数理解析研究所講究録
第1968巻 2015年 16-22

ON THE V-TRANSVERSALITY CONSTRUCTION OF EQUIVARIANT FRAMED MAPS

Masaharu Morimoto

Graduate School of Natural Science and Technology, Okayama University

Abstract. Let G be a finite group. For a pointed G-map $Y^+ \wedge V^\bullet \to V^\bullet$, we introduce the notion of V-transversality, where Y is a compact (smooth) G-manifold, V is a real G-module, $Y^+ = Y \amalg \{y_\infty\}$, and $V^\bullet = V \cup \{v_\infty\}$. Every V-transversal G-map $Y^+ \wedge V^\bullet \to V^\bullet$ gives rise of a G-framed map with the target manifold Y. This yields a one-to-one correspondence from the set of pointed G-homotopy classes to the set of G-framed cobordisms. From this view point, we discuss the Burnside ring of G and the equivariant stable G-cohomotopy group $\omega_G^0(Y)$ which consists of equivalence classes of pointed G-maps $Y^+ \wedge V^\bullet \to V^\bullet$ for $V = \mathbb{C}[G]^m$ $(m \gg 1)$.

1. Introduction

In this article, we report results obtained from discussions with Takashi Matsunaga and Yasuhiro Hara.

Let G be a finite group throughout this paper. A G-framed map $\boldsymbol{f}$ is a pair of a G-map $f : (X, \partial X) \to (Y, \partial Y)$ and a real G-vector bundle isomorphism $b : TX \oplus \varepsilon(\mathbb{R}^m) \to f^*TY \oplus \varepsilon(\mathbb{R}^m)$, where X and Y are compact (smooth) G-manifolds of same dimension. Let $\mathfrak{N}(G, Y)$ be the set of G-framed cobordism classes of G-framed maps with the target manifold Y. Let $\omega_G^0(Y)$ be the equivariant stable cohomotopy group of Y, of which the definition will be given in Section 2. Let V be a real G-module (of finite dimension). In this paper we introduce the notion that a

---

2010 *Mathematics Subject Classification.* Primary 57S17; Secondary 57R85.

Key words and phrases. equivariant transversality, equivariant framed map, the Burnside ring, equivariant stable cohomotopy group.

This research was partially supported by JSPS KAKENHI Grant Number 26400090.

G-map $Y^+ \wedge V^\bullet \to V^\bullet$ is *V-transversal* to 0 in $V^\bullet$, essentially due to T. Petrie [8], cf. [5, §5]. The next result indicates importance of the notion in equivariant surgery theory.

Theorem 1.1. *The V-transversality construction gives a one-to-one correspondence*

$$\Phi : \omega_G^0(Y) \to \mathfrak{N}(G, Y).$$

This is an equivariant version of [4, Theorems B and C].

Let ω_G^n denote the group $\omega_G^n(pt)$. Let $A(G)$ denote the Burnside ring, i.e. the Grothendieck group for the category of finite G-sets. By proofs in equivariant homotopy theory, e.g. [1], [6], we have seen that $A(G)$ is isomorphic to ω_G^0. However the theorem above gives rise of a geometric proof.

Theorem 1.2. *The V-transversality construction yields an isomorphism*

$$\Psi : \omega_G^0 \to A(G).$$

Details of the theorem will be given in Section 4.

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. Let $\mathcal{F}$ be a set of subgroups of G. We call $\mathcal{F}$ *lower closed* if $H \in F$ then $\mathcal{S}(H) \subset \mathcal{F}$. We call $\mathcal{F}$ *conjugation invariant* if $gHg^{-1} \in \mathcal{F}$ holds for all $H \in \mathcal{F}$ and $g \in G$. Hereafter, we suppose that $\mathcal{F}$ is lower closed and conjugacy invariant. Let $\mathfrak{F} = \mathfrak{F}(G, \mathcal{F})$ be the category such that $\mathrm{Obj}(\mathfrak{F})$ is same as $\mathcal{F}$ and $\mathrm{Mor}(\mathcal{F})$ consists of all (H, g, K) with H, $K \in \mathcal{F}$, $g \in G$ and $gHg^{-1} \subset K$. Let Y be a compact G-manifold. If $H \leq K \leq G$ then we have the restriction map $\mathrm{res}_H^K : \omega_K^n(Y) \to \omega_H^n(Y)$. If $H \leq G$ and $g \in G$ then we have the conjugation map $c_g : H \to gHg^{-1}$ and hence the induced map $c_g^* : \omega_{gHg^{-1}}^n(Y) \to \omega_H^n(Y)$. For $(H, g, K) \in \mathrm{Mor}(\mathfrak{F})$, the homomorphism $(H, g, K)^*$ is defined to be the composition

$$\omega_K^n(Y) \xrightarrow{\mathrm{res}^K_{gHg^{-1}}} \omega_{gHg^{-1}}(Y) \xrightarrow{c_g^*} \omega_H^n(Y).$$

Let $\omega_{G,\mathcal{F}}^n(Y)$ denote the inverse limit

$$\text{inv-lim}_{\mathfrak{F}}\, \omega_H^n(Y) \qquad (H \in \mathcal{F}).$$

We set $\omega_{G,\mathcal{F}}^n = \omega_{G,\mathcal{F}}^n(pt)$. It is interesting to study the image and the kernel of the canonical map

$$\mathrm{res}_{\mathcal{F}} : \omega_G^0(Y) \to \omega_{G,\mathcal{F}}^0(Y).$$

Theorem 1.3 (Y. Hara–M. M.). *Let G be a nontrivial nilpotent group and $\mathcal{F} = \mathcal{S}(G) \smallsetminus \{G\}$. The canonical map $\mathrm{res}_{\mathcal{F}} : \omega_G^0 \to \omega_{G,\mathcal{F}}^0$ is surjective if and only if G is a cyclic group of order a prime or a product of distinct primes.*

This implies

Corollary 1.4. *Let G and $\mathcal{F}$ be as above and let Y be a compact G-manifold such that $Y^G \neq \emptyset$. If the canonical map $\mathrm{res}_{\mathcal{F}} : \omega_G^0(Y) \to \omega_{G,\mathcal{F}}^0(Y)$ is surjective then G is a cyclic group of order a prime or a product of distinct primes.*

Let k_G be the product of the primes p such that G contains a normal subgroup N with index p. Here k_G is understood to be 1 if G is a perfect group. The next lemma is essentially due to R. Oliver [7, Lemma 8].

Lemma 1.5. *There exists an element $\gamma_G = [X_1] - [X_2]$ of $A(G)$ such that $|X_1^G| - |X_2^G| = k_G$ and $\mathrm{res}_H^G \gamma_G = 0$ for all $H < G$, where X_1 and X_2 are finite G-sets.*

Proposition 1.6. *Let G be a nontrivial group and $\mathcal{F} = \mathcal{S}(G) \smallsetminus \{G\}$. Then the kernel of $\mathrm{res}_{\mathcal{F}} : \omega_G^0 \to \omega_{G,\mathcal{F}}^0$ is generated by γ_G.*

Corollary 1.7. *Let G and $\mathcal{F}$ be as above. Then $\gamma_G \omega_G^0(Y)$ is contained in the kernel of $\omega_G^0(Y) \to \omega_{G,\mathcal{F}}^0(Y)$.*

2. The equivariant stable cohomotopy group $\omega_G^n(Y)$

Let Y be a compact G-manifold. We denote by Y^+ the disjoint union $Y \amalg \{y_\infty\}$ and the point y_∞ is regarded as the base point of Y^+. Let V be a real G-module. We denote by $V^\bullet$ the one-point compactification $V \cup \{v_\infty\}$ and v_∞ is regarded as the base point of $V^\bullet$. The smash product

$$Y^+ \wedge V^\bullet = \frac{Y^+ \times V^\bullet}{(Y^+ \times v_\infty) \cup (y_\infty \times V^\bullet)}$$

can be regarded as the Thom space of real G-vector bundle $\pi : Y \times V \to Y$ with fiber V.

We denote by $\mathbb{R}$ the 1-dimensional real vector space with trivial G-action. For a finite G-CW complex X and an integer n, we define $\omega_G^n(X)$ by

$$\omega_G^n(X) = \lim_{m \to \infty} [A, B]_0^G$$

where

$$A = \begin{cases} X^+ \wedge M_m^\bullet & (n \geq 0) \\ X^+ \wedge (\mathbb{R}^{|n|} \oplus M_m)^\bullet & (n < 0), \end{cases}$$

$$B = \begin{cases} (\mathbb{R}^n \oplus M_m)^\bullet & (n \geq 0) \\ M_m^\bullet & (n < 0), \end{cases}$$

$$M_m = \mathbb{C}[G]^{\oplus m},$$

and $[-,-]_0^G$ stands for the set of all homotopy classes of maps in the category of pointed G-spaces. We set $\omega_G^n = \omega_G^n(pt)$. Define the map $\deg_H : \omega_G^0 \to \mathbb{Z}$ by

$$\deg_H([f : M_m^\bullet \to M_m^\bullet]) = \deg(f^H : M_m^{H\bullet} \to M_m^{H\bullet}).$$

It is known that the map

$$\prod_{H \in \mathcal{S}(G)} \deg_H : \omega_G^n \to \prod_{H \in \mathcal{S}(G)} \mathbb{Z}$$

is injective and

$$[V^\bullet, V^\bullet]_0^G \cong \omega_G^0$$

via the canonical map whenever $V \supset \mathbb{C}[G]$.

3. V-TRANSVERSALITY OF G-MAPS $Y^+ \wedge V^\bullet \to V^\bullet$

In this section we introduce the notion of *V-transversality* due to T. Petrie [8]. Let V be a real G-module with a G-invariant inner product. For $H \leq G$, V is decomposed into $V^H \oplus V_H$ as real $N_G(H)$-modules, where V^H is the H-fixed point set of V. A base point preserving G-map $\alpha : Y^+ \wedge V^\bullet \to V^\bullet$ is called *V-transversal* to 0 in the target space $V^\bullet$ if the following conditions are fulfilled.

(1) α is smooth on a neighborhood of X.
(2) α is transversal to 0 in V, i.e. $d_x\alpha : T_x(Y \times V) \to T_0V$ is surjective at every $x \in X$.
(3) the normal derivative $\nu_x(\alpha) : V_H \to V_H$ at x, where $H = G_x$, coincides with the identity map $V_H \to V_H$ for every $x \in X$,

where $X = \alpha^{-1}(0)$ and $\nu_x(\alpha)$ is defined to be the composition

$$V_H \xrightarrow{\text{incl}} V = T_{\pi_V(x)}V \xrightarrow{\text{incl}} T_x(Y \times V) \xrightarrow{d_x\alpha} T_0(V) \xrightarrow{\text{proj}} V_H.$$

Lemma 3.1. *Let Y be a compact G-manifold with G-invariant Riemannian metric, A a closed G-subset of Y, V a real G-module, and $\varepsilon : Y^+ \wedge V^\bullet \to \mathbb{R}$ a G-invariant positive function. Let $\alpha : Y^+ \wedge V^\bullet \to V^\bullet$ be a base-point preserving G-map such that $\alpha : A^+ \wedge V^\bullet \to V^\bullet$ is V-transveral to 0 in $V^\bullet$. Then there exists a base-point preserving G-homotopy $H : I \times (Y^+ \wedge V^\bullet) \to V^\bullet$, where $I = [0,1]$, from α to $\beta : Y^+ \wedge V^\bullet \to V^\bullet$ satisfying the following conditions.*

(1) $H(t,(x,v)) = \alpha(x,v)$ *for all* $t \in I$, $x \in A$, *and* $v \in V$.
(2) $d(\alpha(y,v), H(t,(y,v))) < \varepsilon(y,v)$ *for all* $(y,v) \in Y \times V$.
(3) β *is V-transversal to 0 in $V^\bullet$.*

A *G-framed map* $\boldsymbol{f} = (f,b)$ consists of a G-map $f : X \to Y$, where X and Y are compact (smooth) G-manifolds, and a real G-vector bundle isomorphism $b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \to f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^m)$ for some integer $m \geq 0$.

Suppose $\alpha : Y^+ \wedge V^\bullet \to V^\bullet$ is V-transversal to 0 in $V^\bullet$. Then set $X = \alpha^{-1}(0)$ and let $f : X \to Y$ be the composition of the inclusion $j_X : X \to Y \times V$ and the projection $\pi_Y : Y \times V \to Y$. There is a canonical isomorphism

$$T(Y \times V)|_X = (\pi_Y TY \oplus \pi_V TV)|_X = f^*TY \oplus \varepsilon(V).$$

We also have an isomorphism

$$T(Y \times V)|_X = TX \oplus \nu(X, Y \times V) \cong TX \oplus (\alpha|_X)^*\nu(0,V) = TX \oplus \varepsilon(V).$$

Thus we get a G-vector bundle isomorphism

$$\beta : TX \oplus \varepsilon(V) \to f^*TY \oplus \varepsilon(V)$$

such that $(\beta|_x)(x,v) = (x,v)$ $(\in \varepsilon(V))$ for all $x \in X$ and $v \in V_H$, where $H = G_x$. By Lück-Madsen [3, Appendix, Proposition (A2)] and [5, §6], in the case $m > \dim Y$, we obtain a G-vector bundle isomorphism

$$b : TX \oplus \varepsilon(\mathbb{R}^m) \to f^*TY \oplus \varepsilon(\mathbb{R}^m)$$

such that β and b are stably regularly G-homotopic. This procedure obtaining $\boldsymbol{f} = (f,b)$ from α is called the *V-transversality construction* of G-framed maps. The construction may also be called *Pontryagin-Petrie construction* of G-framed maps, cf. [4, §7].

4. The isomorphism $\Psi : \omega_G^0 \to A(G)$

Let $\alpha : V^\bullet \to V^\bullet$ be a base-point preserving G-map such that $V \supset \mathbb{C}[G]$. Then we have $\dim V^G \geq 2$. Suppose α is V-transversal to 0 in $V^\bullet$. The V-transversality construction yields $X = \alpha^{-1}(0)$, $f = \alpha|_X : X \to \{pt\}$, $\beta : \varepsilon(V) \to \varepsilon(V)$ and $b : \varepsilon(V^G) \to \varepsilon(V^G)$ such that $\beta|_x : V \to V$ coincides with $d\alpha_x : V \to V$ for $x \in X$, and β is regularly G-homotopic to $b \oplus id_{\varepsilon(V_G)}$. Decompose the G-set $X = \alpha^{-1}(0)$ into the disjoint union of G-orbits $X_i = Gx_i$, where $i = 1, \ldots, k$; i.e. $X = \coprod_{i=1}^k X_i$. We define

$$\varepsilon(d\alpha_{x_i}) = \begin{cases} 1 & \text{(if } d\alpha_{x_i} \text{ is orientation preserving),} \\ -1 & \text{(if } d\alpha_{x_i} \text{ is orientation reversing).} \end{cases}$$

Set

$$X_+ = \coprod_i Gx_i \ : \ \varepsilon(d\alpha_{x_i}) = 1,$$

$$X_- = \coprod_i Gx_i \ : \ \varepsilon(d\alpha_{x_i}) = -1.$$

Then we obtain an element $[X_+] - [X_-]$ of $A(G)$. The correspondence $[f] \longmapsto [X_+] - [X_-]$ gives the map $\Psi : \omega_G^0 \to A(G)$. We have a canonical one-to-one correspondence $\Xi : \mathfrak{N}(G, pt) \to A(G)$. Moreover the diagram

$$\begin{array}{ccc} \omega_G^0 & \xrightarrow{\Phi} & \mathfrak{N}(G, pt); \\ & {\scriptstyle \Psi} \searrow \quad \swarrow {\scriptstyle \Xi,\ \cong} & \\ & A(G) & \end{array} \qquad \begin{array}{ccc} [\alpha] & \stackrel{\Phi}{\longmapsto} & [f, b] \\ & {\scriptstyle \Psi} \searrow \quad \swarrow {\scriptstyle \Xi} & \\ & [X_+] - [X_-] & \end{array}$$

commutes. Once it was admitted that Φ is a one-to-one correspondence, the map Ψ is bijective and hence an isomorphism.

References

[1] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin-New York, 1987.

[2] E. Laitinen and M. Morimoto, *Finite groups with smooth one fixed point actions on spheres*, Forum Math. **10** (1998), 479–520.

[3] W. Lück and I. Madsen, *Equivariant L-Theory I*, Math. Z. **203** (1990), 503–526.

[4] J. W. Milnor, *Topology from the Differentiable Viewpoint*, Princeton Univ. Press, Princeton, New Jersey, 1965.

[5] M. Morimoto, *Equivariant surgery theory: Construction of equivariant normal maps*, Publ. Res. Inst. Math. Sci. Kyoto Univ. 31 (1995), 145–167.

[6] M. Morimoto, *The Burnside ring revisited*, in: Current Trends in Transformation Groups, A. Bak, M. Morimoto and F. Ushitaki (eds.), K-Monographs in Math. **7**, Kluwer Academic Publ., Dordrecht-Boston, 2002, pp. 129–145.

[7] R. Oliver, *Fixed point sets of groups on finite acyclic complexes*, Comment. Math. Helv. **50** (1975), 155–177.

[8] T. Petrie, *One fixed point actions on spheres I*, Adv. Math. **46** (1982), 3–14.

Graduate School of Natural Science and Technology, Okayama University
3-1-1 Tsushimanaka, Kitaku, Okayama, 700-8530 Japan
E-mail address: `morimoto@ems.okayama-u.ac.jp`