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Some notes on parametric multilevel $q-$Gevrey asymptotics for some linear q-difference-differential equations

By

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Abstract

This manuscript pretends to provide a survey of the work [8], which has been presented in RIMS Symposium Algebraic Analytic Methods in Complex Partial Differential Equations. The concise scheme in these notes aims to give a clear idea on the procedure followed in that work, as well as to clarify the steps underlying in the results in [8].

In the work [8], we study a family of linear $q-$difference-differential equations, under the action of a perturbation parameter ϵ. The procedure leans on a q-analog of an acceleration procedure and a $q-$analog of Ramis-Sibuya theorem in two levels, based on the ideas of the one-level result in [2].

§1. Introduction

This manuscript pretends to provide an abridged slightly modified version of the work [8], which has been presented in RIMS Symposium Algebraic analytic methods in complex partial differential equations. In that work, the problem under study is the family of equations of $q-$difference-differential nature of the shape

$$
\begin{aligned}
Q(\partial_z)\sigma_q u(t,z,\epsilon) &= (\epsilon t)^{d_D}\sigma_q^{\frac{d_D}{k_2}+1}R_D(\partial_z)u(t,z,\epsilon) \\
&\quad + \sum_{\ell=1}^{D-1}\left(\sum_{\lambda\in I_\ell} t^{d_{\lambda,\ell}}\epsilon^{\Delta_{\lambda,\ell}}\sigma_q^{\delta_\ell}c_{\lambda,\ell}(z,\epsilon)R_\ell(\partial_z)u(t,z,\epsilon)\right)+\sigma_q f(t,z,\epsilon).
\end{aligned} \tag{1.1}
$$

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The approach followed lies on the appearance of two different $q-$Gevrey asymptotic behavior of the formal solution linked to two independent respects. On the one hand, the $q-$Gevrey penomenon coming from the structure of the equation itself, but also another $q-$Gevrey order related with the coefficients involved. As a first approach, one is tempted to study an auxiliary problem in the Borel plane directly, following the classical Borel-Laplace method of summability of formal solutions. However, this approach is doomed to failure no matter the Gevrey order we set in the study: the Borel transform of the formal solution might not admit a slow enough $q-$exponential growth along any direction or it might remain as a formal power series of null radius of convergence. The alternative procedure followed is to split the summation procedure in two steps. Firstly, we proceed with a $q-$analog of Borel-Laplace summation method of a lower type, κ and attain the solution by means of an acceleration-like action.

This idea is an adaptation of that in [6], to the $q-$Gevrey case. Also, the idea of concatenating formal and analytic $q-$analogs of Borel and Laplace operators in order to solve $q-$difference equations appears in [1].

The work [8] continues a series of works dedicated to the asymptotic behavior of holomorphic solutions to different kinds of $q-$difference-differential problems involving irregular singularities investigated in [3], [4], [7], [10]. These works can be classified in the branch of studies devoted to study from an analytic point of view of $q-$difference equations and their formal/analytic classification in [18], [11], [12], [13], [14]. It is worth pointing out another approach in the construction of a $q-$analog of summability for formal solutions to inhomogeneous linear $q-$difference-differential equations based on Newton polygon methods, see [16], and also the contribution in the framework of nonlinear $q-$analogs of Briot-Bouquet type partial differential equations, see [19].

§ 2. Description of the problem

We consider the equation (1.1) for its analytic and asymptotic study. In this section, we give a brief description of the elements involved in the equation under study and their precise construction.

Regarding equation (1.1), D, k_1, k_2 are positive integers with $D \geq 3$ and $k_1 < k_2$. We write σ_q^{γ}, for the generalized dilation operator on t variable, $\sigma_q^{\gamma}(f(t)) = f(q^{\gamma}t)$. This definition is assumed to be extended to formal power series. Let κ be given by $1/\kappa = 1/k_1 - 1/k_2$. Assume that I_ℓ is a finite nonempty subset of nonnegative integers whilst δ_ℓ and d_D are positive integers, for every $1 \leq \ell \leq D-1$. We also put $d_{\lambda,\ell} \geq 1$ and $\Delta_{\lambda,\ell} \geq 0$, for every $\lambda \in I_\ell$. We make the assumption that $\delta_1 = 1$ and $\delta_\ell < \delta_{\ell+1}$, for every $1 \leq \ell \leq D-1$. We assume that

$$\Delta_{\lambda,\ell} \geq d_{\lambda,\ell}, \quad \frac{d_{\lambda,\ell}}{k_2} + 1 \geq \delta_\ell, \qquad \frac{d_D - 1}{k_2} + 1 \geq \delta_\ell$$

for every $1 \leq \ell \leq D-1$ and all $\lambda \in I_\ell$. Let $Q, R_\ell \in \mathbb{C}[X]$ with

$$\deg(Q) \geq \deg(R_D) \geq \deg(R_\ell), \quad Q(im) \neq 0, \quad R_D(im) \neq 0,$$

for all $1 \leq \ell \leq D-1$ and $m \in \mathbb{R}$.

We require the existence of an unbounded sector

$$S_{Q,R_D} = \{z \in \mathbb{C} : |z| \geq r_{Q,R_D}, |\arg(z) - d_{Q,R_D}| \leq \eta_{Q,R_D}\},$$

for some $r_{Q,R_D}, \eta_{Q,R_D} > 0$, such that

$$\frac{Q(im)}{R_D(im)} \in S_{Q,R_D}, \quad m \in \mathbb{R}.$$

Let $\varsigma \geq 2$ be an integer. Let $\mathcal{E}_p$ be an open sector with vertex at the origin and radius ϵ_0 for every $0 \leq p \leq \varsigma - 1$ and such that $\mathcal{E}_j \cap \mathcal{E}_k \neq \emptyset$ for every $0 \leq j, k \leq \varsigma - 1$ if and only if $|j-k| \leq 1$ (under the notation $\mathcal{E}_\varsigma := \mathcal{E}_0$) and such that $\cup_{p=0}^{\varsigma-1}\mathcal{E}_p = \mathcal{U}\setminus\{0\}$, for some neighborhood of the origin, $\mathcal{U}$. A family $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ satisfying these properties is known as a good covering in $\mathbb{C}^\star$.

Let $\mathcal{T}$ be an open bounded sector with vertex at 0 and radius $r_\mathcal{T} > 0$. We make the assumption that

(2.1)
$$0 < \epsilon_0, r_\mathcal{T} < 1, \quad \nu + \frac{k_2}{\log(q)}\log(r_\mathcal{T}) < 0, \quad \alpha + \frac{\kappa}{\log(q)}\log(\epsilon_0 r_\mathcal{T}) < 0, \quad \epsilon_0 r_\mathcal{T} \leq q^{(\frac{1}{2}-\nu)/k_2}/2$$

for some $\nu \in \mathbb{R}$.

We consider a family of unbounded sectors $(S_{\mathfrak{d}_p})_{0\leq p\leq\varsigma-1}$ with bisecting direction $\mathfrak{d}_p \in \mathbb{R}$ and a family of open domains $\mathcal{R}^b_{\mathfrak{d}_p} := \mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}} \cap D(0, \epsilon_0 r_\mathcal{T})$, where

$$\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}} = \left\{T \in \mathbb{C}^\star : \left|1 + \frac{e^{i\mathfrak{d}_p}}{T}r\right| > \tilde{\delta}, \quad \text{for every } r \geq 0\right\},$$

for some $\tilde{\delta} > 0$.

We assume $\mathfrak{d}_p$, $0 \leq p \leq \varsigma - 1$, are chosen so that some conditions are satisfied. In order to enumerate them, we denote $q_\ell(m)$ the roots of the polynomial

$$P_m(\tau) = \frac{Q(im)}{(q^{1/k_2})^{k_2(k_2-1)/2}} - \frac{R_D(im)}{(q^{1/k_2})^{\frac{(d_D+k_2)(d_D+k_2-1)}{2}}}\tau^{d_D}.$$

We take an unbounded sector with vertex at 0 and bisecting direction $\mathfrak{d}_p$, $S_{\mathfrak{d}_p}$, $0 \leq p \leq \varsigma - 1$; and we choose $\rho > 0$ such that:

1) There exists $M_1 > 0$ such that $|\tau - q_l(m)| \geq M_1(1 + |\tau|)$ holds for all $m \in \mathbb{R}$, $\tau \in S_{\mathfrak{d}_p} \cup \overline{D}(0, \rho)$, all $0 \leq p \leq \varsigma - 1$ and all $0 \leq l \leq d_D - 1$.

2) There exists $M_2 > 0$ and $l_0 \in \{0,...,d_D-1\}$ such that $|\tau - q_{\ell_0}(m)| \geq M_2|q_{l_0}(m)|$ holds for every $m \in \mathbb{R}$, $\tau \in S_{\mathfrak{d}_p} \cup \overline{D}(0,\rho)$, and all $0 \leq p \leq \varsigma - 1$.
3) For every $0 \leq p \leq \varsigma - 1$ we have $\mathcal{R}^b_{\mathfrak{d}_p} \cap \mathcal{R}^b_{\mathfrak{d}_{p+1}} \neq \emptyset$, and for all $t \in \mathcal{T}$ and $\epsilon \in \mathcal{E}_p$, we have that $\epsilon t \in \mathcal{R}^b_{\mathfrak{d}_p}$. Here we have put $\mathcal{R}^b_{\mathfrak{d}_\varsigma} := \mathcal{R}^b_{\mathfrak{d}_0}$.

The family $\{(\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}})_{0\leq p\leq\varsigma-1}, D(0,\rho), \mathcal{T}\}$ is said to be associated to the good covering $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$. For every $0 \leq p \leq \varsigma - 1$ we study the $q-$difference-differential equation

$$
\begin{aligned}
Q(\partial_z)\sigma_q u^{\mathfrak{d}_p}(t,z,\epsilon) &= (\epsilon t)^{d_D}\sigma_q^{\frac{d_D}{k_2}+1}R_D(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&\quad + \sum_{\ell=1}^{D-1}\left(\sum_{\lambda\in I_\ell} t^{d_{\lambda,\ell}}\epsilon^{\Delta_{\lambda,\ell}}\sigma_q^{\delta_\ell}c_{\lambda,\ell}(z,\epsilon)R_\ell(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\right) + \sigma_q f^{\mathfrak{d}_p}(t,z,\epsilon).
\end{aligned}
\tag{2.2}
$$

We now give some detail on the construction of the elements $c_{\lambda,\ell}$ and $f^{\mathfrak{d}_p}$.

For every $1 \leq \ell \leq D-1$ and $\lambda \in I_\ell$ and every integer $n \geq 0$, we consider functions $m \mapsto C_{\lambda,\ell}(m,\epsilon)$ and $m \mapsto F_n(m,\epsilon)$ belonging to the Banach space $E_{(\beta,\mu)}$, for some $\beta > 0$ and $\mu > \deg(R_D) + 1$. The Banach space $E_{(\beta,\mu)}$ consists of all continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that

$$|h(m)| \leq C_h(1+|m|)^{-\mu}\exp(-\beta|m|), \qquad m \in \mathbb{R},$$

for some $C_h > 0$. The infimum of such $C_h > 0$ defines its norm.

We assume all these functions depend holomorphically on $\epsilon \in D(0,\epsilon_0)$. Moreover, we assume there exist $\tilde{C}_{\lambda,\ell}, C_F > 0$ such that $\|C_{\lambda,\ell}(m,\epsilon)\|_{(\beta,\mu)} \leq \tilde{C}_{\lambda,\ell}$, for every $\epsilon \in D(0,\epsilon_0)$, and also $\|F_n(m,\epsilon)\|_{(\beta,\mu)} \leq C_F\rho^{-n}q^{\frac{n(n-1)}{2k_1}}$, for all $1 \leq \ell \leq D-1$, $\lambda \in I_\ell$, $n \geq 0$ and $\epsilon \in D(0,\epsilon_0)$. Then, we put $c_{\lambda,\ell}(z,\epsilon) = \mathcal{F}^{-1}(m \mapsto C_{\lambda,\ell}(m,\epsilon))(z)$, which, for every $1 \leq \ell \leq D-1$, $\lambda \in I_\ell$, defines a bounded holomorphic function on $H_{\beta'} \times D(0,\epsilon_0)$ for any $0 < \beta' < \beta$. Here, H_β stands for the strip $H_\beta = \{z \in \mathbb{C} : |\Im(z)| < \beta\}$. We assume the formal power series

$$\psi_{k_1}(\tau,m,\epsilon) = \sum_{n\geq 0} F_n(m,\epsilon)\frac{\tau^n}{(q^{1/k_1})^{\frac{n(n-1)}{2}}},$$

which is convergent on the disc $D(0,\rho)$, can be analytically continued with respect to τ as a function $\tau \mapsto \psi^{\mathfrak{d}_p}_{k_1}(\tau,m,\epsilon)$ on an infinite sector $U_{\mathfrak{d}_p}$ of bisecting direction $\mathfrak{d}_p$, and

$$|\psi^{\mathfrak{d}_p}_{k_1}(\tau,m,\epsilon)| \leq \zeta_{\psi_{k_1}}(1+|m|)^{-\mu}e^{-\beta|m|}\exp\left(\frac{k_1\log^2|\tau+\delta|}{2\log(q)} - \alpha\log|\tau+\delta|\right),$$

for all $\tau \in U_{\mathfrak{d}_p} \cup \overline{D}(0,\rho)$ and $m \in \mathbb{R}$, some positive constant $\zeta_{\psi_{k_1}}$ which does not depend on $\epsilon \in D(0,\epsilon_0)$, and some $\delta > 0$.

The $q-$Laplace operator acting on different stages plays two roles in the work: revert Borel operations action from the study of auxiliary problems in the Borel plane, and accelerating the solutions of auxiliary problems. We refer to Definition 3.3 for the definition of $q-$Laplace operator.

One can prove that the function

$$\psi_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon) := \mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_p}(h \mapsto \psi_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon))(\tau) \tag{2.3}$$

is a continuous complex valued function on $(S_{\mathfrak{d}_p} \cup \overline{\mathcal{R}_{\mathfrak{d}_p}^b}) \times \mathbb{R}$, holomorphic with respect to τ on $S_{\mathfrak{d}_p} \cup \mathcal{R}_{\mathfrak{d}_p}^b$ such that $|\psi_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)| \leq \zeta_{\psi_{k_2}}(1+|m|)^{-\mu}e^{-\beta|m|}\exp\left(\frac{k_2\log^2|\tau|}{2\log(q)} - \nu\log|\tau|\right)$, for $\tau \in (S_{\mathfrak{d}_p} \cup \overline{\mathcal{R}_{\mathfrak{d}_p}^b}), m \in \mathbb{R}$, for some $\zeta_{\psi_{k_2}} > 0$, and some $\nu \in \mathbb{R}$. The constant $\zeta_{\psi_{k_2}}$ depends on $\zeta_{\psi_{k_1}}$ so that $\zeta_{\psi_{k_2}}(\zeta_{\psi_{k_1}}) \to 0$ when $\zeta_{\psi_{k_1}}$ tends to 0. One can apply $q-$Laplace transform of order k_2 to the function $\psi_{k_2}^{\mathfrak{d}_p}$ in τ variable and, in direction $\mathfrak{d}_p$, and obtain that the function

$$F^{\mathfrak{d}_p}(T,m,\epsilon) := \mathcal{L}_{q;1/k_2}^{\mathfrak{d}_p}(\tau \mapsto \psi_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon))(T),$$

is a holomorphic function with respect to T variable in the set $\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}} \cap D(0,r_1)$ for any $0 < r_1 \leq q^{\left(\frac{1}{2}-\nu\right)/k_2}/2$.

We define the forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ by

$$f^{\mathfrak{d}_p}(t,z,\epsilon) := \mathcal{F}^{-1}\left(m \mapsto F^{\mathfrak{d}_p}(\epsilon t,m,\epsilon)\right)(z),$$

which turns out to be a bounded holomorphic function defined on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ provided that (2.1) holds. The operator $\mathcal{F}^{-1}$ stands for the inverse Fourier transform (see Proposition 3.6).

§ 3. Review of some formal and analytic transforms

We recall the definitions and main properties of $q-$Borel, $q-$Laplace and Fourier transforms. Throughout this section, $\mathbb{E}$ stands for a complex Banach space. The proofs are omitted and can be found in [15], [1], [9] and [8].

Let $q > 1$ be a real number and $k \geq 1$ be an integer.

Definition 3.1. Let $\hat{a}(T) = \sum_{n\geq 0} a_n T^n \in \mathbb{E}[[T]]$. We define the formal $q-$Borel transform of order k of $\hat{a}(T)$ as the formal power series $\hat{\mathcal{B}}_{q;1/k}(\hat{a}(T))(\tau) = \sum_{n\geq 0} a_n \frac{\tau^n}{(q^{1/k})^{n(n-1)/2}} \in \mathbb{E}[[\tau]]$.

Proposition 3.2. *Let $\sigma \in \mathbb{N}$ and $j \in \mathbb{Q}$. Then, the following formal identity holds*

$$\hat{\mathcal{B}}_{q;1/k}(T^\sigma \sigma_q^j \hat{a}(T))(\tau) = \frac{\tau^\sigma}{(q^{1/k})^{\sigma(\sigma-1)/2}}\sigma_q^{j-\frac{\sigma}{k}}\left(\hat{\mathcal{B}}_{q;1/k}(\hat{a}(T))(\tau)\right),$$

for every $\hat{a}(T) \in \mathbb{E}[[T]]$.

The $q-$Laplace transform of order $k > 0$ extends that used in [3] for $k = 1$, and introduced in the work [17]. It provides a continuous $q-$analog for the formal inverse of $\hat{\mathcal{B}}_{q;1/k}$ developed in [1]. The associated kernel of the $q-$Laplace operator is the Jacobi theta function of order k, $\Theta_{q^{1/k}}(x) = \sum_{n\in\mathbb{Z}} q^{-\frac{n(n-1)}{2k}} x^n$, for $x \in \mathbb{C}^{\star}$, $m \in \mathbb{Z}$. As a direct consequence of Lemma 4.1 in [3], extended for any value of k, Jacobi theta function of order k satisfies that for every $\tilde{\delta} > 0$ there exists a positive constant $C_{q,k}$ not depending on $\tilde{\delta}$, such that $\left|\Theta_{q^{1/k}}(x)\right| \geq C_{q,k}\tilde{\delta}\exp\left(\frac{k}{2}\frac{\log^2|x|}{\log(q)}\right)|x|^{1/2}$, for every $x \in \mathbb{C}^{\star}$ verifying $|1 + xq^{\frac{m}{k}}| > \tilde{\delta}$, for all $m \in \mathbb{Z}$. This last property is crucial in order for the $q-$Laplace transform of order k to be well-defined.

Definition 3.3. Let $\rho > 0$ and U_d be an unbounded sector with vertex at 0 and bisecting direction $d \in \mathbb{R}$. Let $f : D(0,\rho) \cup U_d \to \mathbb{E}$ be a holomorphic function, continuous on $\overline{D}(0,\rho)$ such that there exist constants $K > 0$ and $\alpha \in \mathbb{R}$ with

$$\|f(x)\|_{\mathbb{E}} \leq K\exp\left(\frac{k}{2}\frac{\log^2|x|}{\log(q)} + \alpha\log|x|\right)$$

for every $x \in U_d$, $|x| \geq \rho$ and

$$\|f(x)\|_{\mathbb{E}} \leq K$$

for all $x \in \overline{D}(0,\rho)$. Take $\gamma \in \mathbb{R}$ such that $e^{i\gamma} \in U_d$. We put $\pi_{q^{1/k}} = \frac{\log(q)}{k}\prod_{n\geq 0}(1 - \frac{1}{q^{\frac{n+1}{k}}})^{-1}$, and define the $q-$Laplace transform of order k of f in direction γ as

$$\mathcal{L}^{\gamma}_{q;1/k}(f(x))(T) = \frac{1}{\pi_{q^{1/k}}}\int_{L_\gamma}\frac{f(u)}{\Theta_{q^{1/k}}\left(\frac{u}{T}\right)}\frac{du}{u},$$

where L_γ stands for the set $\mathbb{R}_+e^{i\gamma} := \{te^{i\gamma} : t \in (0,\infty)\}$.

Lemma 3.4. *Let $\tilde{\delta} > 0$. Under the hypotheses of Definition 3.3, $\mathcal{L}^{\gamma}_{q;1/k}(f(x))(T)$ defines a bounded and holomorphic function on the domain $\mathcal{R}_{\gamma,\tilde{\delta}} \cap D(0,r_1)$ for any $0 < r_1 \leq q^{(\frac{1}{2}-\alpha)/k}/2$. The value of $\mathcal{L}^{\gamma}_{q;1/k}(f(x))(T)$ does not depend on the choice of γ under the condition $e^{i\gamma} \in S_d$ due to Cauchy formula.*

Proposition 3.5. *Let f be a function satisfying the properties in Definition 3.3, and $\tilde{\delta} > 0$. Then, for every $\sigma \geq 0$ one has*

$$T^{\sigma}\sigma_q^j(\mathcal{L}^{\gamma}_{q;1/k}f(x))(T) = \mathcal{L}^{\gamma}_{q;1/k}\left(\frac{x^{\sigma}}{(q^{1/k})^{\sigma(\sigma-1)/2}}\sigma_q^{j-\frac{\sigma}{k}}f(x)\right)(T),$$

for every $T \in \mathcal{R}_{\gamma,\tilde{\delta}} \cap D(0,r_1)$, where $0 < r_1 \leq q^{(\frac{1}{2}-\alpha)/k}/2$.

We are also making use of Fourier transform and some of its properties, in the spirit of [5, 9].

Proposition 3.6. *Take $\mu > 1, \beta > 0$ and let $f \in E_{(\beta,\mu)}$. The inverse Fourier transform is defined by $\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(m) \exp(ixm)\, dm$, for $x \in \mathbb{R}$, which can be extended to an analytic function on the strip H_β. Let $\phi(m) = imf(m) \in E_{(\beta,\mu-1)}$. Then, we have $\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z)$, for every $z \in H_\beta$.*

Let $g \in E_{(\beta,\mu)}$ and let $\psi(m) = \frac{1}{(2\pi)^{1/2}}(f * g)(m)$, the convolution product of f and g, for all $m \in \mathbb{R}$. The function ψ is an element of $E_{(\beta,\mu)}$. Moreover, we have $\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z)$, for every $z \in H_\beta$.

§ 4. Sketch of the procedure

This section is the main core of these notes. We aim to clarify the procedure followed in the construction of the analytic solutions of the main problem under study. For the sake of clarity, we focus on the steps in the construction rather than giving detail on the technical and cumbersome constructions, which can all be found in [8].

Let us consider the main equation (2.2) for each $0 \leq p \leq \varsigma - 1$. We omit the subindex p for the sake of clarity, refering to d for the direction $\mathfrak{d}_p$, $\mathcal{E}_d$ for $\mathcal{E}_p$ and f for $f^{\mathfrak{d}_p}$. We apply Fourier transform at both sides of the equation and then the formal q–Borel transformation of order k_1. Equation (2.2) is transformed into the auxiliary q–difference-convolution equation (**Auxiliary equation 1**)

$$
\begin{aligned}
Q(im)\frac{\tau^{k_1}}{(q^{1/k_1})^{k_1(k_1-1)/2}} w^d_{k_1}(\tau,m,\epsilon) &= \frac{\tau^{d_D+k_1}}{(q^{1/k_1})^{(d_D+k_1)(d_D+k_1-1)/2}} \sigma_q^{-d_D/\kappa} R_D(im) w^d_{k_1}(\tau,m,\epsilon) \\
+\sum_{\ell=1}^{D-1}\left(\sum_{\lambda\in I_\ell} \frac{\epsilon^{\Delta_{\lambda,\ell}-d_{\lambda,\ell}}\tau^{d_{\lambda,\ell}+k_1}}{(q^{1/k_1})^{(d_{\lambda,\ell}+k_1)(d_{\lambda,\ell}+k_1-1)/2}} \sigma_q^{\delta_\ell-\frac{d_{\lambda,\ell}}{k_1}-1} \right.&\left.\frac{1}{(2\pi)^{1/2}} (C_{\lambda,\ell}(m,\epsilon) *^{R_\ell} w^d_{k_1}(\tau,m,\epsilon))\right) \\
&+ \frac{\tau^{k_1}}{(q^{1/k_1})^{k_1(k_1-1)/2}} \psi_{k_1}(\tau,m,\epsilon).
\end{aligned}
\tag{4.1}
$$

Here, we denote the convolution product

$$
h_1(m) *^Q h_2(m) := \int_{-\infty}^{\infty} h_1(m-m_1) Q(im_1) h_2(m_1) dm_1, \quad m \in \mathbb{R},
$$

for any continuous functions $h_j : \mathbb{R} \to \mathbb{C}$, $j = 1, 2$.

By means of a fixed point argument in appropriate Banach spaces of functions, we get the existence of a function $w^d_{k_1}$, continuous on $U_d \cup D(0,\rho) \times \mathbb{R}$, holomorphic with respect to τ on $U_d \cup D(0,\rho)$, which solves (4.1) and satisfies

$$
|w^d_{k_1}(\tau,m,\epsilon)| \leq C_1(1+|m|)^{-\mu} e^{-\beta|m|} \exp\left(\frac{\kappa \log^2|\tau+\delta|}{2\log(q)} + \alpha \log|\tau+\delta|\right),
$$

for every $\tau \in U_d \cup D(0,\rho)$, all $\epsilon \in \mathcal{E}_p$ and some $C_1 > 0$. The function w_{k_1} is holomorphic on $D(0,\epsilon_0)$ with respect to the perturbation parameter.

We apply Fourier transform to the main equation and then formal $q-$Borel transform of order k_2. This procedure concludes with a second auxiliary equation (**Auxiliary equation 2**):

$$Q(im)\frac{\tau^{k_2}}{(q^{1/k_2})^{k_2(k_2-1)/2}}\hat{w}_{k_2}(\tau,m,\epsilon) = R_D(im)\frac{\tau^{d_D+k_2}}{(q^{1/k_2})^{(d_D+k_2)(d_D+k_2-1)/2}}\hat{w}_{k_2}(\tau,m,\epsilon)$$
$$+\sum_{\ell=1}^{D-1}\left(\sum_{\lambda\in I_\ell}\frac{\epsilon^{\Delta_{\lambda,\ell}-d_{\lambda,\ell}}\tau^{d_{\lambda,\ell}+k_2}}{(q^{1/k_2})^{(d_{\lambda,\ell}+k_2)(d_{\lambda,\ell}+k_2-1)/2}}\sigma_q^{\delta_\ell-\frac{d_{\lambda,\ell}}{k_2}-1}\frac{1}{(2\pi)^{1/2}}(C_{\lambda,\ell}(m,\epsilon)*^{R_\ell}\hat{w}_{k_2}(\tau,m,\epsilon))\right)$$
$$+\frac{\tau^{k_2}}{(q^{1/k_2})^{k_2(k_2-1)/2}}\hat{\psi}_{k_2}(\tau,m,\epsilon),$$

As it was pointed out in the introduction, such equation can, in principle, only provide formal solutions due to the forcing term is only guaranteed to be formal. However, one can substitute $\hat{\psi}_{k_2}$ by ψ_{k_2}, as defined in (2.3), to arrive at a novel auxiliary equation (**Auxiliary equation 2'**), which can be analytically solved.

$$Q(im)\frac{\tau^{k_2}}{(q^{1/k_2})^{k_2(k_2-1)/2}}w_{k_2}(\tau,m,\epsilon) = R_D(im)\frac{\tau^{d_D+k_2}}{(q^{1/k_2})^{(d_D+k_2)(d_D+k_2-1)/2}}w_{k_2}(\tau,m,\epsilon)$$
$$+\sum_{\ell=1}^{D-1}\left(\sum_{\lambda\in I_\ell}\frac{\epsilon^{\Delta_{\lambda,\ell}-d_{\lambda,\ell}}\tau^{d_{\lambda,\ell}+k_2}}{(q^{1/k_2})^{(d_{\lambda,\ell}+k_2)(d_{\lambda,\ell}+k_2-1)/2}}\sigma_q^{\delta_\ell-\frac{d_{\lambda,\ell}}{k_2}-1}\frac{1}{(2\pi)^{1/2}}(C_{\lambda,\ell}(m,\epsilon)*^{R_\ell}w_{k_2}(\tau,m,\epsilon))\right)$$
$$+\frac{\tau^{k_2}}{(q^{1/k_2})^{k_2(k_2-1)/2}}\psi_{k_2}(\tau,m,\epsilon),$$

Indeed, the solution of the previous problem, $w_{k_2}(\tau,m,\epsilon)$, is holomorphic with respect to τ on $S_d \cup \mathcal{R}_d^b$, and holomorphic with respect to τ on $S_d \cup \mathcal{R}_d^b$. There exists $C_2 > 0$ such that

$$|w_{k_2}(\tau,m\epsilon)| \leq C_2(1+|m|)^{-\mu}e^{-\beta|m|}\exp\left(\frac{k_2\log^2|\tau|}{2\log(q)}+\nu\log|\tau|\right),$$

for every $\tau \in S_d \cup \mathcal{R}_d^b$, $m \in \mathbb{R}$, $\epsilon \in D(0,\epsilon_0)$. Moreover, this function is holomorphic with respect to ϵ in its domain of definition. It can be obtained by a fixed point argument in appropriate Banach spaces of functions.

At this point, one may observe that two different $q-$Gevrey levels are distinguished. A first level in the Borel plane, in which $w_{k_1}^d$ lies; and a second level in the Borel plane in which w_{k_2} remains. The fist funtion has been obtained after Borel action of order $1/k_1$, whilst the second one is only at $1/k_2$ depth. It is natural to expect some Borel-like relationship of order $1/\kappa = 1/k_1 - 1/k_2 > 0$ among them. As a matter of fact, one gets the following result linking both functions.

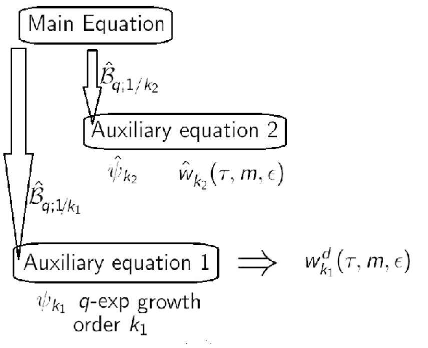

Figure 1. Plan of the procedure (I)

Proposition 4.1. *For every $\tilde{\delta} > 0$, the function*

$$\tau \mapsto \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon)) := \mathcal{L}^d_{q;1/\kappa}\left(h \mapsto w^d_{k_1}(h,m,\epsilon)\right)(\tau)$$

defines a bounded holomorphic function in $\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)$ with $0 \le r_1 \le q^{(1/2-\alpha)/\kappa}/2$. Moreover, it holds that

$$\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon) = w_{k_2}(\tau,m,\epsilon)$$

for every $\tau \in S^b_d$, $m \in \mathbb{R}$ and $\epsilon \in D(0,\epsilon_0)$, where $\tilde{\rho} > 0$ and S^b_d is a finite sector of bisecting direction d.

The importance of this result lies on the fact that the domain of definition of $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$ with respect to τ can be extended to an infinite sector of bisecting direction d, with appropriate $q-$Gevrey growth in order to be able to apply $q-$Laplace transform. We adopt the notation $w^d_{k_2}$ for the extension of w_{k_2} to such infinite sector.

We conclude with the acceleration of the solution of the Auxiliary equation II', by means of $q-$Laplace transformation of order $1/k_2$ and then the inverse Fourier transform. The expression of the solution of (2.2) is given by

$$u^{\mathfrak{d}_p}(t,z,\epsilon) = \frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{L_{\gamma_p}}\frac{w^{\mathfrak{d}_p}_{k_2}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\frac{du}{u}\exp(izm)dm,$$

which turns out to be a holomorphic function on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, for every $0 \le p \le \varsigma - 1$. A scheme of the procedure is represented in Figures 1-3.

The existence of a formal solution of the main problem is made by means of a $q-$analog of Ramis-Sibuya theorem in two levels. The asymptotic analysis of the solutions is stated in terms of the so-called $q-$Gevrey asymptotic expansions.

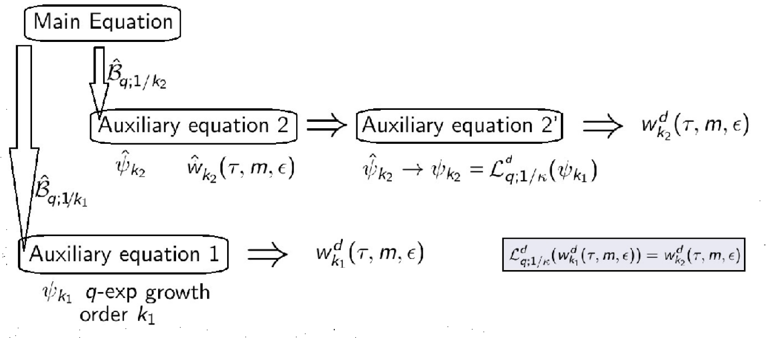

Figure 2. Plan of the procedure (II)

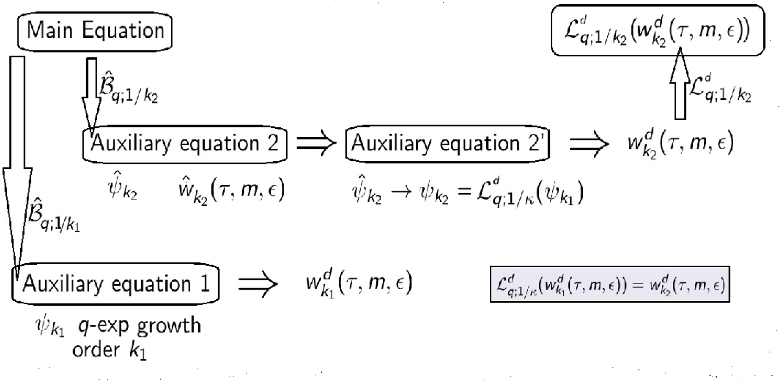

Figure 3. Plan of the procedure (III)

Definition 4.2. Let V be a bounded open sector with vertex at 0 in $\mathbb{C}$. Let $(\mathbb{F}, ||\cdot||_{\mathbb{F}})$ be a complex Banach space. Let $q \in \mathbb{R}$ with $q > 1$ and let k be a positive integer. We say that a holomorphic function $f : V \to \mathbb{F}$ admits the formal power series $\hat{f}(\epsilon) = \sum_{n\geq 0} f_n \epsilon^n \in \mathbb{F}[[\epsilon]]$ as its $q-$Gevrey asymptotic expansion of order $1/k$ if for every open subsector U with $(\overline{U} \setminus \{0\}) \subseteq V$, there exist $A, C > 0$ such that

$$\left\| f(\epsilon) - \sum_{n=0}^{N} f_n \epsilon^n \right\|_{\mathbb{F}} \leq C A^{N+1} q^{\frac{N(N+1)}{2k}} |\epsilon|^{N+1},$$

for every $\epsilon \in U$, and $N \geq 0$.

We prove that the difference of two solutions $u^{\mathfrak{d}_p}$ and $u^{\mathfrak{d}_{p+1}}$ in the intersection of their domain of definition with respect to the perturbation parameter is asymptotically null on certain Banach spaces of functions.

Let $\mathbb{F}$ be the Banach space of holomorphic and bounded functions defined on $\mathcal{T}\times H_{\beta'}$, with the supremum norm.

Lemma 4.3. *There exists a formal power series* $\hat{f}(t,z,\epsilon)=\sum_{m\geq 0} f_m \frac{\epsilon^m}{m!}$*, with* $f_m \in \mathbb{F}$ *for every* $m\geq 0$ *which is the common* $q-$*Gevrey asymptotic expansion of order* $1/k_1$ *on* $\mathcal{E}_p$ *of the function* $f^{\mathfrak{f}_p}$*, seen as holomorphic functions from* $\mathcal{E}_p$ *to* $\mathbb{F}$*, for all* $0\leq p\leq \varsigma-1$*.*

More precisely, the main result of the work [8] reads as follows:

Theorem 4.4. *If* $1/r_{Q,R_D}$*,* $\tilde{C}_{\lambda,\ell}$ *and* C_F *are small enough, then there exists a formal power series*

$$\hat{u}(t,z,\epsilon)=\sum_{m\geq 0} h_m(t,z)\frac{\epsilon^m}{m!}\in\mathbb{F}[[\epsilon]],$$

formal solution of the equation

$$\begin{aligned} Q(\partial_z)\sigma_q\hat{u}(t,z,\epsilon) &= (\epsilon t)^{d_D}\sigma_q^{\frac{d_D}{k_2}+1}R_D(\partial_z)\hat{u}(t,z,\epsilon) \\ &\quad+\sum_{\ell=1}^{D-1}\left(\sum_{\lambda\in I_\ell} t^{d_{\lambda,\ell}}\epsilon^{\Delta_{\lambda,\ell}}\sigma_q^{\delta_\ell}c_{\lambda,\ell}(z,\epsilon)R_\ell(\partial_z)\hat{u}(t,z,\epsilon)\right)+\sigma_q\hat{f}(t,z,\epsilon). \end{aligned}$$

Moreover, $\hat{u}(t,z,\epsilon)$ *turns out to be the common* $q-$*Gevrey asymptotic expansion of order* $1/k_1$ *on* $\mathcal{E}_p$ *of the function* $u^{\mathfrak{d}_p}$*, seen as holomorphic function from* $\mathcal{E}_p$ *into* $\mathbb{F}$*, for* $0\leq p\leq\varsigma-1$*. In addition to that,* $\hat{u}$ *is of the form*

$$\hat{u}(t,z,\epsilon)=a(t,z,\epsilon)+\hat{u}_1(t,z,\epsilon)+\hat{u}_2(t,z,\epsilon),$$

where $a(t,z,\epsilon)\in\mathbb{F}\{\epsilon\}$ *and* $\hat{u}_1(t,z,\epsilon),\hat{u}_2(t,z,\epsilon)\in\mathbb{F}[[\epsilon]]$ *and such that for every* $0\leq p\leq\varsigma-1$*, the function* $u^{\mathfrak{d}_p}$ *can be written in the form*

$$u^{\mathfrak{d}_p}(t,z,\epsilon)=a(t,z,\epsilon)+u_1^{\mathfrak{d}_p}(t,z,\epsilon)+u_2^{\mathfrak{d}_p}(t,z,\epsilon),$$

where $\epsilon\mapsto u_1^{\mathfrak{d}_p}(t,z,\epsilon)$ *is a* $\mathbb{F}$*-valued function that admits* $\hat{u}_1(t,z,\epsilon)$ *as its* $q-$*Gevrey asymptotic expansion of order* $1/k_1$ *on* $\mathcal{E}_p$ *and also* $\epsilon\mapsto u_2^{\mathfrak{d}_p}(t,z,\epsilon)$ *is a* $\mathbb{F}$*-valued function that admits* $\hat{u}_2(t,z,\epsilon)$ *as its* $q-$*Gevrey asymptotic expansion of order* $1/k_2$ *on* $\mathcal{E}_p$*.*

For the application of the previous result to some factorized problem, we refer to Section 7 in [8].

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