to e_1 \in \mathrm{Hom}_{A^e}(R_{(i,0,0)}, A), \\ &e_{2,(i,0,0)} : \varepsilon_{(i,0,0),(2,2)} \to e_2 \in \mathrm{Hom}_{A^e}(R_{(i,0,0)}, A) \text{ for } i \text{ is even.} \end{aligned}$$

with the following Yoneda product:

$$(e_{1,(i,0,0),(0,i)} + e_{2,(i,0,0)}) \circ (e_{1,(i',0,0),(0,i')} + e_{2,(i',0,0)}) = (e_{1,(i+i',0,0),(0,i+i')} + e_{2,(i+i',0,0)}).$$

So we have the following result.

Theorem 2.2. *The Hochsxhild cohomology ring of A_1 modulo nilpotence is the polynomial ring.*

$$\mathrm{HH}^*(A)/\mathcal{N} = k[e_{1,(2,0,0),(0,2)} + e_{2,(2,0,0)}]$$

We conjecture that the projective bimodule resolution of the finite dimensional algebra with quantum-like relations and monomial relations is given by the total complex of the complexes depending on the relations.

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