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NOTE ON STABILITY OF AN SIRS EPIDEMIC MODEL

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ABSTRACT. In this paper we derive a scalar delay differential equation from an epidemic model with waning immunity. The model is formulated as a system of delay differential equations. The characteristic equation is computed. We visualize the stability condition for an endemic equilibrium in a two-parameter plane.

1. INTRODUCTION

In [9] we consider periodic outbreak of mycoplasma pneumoniae in Japan. Minor variation of the immunity period is shown to be essential in order to explain the infectious disease dynamics. Our agenda includes mathematical studies for periodic solutions of the mathematical model employed in [9]. This note is a preliminary study for the project. The model in [9] takes the form of SEIRS type epidemic model with a gamma distribution which the immunity period follows. In this paper for mathematical analysis we consider SIRS type epidemic model with fixed immunity period (such that the variance of the immune period is zero). Let us consider the following SIRS epidemic model

$$\frac{d}{dt}S(t) = -\beta S(t)I(t) + \gamma I(t-\tau), \tag{1.1a}$$

$$\frac{d}{dt}I(t) = \beta S(t)I(t) - \gamma I(t), \tag{1.1b}$$

$$\frac{d}{dt}R(t) = \gamma I(t) - \gamma I(t-\tau), \tag{1.1c}$$

where the total population is fixed as

$$S(t) + I(t) + R(t) = 1, \ t \geq -\tau \tag{1.2}$$

and recovered population satisfies

$$R(t) = \gamma \int_0^\tau I(t-s)ds, \ t \geq 0. \tag{1.3}$$

Here $S(t)$, $I(t)$ and $R(t)$ respectively denote the fraction of susceptible, infective and recovered populations at time t. The model has three parameters: transmission coefficient $\beta > 0$, the recovery rate $\gamma > 0$ and the immune period $\tau > 0$. We also refer the papers [6, 10, 4] for analyses of similar SIRS epidemic models. See also [5] for detail of compartmental model in epidemiology. The model (1.1) appears in the paper [5] and we here review the stability analysis.

From (1.2) and (1.3) we get

$$S(t) = 1 - I(t) - \gamma \int_0^\tau I(t-s)ds$$

then we obtain a scalar delay differential equation:

$$\frac{d}{dt}I(t) = I(t)\left\{\beta\left(1 - I(t) - \gamma\int_0^\tau I(t-s)ds\right) - \gamma\right\}. \tag{1.4}$$

The basic reproduction number is given as

$$R_0 := \frac{\beta}{\gamma}.$$

It is assumed that $R_0 > 1$ holds so that (1.4) has a positive equilibrium:

$$I_e := \frac{1 - \frac{1}{R_0}}{1 + \gamma\tau}.$$

Remark 1. $\gamma\tau = \frac{\tau}{\frac{1}{\gamma}}$ denotes the fraction of the immunity period over the expected infectious period, which may be a large parameter. Note that

$$\gamma\tau > 1 \leftrightarrow \tau > \frac{1}{\gamma}$$

implies that the immunity period is longer than the expected infectious period.

We normalize the equation (1.4) defining

$$x(t) := \frac{I(t)}{I_e} - 1$$

and subsequently consider the nondimensional time $u = \frac{t}{\tau}$. Abusing notation we finally obtain

$$\frac{d}{dt}x(t) = -p\left(x(t) + 1\right)\left(x(t) + \eta\int_0^1 x(t-s)ds\right), \tag{1.5}$$

where

$$p := \frac{\gamma\tau}{1 + \gamma\tau}(R_0 - 1), \tag{1.6a}$$

$$\eta := \gamma\tau. \tag{1.6b}$$

Initial condition for (1.5) is

$$x(\theta) = \psi(\theta) \geq -1, \;\; \theta \in [-1, 0],$$

excluding the constant function $\psi(\theta) \equiv -1, \;\; \theta \in [-1, 0]$.

Remark 2. To apply the time transformation we introduce a nondimensional time $u = \frac{t}{\tau}$ and define

$$\tilde{x}(u) := \tilde{x}(\frac{t}{\tau}) = x(t).$$

Then one can see $\frac{d}{du}\tilde{x}(u) = \tau\frac{d}{dt}x(t)$ and

$$\int_0^\tau x(t-s)ds = \int_0^\tau \tilde{x}(u - \frac{s}{\tau})ds = \tau\int_0^1 \tilde{x}(u-\theta)d\theta.$$

2. Stability analysis

Applying the fluctuation lemma we can obtain a global stability result. There seems to be no other results for global stability. See also the global stability condition by the fluctuation lemma in [7].

Theorem 3. *Let us assume that $\eta < 1$ holds. Then the trivial equilibrium of (1.5) is globally attractive.*

Proof. Let us write

$$\overline{x} = \limsup_{t\to\infty} x(t), \ \ \underline{x} = \liminf_{t\to\infty} x(t).$$

Consider a sequence such that $x(t_n) \to \overline{x}$ as $n \to \infty$ with $x'(t_n) \geq 0$. Then we get

$$0 \geq x(t_n) + \eta \int_0^1 x(t_n - s)ds.$$

Taking the limit and estimating the second term of the right hand side from below by $\underline{x}$, we get

$$0 \geq \overline{x} + \eta\underline{x}.$$

Similarly, considering a sequence which tends to $\underline{x}$, we get

$$0 \leq x(u_n) + \eta \int_0^1 x(u_n - s)ds.$$

Thus

$$0 \leq \underline{x} + \eta\overline{x}.$$

Therefore it holds

$$\overline{x} + \eta\underline{x} \leq 0 \leq \underline{x} + \eta\overline{x},$$

thus

$$(\overline{x} - \underline{x}) \leq \eta\,(\overline{x} - \underline{x})\,.$$

Since $\eta < 1$ is assumed, we obtain $\underline{x} = \overline{x}$. It is easy to see that $\underline{x} = \overline{x} = 0$ follows. □

If $\eta < 1$ then the trivial equilibrium is shown to be asymptotically stable in the following section.

2.1. **Linearized stability analysis.** To analyze asymptotic stability of the trivial equilibrium of (1.5) we derive the characteristic equation. The characteristic equation is computed as

$$\lambda = -p\left(1 + \eta\int_0^1 e^{-\lambda s}ds\right), \ \lambda \in \mathbb{C}. \tag{2.1}$$

We analyze the characteristic equation (2.1) following Chapter XI of [2]. See also [3, 1, 8] for analysis of characteristic equations of delay equations. One can see that $\lambda = 0$ is a root of (2.1) if $p = 0$ holds, where the transcritical bifurcation occurs (as p increases). Substituting $\lambda = i\omega$, $\omega \in \mathbb{R}$ we get

$$0 = 1 + \eta\int_0^1 \cos(\omega s)\,ds, \tag{2.2}$$

$$\omega = p\eta\int_0^1 \sin(\omega s)\,ds. \tag{2.3}$$

From (2.2) one has

$$\eta = \frac{-1}{\int_0^1 \cos(\omega s)\, ds} = -\frac{\omega}{\sin\omega}.$$

Then p is determined from (2.3) as

$$p = \frac{\omega}{\eta \int_0^1 \sin(\omega s)\, ds} = -\frac{\omega \sin\omega}{1-\cos\omega}.$$

For $n \in \mathbb{N}_+$ let

$$I_n := ((2n+1)\pi, 2(n+1)\pi).$$

The parametric curve

$$(\eta(\omega), p(\omega)) = \left(-\frac{\omega}{\sin\omega}, -\frac{\omega\sin\omega}{1-\cos\omega}\right), \ \omega \in I_n \tag{2.4}$$

depicts the condition where the characteristic equation (2.1) has a conjugated pair of purely imaginary roots $\lambda = \pm i\omega$, $\omega \in I_n$. One can easily see that

$$(\eta(\omega), p(\omega)) \in \mathbb{R}^2_+, \ \omega \in I_n.$$

The parametric curve (2.4) can be translated in terms of R_0 and $\gamma\tau$ using the relation (1.6). We get the following condition

$$R_0 - 1 = -\frac{\omega\sin\omega}{1-\cos\omega}\left(1 - \frac{\sin\omega}{\omega}\right),$$
$$\gamma\tau = -\frac{\omega}{\sin\omega},$$

where the characteristic equation (2.1) has purely imaginary roots.

3. Discussion

We here sketch the stability analysis for the SIRS epidemic model with delay. Although the model equation has a simple looking, it exhibits destabilization of the endemic equilibrium and has a periodic solutions via Hopf bifurcation. In the paper [9] we discuss a role of the minor variation of the immunity period in the periodic epidemic cycle seen in a childhood disease, in particular, for small R_0. See also [8]. The author study periodicity and uniqueness of a periodic solution of the equation (1.5) in the collaboration with G. Kiss, G. Vas and R. Omori.

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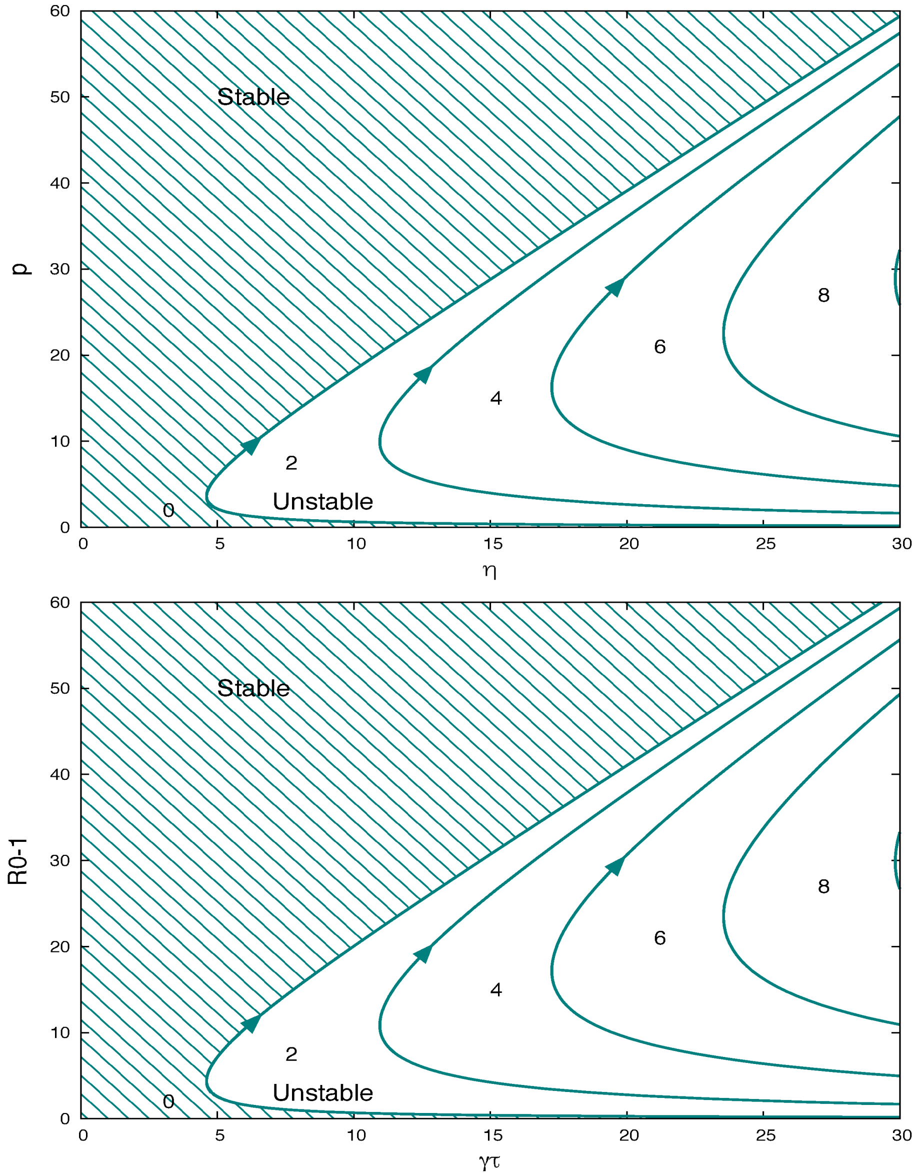

FIGURE 2.1. Stability boundary of the endemic equilibrium in (η, p) plane (above) and in $(\gamma\tau,\ R_0 - 1)$ plane (below). The number depicts the number of roots of the characteristic equation in the right half complex plane. The arrows in the curves indicate the direction of increasing ω.

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