

# Unbounded solutions to some reaction-diffusion-ODE systems modeling pattern formation

Kanako Suzuki

College of Science, Ibaraki University,  
2-1-1 Bunkyo, Mito 310-8512, Japan  
kanako.suzuki.sci2@vc.ibaraki.ac.jp

## 1 Introduction

The mechanism of pattern formation is one of the most interesting subjects in mathematical biology. A. M. Turing proposed a notion of the *diffusion-driven instability* in the seminal paper [13]. It means that a reaction between two chemicals with different diffusion rates may cause a destabilization of a spatially homogeneous state, thus leading to the formation of nontrivial spatial structure. This is a bifurcation that arises in a reaction-diffusion system, when there exists a spatially homogeneous stationary solution which is asymptotically stable with respect to spatially homogeneous perturbations but unstable to spatially heterogeneous perturbations. Models with the diffusion-driven instability describe a process of a destabilization of stationary spatially homogeneous steady states and evolution of the system towards spatially heterogeneous steady states.

Recently, the diffusion-driven instability has been observed in models describing a coupling of cell-localized processes with a cell-to-cell communication via diffusion. Such models are of a form of systems consisting of a single ordinary differential equation coupled with a reaction-diffusion equation:

$$u_t = f(u, v), \quad v_t = D\Delta v + g(u, v), \quad (1.1)$$

such as in Refs. [4, 8, 10, 12]. We call the system in the form of (1.1) *reaction-diffusion-ODE system*. Simulations of different models of this type indicate a formation of dynamical, multimodal, and apparently irregular and unbounded structures, the shape of which depends strongly on initial conditions [1, 9, 10, 12].

A scalar reaction-diffusion equation (in a bounded, convex domain and the Neumann boundary conditions) cannot exhibit stable spatially heterogeneous patterns. Coupling it to an ODE fulfilling the following *autocatalysis* condition at the equilibrium  $(\bar{u}, \bar{v})$

$$f_u(\bar{u}, \bar{v}) > 0 \quad (1.2)$$

leads to the diffusion-driven instability. However, in such a case, all regular Turing patterns are unstable, because the same mechanism which destabilizes constant

solutions, destabilizes also all continuous spatially heterogeneous stationary solutions, [5, 6]. This instability result holds also for discontinuous patterns in case of a specific class of nonlinearities, see also [5, 6].

In this paper, we present two examples of (1.1) to understand the dynamics of non-constant solutions of the reaction-diffusion-ODE systems exhibiting the diffusion-driven instability. In both cases, we show that they have solutions which become unbounded (blow up) in a finite time.

This is a joint work with A. Marciniak-Czochra (University of Heidelberg), G. Karch (University of Wroclaw) and J. Zienkiewicz (University of Wroclaw).

We begin our study by stating a result on the existence and boundedness of a solution to the initial boundary value problem for (1.1).

## 2 Existence of solutions

We consider the following system

$$u_t = f(u, v), \quad \text{for } x \in \overline{\Omega}, \quad t > 0, \quad (2.1)$$

$$v_t = \Delta v + g(u, v) \quad \text{for } x \in \Omega, \quad t > 0 \quad (2.2)$$

in a bounded domain  $\Omega \subset \mathbb{R}^n$  for  $n \geq 1$ , with a  $C^2$ -boundary  $\partial\Omega$ , supplemented with the Neumann boundary condition

$$\partial_\nu v = 0 \quad \text{for } x \in \partial\Omega, \quad t > 0, \quad (2.3)$$

where  $\partial_\nu = \frac{\partial}{\partial \nu}$  and  $\nu$  denotes the unit outer normal vector to  $\partial\Omega$ , and with initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \quad (2.4)$$

The nonlinearities  $f = f(u, v)$  and  $g = g(u, v)$  are arbitrary  $C^3$ -functions. Notice that equation (2.2) may contain an arbitrary diffusion coefficient which, however, can be rescaled and assumed to be equal to one.

**Theorem 2.1** (Local-in-time solution). *Assume that  $u_0, v_0 \in L^\infty(\Omega)$ . Then, there exists  $T = T(\|u_0\|_\infty, \|v_0\|_\infty) > 0$  such that the initial-boundary value problem (2.1)–(2.4) has a unique local-in-time mild solution  $u, v \in L^\infty([0, T], L^\infty(\Omega))$ .*

We recall that a mild solution of problem (2.1)–(2.4) is a pair of measurable functions  $u, v : [0, T] \times \overline{\Omega} \mapsto \mathbb{R}$  satisfying the following system of integral equations

$$u(x, t) = u_0(x) + \int_0^t f(u(x, s), v(x, s)) ds, \quad (2.5)$$

$$v(x, t) = e^{t\Delta} v_0(x) + \int_0^t e^{(t-s)\Delta} g(u(x, s), v(x, s)) ds, \quad (2.6)$$

where  $e^{t\Delta}$  is the semigroup of linear operators generated by Laplacian with the Neumann boundary condition. Since our nonlinearities  $f = f(u, v)$  and  $g = g(u, v)$  are locally Lipschitz continuous, to construct a local-in-time unique solution of system (2.5)–(2.6), it suffices to apply the Banach fixed point theorem.

If  $u_0$  and  $v_0$  are more regular, *i.e.* if for some  $\alpha \in (0, 1)$  we have  $u_0 \in C^\alpha(\overline{\Omega})$ ,  $v_0 \in C^{2+\alpha}(\overline{\Omega})$  and  $\partial_\nu v_0 = 0$  on  $\partial\Omega$ , then the mild solution of problem (2.1)–(2.4) is smooth and satisfies  $u \in C^{1,\alpha}([0, T] \times \overline{\Omega})$  and  $v \in C^{1+\alpha/2, 2+\alpha}([0, T] \times \overline{\Omega})$ .

### 3 Blowup solutions

Throughout this section, we let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a sufficiently regular boundary  $\partial\Omega$ . The unit outer normal vector to  $\partial\Omega$  is denoted by  $\nu$ , and let  $\partial_\nu = \frac{\partial}{\partial \nu}$ .

#### 3.1 Resource-consumer type reaction

We consider the following system of equations

$$u_t = -au + u^p f(v), \quad \text{for } x \in \overline{\Omega}, \quad t > 0, \quad (3.1)$$

$$v_t = D\Delta v - bv - u^p f(v) + \kappa \quad \text{for } x \in \Omega, \quad t > 0, \quad (3.2)$$

where  $D > 0$ ,  $p > 1$ ,  $a, b \in (0, \infty)$  and  $\kappa \in [0, \infty)$ . In equations (3.1)–(3.2), an arbitrary function  $f = f(v)$  satisfies

$$f \in C^1([0, \infty)), \quad f(v) > 0 \quad \text{for } v > 0, \quad \text{and } f(0) = 0. \quad (3.3)$$

We supplement system (3.1)–(3.2) with the homogeneous Neumann boundary condition for  $v$ :

$$\partial_\nu v = 0 \quad \text{for } x \in \partial\Omega, \quad t > 0, \quad (3.4)$$

and with bounded, nonnegative, and continuous initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{for } x \in \Omega. \quad (3.5)$$

When  $u$  has a diffusion term on the right-hand side of (3.1), the model (3.1)–(3.5) can be found in literature in context of several applications. Let us mention a few of them. For  $p = 2$ ,  $f(v) = v$ , and suitably chosen coefficients, we obtain either the, so-called, *Brussellator* appearing in the modeling of chemical morphogenetic processes, the *Gray-Scott model* (also known as a *model of glycolysis*, or the *Schnackenberg model*).

Nonnegative solutions to the following initial value problem for the system of ordinary differential equations:

$$\frac{d}{dt}\bar{u} = -a\bar{u} + \bar{u}^p f(\bar{v}), \quad \frac{d}{dt}\bar{v} = -b\bar{v} - \bar{u}^p f(\bar{v}) + \kappa, \quad (3.6)$$

$$\bar{u}(0) = \bar{u}_0 \geq 0, \quad \bar{v}(0) = \bar{v}_0 \geq 0. \quad (3.7)$$

are global-in-time and bounded on  $[0, \infty)$ .

A behavior of solutions the system of ODEs from (3.6) depends essentially on its parameters. Let  $p = 2$  and  $f(v) = v$ . For  $a > 0$  and  $b > 0$ , this particular system has the trivial stationary nonnegative solution  $(\bar{u}, \bar{v}) = (0, \kappa/b)$  which is an asymptotically stable solution. If, moreover,  $\kappa^2 > 4a^2b$ , we have two other nontrivial nonnegative stationary solutions which satisfy the following system of equations

$$\bar{u} = \frac{a}{\bar{v}} \quad \text{and} \quad -b\bar{v} - \frac{a^2}{\bar{v}} + \kappa = 0.$$

Every such a constant nontrivial and *stable* solution of ODEs is an *unstable* solution of the reaction-diffusion-ODE problem (3.1)-(3.5), which means that it has the diffusion-driven instability due to the autocatalysis  $f_u(\bar{u}, \bar{v}) = -a + 2\bar{u}\bar{v} = a > 0$ .

We show that there are non-constant initial conditions such that the corresponding solution to the reaction-diffusion-ODE problem (3.1)-(3.5) blows up at one point and in a finite time.

Here, without loss of generality, we assume that  $0 \in \Omega$ , where  $\Omega \subset \mathbb{R}^n$  is an arbitrary bounded domain with a smooth boundary, and we rescale system (3.1)-(3.2) in such a way that the diffusion coefficient in equation (3.2) is equal to one.

In the following theorem, we prove that if  $u_0$  is concentrated around an arbitrary point  $x_0 \in \Omega$  (we choose  $x_0 = 0$ , for simplicity) and if  $v_0(x) = \bar{v}_0$  is a constant function, then the corresponding solution to problem (3.1)-(3.5) blows up in a finite time.

**Theorem 3.1.** *Assume that  $f \in C^1([0, \infty))$  satisfies  $\inf_{v \geq R} f(v) > 0$  for each  $R > 0$ . Let  $p > 1$  and  $a, b, \kappa \in (0, \infty)$  be arbitrary. There exist numbers  $\alpha \in (0, 1)$ ,  $\varepsilon > 0$ , and  $R_0 > 0$  (depending on parameters of problem (3.1)-(3.5) and determined in the proof) such that if initial conditions  $u_0, v_0 \in C(\bar{\Omega})$  satisfy*

$$0 < u_0(x) < \left( u_0(0)^{1-p} + 2\varepsilon^{-(p-1)}|x|^\alpha \right)^{-\frac{1}{p-1}} \quad \text{for all } x \in \Omega \quad (3.8)$$

$$u_0(0) \geq \left( \frac{a}{(1 - e^{(1-p)a})F_0} \right)^{\frac{1}{p-1}}, \quad \text{where } F_0 = \inf_{v \geq R_0} f(v), \quad (3.9)$$

$$v_0(x) \equiv \bar{v}_0 > R_0 > 0 \quad \text{for all } x \in \Omega, \quad (3.10)$$

then the corresponding solution to problem (3.1)-(3.5) blows up at certain time  $T_{max} \leq 1$ . Moreover, the following uniform estimates are valid

$$0 < u(x, t) < \varepsilon|x|^{-\frac{\alpha}{p-1}} \quad \text{and} \quad v(x, t) \geq R_0 \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.11)$$

Total mass  $\int_{\Omega} (u(x, t) + v(x, t)) dx$  of each nonnegative solution to the reaction-diffusion problem (3.1)-(3.5) with  $D \geq 0$  does not blow up, and stays uniformly

bounded in  $t > 0$ . Indeed, we obtain that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (u(x, t) + v(x, t)) \, dx &= - \int_{\Omega} (au(x, t) + bv(x, t)) \, dx + \int_{\Omega} \kappa \, dx \\ &\leq - \min\{a, b\} \int_{\Omega} (u(x, t) + v(x, t)) \, dx + \kappa|\Omega|. \end{aligned}$$

Theorem 3.1 shows that this *a priori* estimate is not sufficient to prevent the blow-up of solutions in a finite time.

### 3.1.1 Idea for proof of Theorem 3.1

We would like to give a sketch of the proof of Theorem 3.1. There are more details in [7].

It is an important key to solve the equation (3.1) with respect to  $u(x, t)$ , which leads to the following formula for all  $(x, t) \in \Omega \times [0, T_{max})$ :

$$u(x, t) = \frac{e^{-at}}{\left( \frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t f(v(x, s)) e^{(1-p)as} \, ds \right)^{\frac{1}{p-1}}}. \quad (3.12)$$

Thus, it is clear that if we have an uniform lower bound for  $v(x, t)$  and the initial condition satisfies (3.9), then the equation (3.12) leads to the following lower bound

$$u(x, t) \geq \frac{e^{-at}}{\left( \frac{1}{u_0(x)^{p-1}} - (1 - e^{(1-p)at}) a^{-1} F_0 \right)^{\frac{1}{p-1}}}. \quad (3.13)$$

This implies that  $u(0, t)$  blows up in finite time because the right-hand side of inequality (3.13) for  $x = 0$  blows up at some  $t \leq 1$  under the assumption (3.9).

Therefore, it is sufficient to show the existence of a lower bound for  $v$  for all  $(x, t) \in \Omega \times [0, T_{max})$  in order to finish the proof of Theorem 3.1. We have the following lemma.

**Lemma 3.2.** *Assume that  $v(x, t)$  is a solution of the reaction-diffusion equation (3.2) with an arbitrary function  $u(x, t)$  and with a constant initial condition satisfying  $v_0(x) \equiv \bar{v}_0 > 0$ . Suppose that there are numbers  $\varepsilon > 0$  and*

$$\alpha \in \left( 0, \frac{2(p-1)}{p} \right) \quad \text{if } n \geq 2 \quad \text{and} \quad \alpha \in \left( 0, \frac{p-1}{p} \right) \quad \text{if } n = 1 \quad (3.14)$$

such that

$$0 < u(x, t) < \varepsilon |x|^{-\frac{\alpha}{p-1}} \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.15)$$

Then, there is an explicit number  $C_0 > 0$  independent of  $\varepsilon$  such that for all  $\varepsilon > 0$  we have

$$v(x, t) \geq \min \left\{ \bar{v}_0, \frac{\kappa}{b} \right\} - \varepsilon^p C_0 \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.16)$$

*Proof of Lemma 3.2.* Let  $z(t)$  be a solution of the problem

$$z_t = \Delta z - bz + \kappa, \quad z(x, 0) = \bar{v}_0 \quad (3.17)$$

with the homogeneous Neumann boundary conditions. Then, we can rewrite equation (3.2) in the integral form

$$v(t) = z(t) - \int_0^t e^{(t-s)(\Delta-bI)} (u^p f(v))(s) ds. \quad (3.18)$$

Here, we have a lower bound for  $z(t)$ :

$$z(t) = e^{-bt}\bar{v}_0 + \frac{\kappa}{b} (1 - e^{-bt}) \geq \min \left\{ \bar{v}_0, \frac{\kappa}{b} \right\} \quad \text{for all } t \in [0, T_{max}]. \quad (3.19)$$

Moreover, it is easy to see that there exists an upper bound for  $v(x, t)$ :

$$0 \leq v(x, t) \leq \max \left\{ \|v_0\|_\infty, \frac{\kappa}{b} \right\} \equiv R_1 \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}]. \quad (3.20)$$

Therefore, we compute the  $L^\infty$ -norm of equation (3.18) using (3.19) and (3.20), as well as the *a priori* assumption on  $u$  in (3.15) to obtain that

$$\begin{aligned} v(x, t) &\geq z(t) - \int_0^t \|e^{(t-s)(\Delta-b)} (u^p f(v))(s)\|_\infty ds \\ &\geq \min \left\{ \bar{v}_0, \frac{\kappa}{b} \right\} - \varepsilon^p C_q \left( \sup_{0 \leq v \leq R_1} f(v) \right) \int_0^t \left( 1 + (t-s)^{-\frac{n}{2q}} \right) \| |x|^{-\frac{\alpha p}{p-1}} \|_q ds. \end{aligned} \quad (3.21)$$

Here, we have used the following well-known estimate

$$\|e^{t(\Delta-bI)} w_0\|_\infty \leq C_q \left( 1 + t^{-\frac{n}{2q}} \right) \|w_0\|_q \quad \text{for all } t > 0, \quad (3.22)$$

which is satisfied for each  $w_0 \in L^q(\Omega)$ , each  $q \in [1, \infty]$ , and with a constant  $C_q = C(q, n, \Omega)$  independent of  $w_0$  and of  $t$ .

Hence, choosing  $n/2 < q < n(p-1)/(\alpha p)$  to have  $n/(2q) < 1$  and  $|x|^{-\frac{\alpha p}{p-1}} \in L^q(\Omega)$ , we finish the proof of this lemma.  $\square$

By Lemma 3.2, the proof of Theorem 3.1 can be finished by showing an estimate (3.15). To do so, we need the following result on the Hölder continuity of  $v(x, t)$ .

**Lemma 3.3.** *Let  $v(x, t)$  be a nonnegative solution of the problem*

$$v_t = \Delta v - bv - u^p f(v) + \kappa \quad \text{for } x \in \Omega, \quad t \in [0, T_{max}] \quad (3.23)$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times [0, T_{max}], \quad (3.24)$$

$$v(x, 0) = \bar{v}_0 \quad \text{for } x \in \Omega, \quad t \in [0, T_{max}], \quad (3.25)$$

where  $\bar{v}_0$  is a positive constant and  $u(x, t)$  is a nonnegative function. There exists a constant  $\alpha \in (0, 1)$  satisfying also (3.14), such that if the a priori estimate (3.15) for  $u(x, t)$  holds true with some  $\varepsilon > 0$ , then

$$|v(x, t) - v(y, t)| \leq \varepsilon^p C |x - y|^\alpha \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}),$$

where the constant  $C > 0$  is independent of  $\varepsilon$ .

This lemma follows from a classical result on the Hölder continuity of solutions to the inhomogeneous heat equation.

*Proof of Theorem 3.1.* By assumption (3.8), we have  $0 < u_0(x) < \varepsilon |x|^{-\frac{\alpha}{p-1}}$  for all  $x \in \Omega$ . Suppose that there exists  $T_1 \in (0, 1)$  such that the solution of problem (3.1)-(3.5) exists on the interval  $[0, T_1]$  and satisfies

$$\sup_{x \in \Omega} |x|^{\frac{\alpha}{p-1}} u(x, t) < \varepsilon \quad \text{for all } t < T_1, \quad \sup_{x \in \Omega} |x|^{\frac{\alpha}{p-1}} u(x, T_1) = \varepsilon. \quad (3.26)$$

First, we estimate the denominator of the fraction in (3.12) using assumption (3.8) as follows

$$\begin{aligned} & \frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t f(v(x, s)) e^{(1-p)as} ds \\ & \geq 2\varepsilon^{1-p} |x|^\alpha + \frac{1}{u_0(0)^{p-1}} - (p-1) \int_0^t f(v(0, s)) e^{(1-p)as} ds \\ & \quad + (p-1) \int_0^t (f(v(0, s)) - f(v(x, s))) e^{(1-p)as} ds. \end{aligned} \quad (3.27)$$

By the definition of  $T_{max}$  and formula (3.12), we immediately obtain

$$\frac{1}{u_0(0)^{p-1}} - (p-1) \int_0^t f(v(0, s)) e^{(1-p)as} ds > 0 \quad \text{for all } t \in [0, T_{max}). \quad (3.28)$$

Next, we use our hypothesis (3.26) together with the Hölder continuity of  $v(x, t)$  established in Lemma 3.3 to find constants  $C > 0$  and  $\alpha \in (0, 1)$ , the both independent of  $\varepsilon \geq 0$ , such that

$$(p-1) \int_0^t |f(v(0, s)) - f(v(x, s))| e^{(1-p)as} ds \leq \varepsilon^p C a^{-1} |x|^\alpha \quad (3.29)$$

for all  $(x, t) \in \Omega \times [0, T_1]$ . Consequently, we obtain the lower bound for the denominator in (3.12)

$$\frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t f(v(x, s)) e^{(1-p)as} ds \geq (2\varepsilon^{-(p-1)} - \varepsilon^p C a^{-1}) |x|^\alpha \quad (3.30)$$

for all  $(x, t) \in \Omega \times [0, T_1]$ . Finally, we choose  $\varepsilon > 0$  so small that  $2\varepsilon^{-(p-1)} - \varepsilon^p C a^{-1} > \varepsilon^{-(p-1)}$  and we substitute estimate (3.30) in equation (3.12) to obtain

$$0 < u(x, t) \leq \frac{e^{-at}}{\left( (2\varepsilon^{-(p-1)} - \varepsilon^p C a^{-1}) |x|^\alpha \right)^{\frac{1}{p-1}}} < \frac{\varepsilon}{|x|^{\frac{\alpha}{p-1}}} \quad \text{for all } (x, t) \in \Omega \times [0, T_1].$$

This inequality for  $t = T_1$  contradicts our hypothesis (3.26).  $\square$

### 3.2 Activator-inhibitor type reaction

We consider the following initial-boundary value problem for a reaction-diffusion-ODE system:

$$u_t = -au + \frac{u^p}{v^q}, \quad \text{for } x \in \bar{\Omega}, \quad t > 0, \quad (3.31)$$

$$v_t = D\Delta v - bv + \gamma \frac{u^r}{v^s} \quad \text{for } x \in \Omega, \quad t > 0, \quad (3.32)$$

supplemented with the the initial data  $u_0, v_0 \in C(\bar{\Omega})$  such that

$$u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) > 0 \quad \text{for all } x \in \bar{\Omega} \quad (3.33)$$

and with the Neumann boundary conditions for  $v$ ;

$$\partial_\nu v = 0 \quad \text{for } x \in \partial\Omega, \quad t > 0. \quad (3.34)$$

Here,  $D > 0$ ,  $a, b, \gamma$  are nonnegative constants, and the nonlinearity exponents in (3.31)-(3.32) satisfy

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0. \quad (3.35)$$

From the initial conditions, we have  $\inf u_0 \equiv \inf_{x \in \Omega} u_0(x) > 0$  and  $\inf v_0 \equiv \inf_{x \in \Omega} v_0(x) > 0$ .

In the following, for simplicity of notation, we use the quantities

$$f_{0,T} \equiv \inf_{t \in [0,T]} e^{\alpha(1-p+q)t} \quad \text{and} \quad g_{1,T} \equiv \sup_{t \in [0,T]} e^{b(1-r+s)t}. \quad (3.36)$$

For  $p > 1$ , it is easy to see that the reaction-diffusion-ODE system (3.31)-(3.32) has the diffusion-driven instability at a constant steady state. When the right-hand side of the equation (3.31) has a diffusion term and the exponents satisfy

$$0 < \frac{p-1}{r} < \frac{q}{s+1}, \quad (3.37)$$

the system (3.31)-(3.32) is an activator-inhibitor system proposed by Gierer and Meinhardt. It has been widely used to model various biological pattern formations.

First, we consider the the *kinetic system* of ordinary differential equations associated with (3.31)–(3.32):

$$\frac{d}{dt}\bar{u} = -a\bar{u} + \frac{\bar{u}^p}{\bar{v}^q}, \quad \frac{d}{dt}\bar{v} = -b\bar{v} + \gamma\frac{\bar{u}^r}{\bar{v}^s}. \quad (3.38)$$

When  $a = b = \gamma = 1$ , it turns out that this dynamics already exhibits various kinds of interesting behaviors including the convergence to the equilibria  $(0, 0)$  and  $(1, 1)$ , periodic solutions, unbounded oscillating global solutions, and a blowup of solutions in finite time [11]. In particular, if inequalities (3.37) and  $p - 1 \leq r$  are satisfied, then solutions of (3.38) are global-in-time, while there are solutions blowing up in finite time under the conditions (3.37) and  $p - 1 > r$ . Thus, our Theorem 3.4 shows that the diffusion of the inhibitor described by  $v(x, t)$  induces a *blowup* of the space-inhomogeneous and non-diffusing activator  $u(x, t)$  – also in the case when space-homogeneous solutions are global-in-time.

In the following, without loss of generality, we assume that  $0 \in \Omega$ . Moreover, system (3.31)–(3.32) is rescaled in such a way so that the diffusion coefficient in equation (3.32) is equal to one.

We prove that if  $u_0$  is sufficiently well concentrated around an arbitrary point  $x_0 \in \Omega$  (here, for simplicity of notation, we choose  $x_0 = 0$ ), if  $v_0$  is a constant function, and if  $\gamma > 0$  is sufficiently small then the corresponding solution to problem (3.31)–(3.34) blows up in a finite time  $T_{max} > 0$ , *without additional restrictions* on the exponents in nonlinearities.

**Theorem 3.4.** *Assume the nonlinearity exponents satisfy (3.35) and let  $T > 0$  be arbitrary. Suppose that  $0 \in \Omega$  and*

- *there exists a number*

$$\alpha \in \left(0, \frac{2(p-1)}{r}\right) \quad \text{if } n \geq 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{r}\right) \quad \text{if } n = 1$$

*such that  $u_0 \in C(\bar{\Omega})$  satisfies*

$$0 < u_0(x) \leq \frac{1}{(u_0(0)^{1-p} + 2|x|^\alpha)^{\frac{1}{p-1}}} \quad \text{for all } x \in \Omega, \quad (3.39)$$

- *$v(x, 0) = \bar{v}_0$  is a constant such that*

$$0 < \bar{v}_0 < R_0 \equiv \left(T(p-1)f_{0,T}(\inf_{x \in \Omega} u_0(x))^{p-1}\right)^{\frac{1}{q}} \quad \text{for all } x \in \Omega, \quad (3.40)$$

- *$\gamma \in [0, \gamma_0)$ , where  $\gamma_0 = \gamma_0(u_0, \bar{v}_0, T, p, q, r, s, n)$  is a certain number determined in the proof.*

*Then the corresponding solution to problem (3.31)–(3.34) blows up at some  $T_{max} \leq T$ . Moreover, the following uniform estimates are valid*

$$0 < u(x, t) < |x|^{-\frac{\alpha}{p-1}} \quad \text{and} \quad 0 < v(x, t) < R_0 \quad (3.41)$$

*for all  $(x, t) \in \Omega \times [0, T_{max})$ .*

### 3.2.1 Idea for proof of Theorem 3.4

To show that some solutions to problems (3.31)-(3.34) blow up in a finite time, we first notice that if  $(u(x, t), v(x, t))$  is their solution, then the functions  $u(x, t)e^{at}$  and  $v(x, t)e^{bt}$  satisfy the following boundary-value problem

$$u_t = \frac{u^p}{v^q} f(t) \quad \text{for } x \in \bar{\Omega}, \quad t > 0, \quad (3.42)$$

$$v_t = D\Delta v + \gamma \frac{u^r}{v^s} g(t) \quad \text{for } x \in \Omega, \quad t > 0, \quad (3.43)$$

$$\partial_\nu v = 0 \quad \text{for } x \in \partial\Omega, \quad t > 0, \quad (3.44)$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (3.45)$$

where

$$f(t) = e^{a(1-p+q)t} \quad \text{and} \quad g(t) = e^{b(1-r+s)t}. \quad (3.46)$$

Obviously, it suffices to prove a blowup of solutions to the new problem (3.42)-(3.45).

In the following, we would like to give a sketch of the proof of Theorem 3.4. There are more details in [2].

First, we note that, for every nonnegative  $u_0, v_0 \in C(\bar{\Omega})$ ,  $u(x, t)$  and  $v(x, t)$  satisfy

$$u(x, t) \geq \inf u_0 \quad \text{and} \quad v(x, t) \geq \inf v_0 \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.47)$$

Now, for a given function  $v(x, t)$ , we solve equation (3.31) with respect to  $u(x, t)$  to obtain the explicit formula for  $u(x, t)$ :

$$u(x, t) = \frac{1}{\left( \frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(x, \tau)^q} d\tau \right)^{\frac{1}{p-1}}}. \quad (3.48)$$

From the assumption (3.39) for  $u_0(x)$ , we can obtain the blowup of  $u(x, t)$  at  $x = 0$  in finite time if  $v(x, t)$  is uniformly bounded from above. Next lemma shows that an upper bound for  $v(x, t)$  leads to the blowup of  $u(x, t)$  in a finite time indeed.

**Lemma 3.5.** *Let  $(u(x, t), v(x, t))$  be a nonnegative solution to (3.42)-(3.45) with  $\varepsilon \geq 0$  and  $D > 0$ . Suppose that for some constant  $T > 0$  we have*

$$0 < v(x, t) < R_0 = \left( T(p-1)f_{0,T}(\inf u_0)^{p-1} \right)^{\frac{1}{q}} \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.49)$$

*Then  $u(x, t)$  blows up at certain  $T_{max} \leq T$ .*

*Proof of Lemma 3.5.* Applying the comparison principle to equation (3.42), we obtain the estimate

$$u(x, t) \geq \bar{u}_1(t) \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}), \quad (3.50)$$

where  $\bar{u}_1 = \bar{u}_1(t)$  is the solution of the Cauchy problem

$$\frac{d}{dt} \bar{u}_1 = \frac{\bar{u}_1^p}{R_0^q} f_{0,T}, \quad \bar{u}_1(0) = \inf u_0. \quad (3.51)$$

The function  $\bar{u}_1$  may be computed explicitly:

$$\bar{u}_1(t) = \frac{1}{((\inf u_0)^{1-p} - t(p-1)R_0^{-q} f_{0,T})^{\frac{1}{p-1}}}. \quad (3.52)$$

Recalling the definition of the number  $R_0$  in (3.49), we obtain that  $\bar{u}_1(t)$  blows up at  $t = T$ , which due to inequality (3.50) implies that  $T_{max} \leq T$ .  $\square$

From Lemma 3.5, it is sufficient to obtain an upper bound for  $v(x, t)$  to finish the proof of Theorem 3.4. The following lemma shows that a priori estimate for  $u(x, t)$  similar to (3.15) is important to lead to the upper bound for  $v(x, t)$ .

**Lemma 3.6.** *Let  $u(x, t)$  and  $v(x, t)$  be a solution to problem (3.31)-(3.33). Suppose that there is a number*

$$\alpha \in \left(0, \frac{2(p-1)}{r}\right) \quad \text{if } n \geq 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{r}\right) \quad \text{if } n = 1 \quad (3.53)$$

*such that, a priori, the following inequality holds true*

$$0 < u(x, t) < |x|^{-\frac{\alpha}{p-1}} \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.54)$$

*Then, there is an explicit number  $C_0 > 0$  such that for all  $\gamma \geq 0$  we have*

$$\|v(t)\|_\infty \leq \|v_0\|_\infty + \gamma C_0 \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}). \quad (3.55)$$

*Proof of Lemma 3.6.* We use the following integral formulation of equation (3.32)

$$v(t) = e^{t\Delta} v_0 + \gamma \int_0^t e^{(t-\tau)\Delta} \left( \frac{u^r(\tau)}{v^s(\tau)} g(\tau) \right) d\tau. \quad (3.56)$$

Here, we recall the following well-known estimates for the heat semigroup which are valid for all  $t > 0$ ,  $D > 0$ , and all  $w_0 \in L^\infty(\Omega)$ :

$$\|e^{tD\Delta} w_0\|_\infty \leq \|w_0\|_\infty \quad \text{and} \quad \|e^{tD\Delta} w_0\|_\infty \leq C_\ell (1 + t^{-\frac{n}{2\ell}}) \|w_0\|_\ell \quad (3.57)$$

for each  $\ell \in [1, \infty]$ , with a constant  $C_\ell = C(\ell, n, D, \Omega)$  independent of  $w_0$  and of  $t$ .

Now, we compute the  $L^\infty$ -norm of equation (3.56). Using inequalities (3.57), the lower bound of  $v$  in (3.47) as well as the *a priori* assumption on  $u$  in (3.54) we obtain the estimate

$$\begin{aligned} \|v(t)\|_\infty &\leq \|v_0\|_\infty + \gamma \int_0^t \left\| e^{(t-\tau)\Delta} \left( \frac{u^r(\tau)}{v^s(\tau)} g(\tau) \right) \right\|_\infty d\tau \\ &\leq \|v_0\|_\infty + \gamma C_\ell (\inf v_0)^{-s} g_{1,T} \int_0^t (1 + (t-\tau)^{-\frac{n}{2\ell}}) \left\| |x|^{-\frac{\alpha r}{p-1}} \right\|_\ell d\tau, \end{aligned} \quad (3.58)$$

where the constant  $g_{1,T}$  is defined in (3.36). Here, we choose  $n/2 < \ell < n(p-1)/(\alpha r)$  to have  $n/(2\ell) < 1$  and  $|x|^{-\frac{\alpha r}{p-1}} \in L^\ell(\Omega)$  to finish the proof of lemma.  $\square$

We can show the Hölder continuity of  $v$ , which is similar to Lemma 3.3. Indeed, there exists a constant  $\alpha \in (0, 1)$  satisfying also (3.53) and a number  $C > 0$ , the both independent of  $\gamma > 0$ , such that

$$|v(x, t) - v(y, t)| \leq \gamma C |x - y|^\alpha \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}].$$

We are ready to prove a result on the one-point blowup of solutions to the reaction-diffusion-ODE problem (3.31)-(3.34).

*Proof of Theorem 3.4.* Let  $(u(x, t), v(x, t))$  be a solution to the problem (3.42)-(3.45). By Lemmas 3.5 and 3.6, it suffices to show the following estimate

$$0 < u(x, t) < |x|^{-\frac{\alpha}{p-1}} \quad \text{for all } (x, t) \in \Omega \times [0, T_{max}], \quad (3.59)$$

under the assumption that  $\gamma > 0$  is sufficiently small. Let  $T > 0$  be a number such that inequality (3.40) holds true.

By assumption (3.39), we have  $0 < u_0(x) < |x|^{-\frac{\alpha}{p-1}}$  for all  $x \in \Omega$ , hence, by a continuity argument, inequality (3.59) is satisfied on a certain initial time interval. Suppose that there exists  $T_1 \in (0, \min\{T_{max}, T\})$  such that the solution of problem (3.42)-(3.45) exists on the interval  $[0, T_1]$  and satisfies

$$\sup_{x \in \Omega} |x|^{-\frac{\alpha}{p-1}} u(x, t) < 1 \quad \text{for all } t < T_1, \quad \sup_{x \in \Omega} |x|^{-\frac{\alpha}{p-1}} u(x, T_1) = 1. \quad (3.60)$$

We are going to use the explicit formula (3.48) for  $u(x, t)$  and the Hölder continuity of  $v(x, t)$  to obtain a contradiction with equality (3.60).

First, notice that assumption (3.39) can be written as  $u_0(x)^{1-p} \geq 2|x|^\alpha + u_0(0)^{1-p}$ . Thus, we may estimate the denominator of the fraction in (3.48) using this assumption as follows

$$\begin{aligned} &\frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(x, \tau)^q} d\tau \\ &\geq 2|x|^\alpha + \frac{1}{u_0(0)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(0, \tau)^q} d\tau \\ &\quad + (p-1) \int_0^t \left( \frac{1}{v(0, \tau)^q} - \frac{1}{v(x, \tau)^q} \right) f(\tau) d\tau. \end{aligned} \quad (3.61)$$

By the definition of  $T_{max}$  and due to formula (3.48), we immediately obtain

$$\frac{1}{u_0(0)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(0, \tau)^q} d\tau > 0 \quad \text{for all } t \in [0, T_{max}]. \quad (3.62)$$

Next, we use our hypotheses (3.60) implying estimate (3.55) and the Hölder continuity of  $v(x, t)$  as well as the lower bound of  $v(x, t)$  in (3.47), to find constants  $C > 0$  and  $\alpha \in (0, 1)$ , satisfying also (3.53), such that the following inequality is satisfied:

$$(p-1) \int_0^t \left| \frac{1}{v(0, \tau)^q} - \frac{1}{v(x, \tau)^q} \right| f(\tau) d\tau \leq \gamma C(T) |x|^\alpha \quad (3.63)$$

for all  $(x, t) \in \Omega \times [0, T_1]$ . Consequently, applying inequalities (3.62) and (3.63) in (3.61) we obtain the following lower bound for the denominator in (3.48)

$$\frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(x, \tau)^q} d\tau \geq (2 - \gamma C) |x|^\alpha \quad (3.64)$$

for all  $(x, t) \in \Omega \times [0, T_1]$ . Finally, we choose  $\gamma > 0$  so small that  $\gamma C < 1$  (hence  $2 - \gamma C > 1$ ) and we substitute estimate (3.64) into equation (3.52) to obtain

$$0 < u(x, t) \leq \frac{1}{\left( (2 - \gamma C) |x|^\alpha \right)^{\frac{1}{p-1}}} < \frac{1}{|x|^{\frac{\alpha}{p-1}}} \quad \text{for all } (x, t) \in \Omega \times [0, T_1].$$

This inequality for  $t = T_1$  contradicts our hypothesis (3.60). □

## Acknowledgments

The author acknowledges JSPS the Grant-in-Aid for Scientific Research (C) 26400156.

## References

- [1] S. Härtling and A. Marciniak-Czochra, *Spike patterns in a reaction-diffusion-ode model with Turing instability*, Math. Methods in the Applied Sciences (2013), to appear. preprint, arXiv:1303.5362 [math.AP]
- [2] G. Karch, K. Suzuki, J. Zienkiewicz, *Finite-time blowup of solutions to some activator-inhibitor systems*, to appear in DCDS-A.
- [3] M. D. Li, S. H. Chen, and Y. C. Qin, Boundedness and blow up for the general activator-inhibitor model, *Acta Math. Appl. Sinica (English Ser.)*, **11** (1995), 59–68.

- [4] A. Marciniak-Czochra, Receptor-based models with diffusion-driven instability for pattern formation in *Hydra*. *J. Biol. Sys.* **11** (2003) 293–324 .
- [5] A. Marciniak-Czochra, G. Karch, & K. Suzuki, *Unstable patterns in reaction-diffusion model of early carcinogenesis*, *J. Math. Pures Appl.* (9) **99** (2013), 509–543
- [6] A. Marciniak-Czochra, G. Karch, and K. Suzuki *Unstable patterns in autocatalytic reaction-diffusion-ODE systems*, (2013), arXiv:1301.2002 [math.AP].
- [7] A. Marciniak-Czochra, G. Karch, K. Suzuki, J. Zienkiewicz, *Diffusion-driven blowup of nonnegative solutions to reaction-diffusion-ODE systems*, to appear in *Differential and Integral Equations*.
- [8] A. Marciniak-Czochra, M. Kimmel, *Dynamics of growth and signaling along linear and surface structures in very early tumors*, *Comput. Math. Methods Med.* **7** (2006), 189–213.
- [9] A. Marciniak-Czochra, M. Kimmel, *Modelling of early lung cancer progression: influence of growth factor production and cooperation between partially transformed cells*, *Math. Models Methods Appl. Sci.* **17** (2007), suppl., 1693–1719.
- [10] A. Marciniak-Czochra, M. Kimmel, *Reaction-diffusion model of early carcinogenesis: the effects of influx of mutated cells*, *Math. Model. Nat. Phenom.* **3** (2008), 90–114.
- [11] W.-M. Ni, K. Suzuki, and I. Takagi, The dynamics of a kinetic activator-inhibitor system, *J. Differential Equations*, **229** (2006), 426–465.
- [12] K. Pham, A. Chauviere, H. Hatzikirou, X. Li, H.M.. Byrne, V. Cristini, J. Lowengrub, *Density-dependent quiescence in glioma invasion: instability in a simple reaction-diffusion model for the migration/proliferation dichotomy*, *J. Biol. Dyn.* **6** (2011) 54–71.
- [13] A. M. Turing, *The chemical basis of morphogenesis*, *Phil. Trans. Roy. Soc. B* **237** (1952), 37–72.