数理解析研究所講究録
第2023巻 2017年 1-14

Anomalous localized resonance on smooth domains using spectral properties of the Neumann-Poincaré operators

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1 Introduction

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^d$, $d=2,3$. We assume that Ω is occupied with a material which has the dielectric constant $\epsilon_c<0$ with dissipation $\delta>0$ and the matrix $\mathbb{R}^d\setminus\overline{\Omega}$ has the dielectric constant $\epsilon_m>0$. So the total distribution of the dielectric constant is written as

$$\epsilon=\begin{cases}\epsilon_c+i\delta, & \text{in } \Omega,\\ \epsilon_m, & \text{in } \mathbb{R}\setminus\overline{\Omega},\end{cases}$$

which is called the plasmonic structure.

We also assume that Ω is diametrically small and there exists the polarizable dipole source $a\cdot\nabla\delta_z$ outside Ω, where $a\in\mathbb{R}^d$ is a constant vector and δ_z is the Dirac mass at $z\in\mathbb{R}^d\setminus\overline{\Omega}$. Then we consider the following dielectric equation under the quasi-static approximation:

$$\begin{cases}\nabla\cdot\epsilon\nabla u=a\cdot\nabla\delta_z & \text{in } \mathbb{R}^d,\\ u(x)=O(|x|^{1-d}) & \text{as } |x|\to\infty.\end{cases}\tag{1}$$

Let u_δ be the solution of (1). The resonance is characterized by the blow-up of $\|\nabla u_\delta\|_{L^2(\Omega)}$:

$$\|\nabla u_\delta\|_{L^2(\Omega)}\to\infty \quad \text{as } \delta\to 0.\tag{2}$$

In particular, we are interested in anomalous localized resonance, which is characterized as follows:

1. $E_\delta:=\delta\|\nabla u_\delta\|^2_{L^2(\Omega)}\to\infty$ as $\delta\to 0$,

2. there exist $R>0$ and $C>0$ such that $|u_\delta(x)|<C$ for $|x|>R$.

Anomalous localized resonance (ALR) is discovered in [13], and applied to cloaking by anomalous localized resonance (CALR) [11]. There are many results on this subject; see e.g. [1] and the references therein. So far, however, ALR has been mainly studied in the core-shell structure.

Here we consider simply connected structure for Ω, and show that ALR occurs on ellipse in two dimensions; on the other hand, it does not occur on ball in three dimensions. We emphasize that in [12] the authors consider the plasmonic structure on disk in two dimensions and showed that the complete resonance occurs.

2 Neumann-Poincaré operator and symmetrization

Let Γ be the fundamental solution to the Laplacian on $\mathbb{R}^d$, $d=2,3$, which is given by

$$\Gamma(x)=\begin{cases}\dfrac{1}{2\pi}\ln|x|, & d=2,\\ -\dfrac{1}{4\pi}|x|^{-1}, & d=3.\end{cases}$$

The single layer potential on $\partial\Omega$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x)=\int_{\partial\Omega}\Gamma(x-y)\varphi(y)\mathrm{d}\sigma(y),\quad x\in\mathbb{R}^d.$$

and the Neumann-Poincaré (NP) operator by

$$\mathcal{K}_{\partial\Omega}[\varphi](x)=\int_{\partial\Omega}\frac{\partial}{\partial\nu_y}\Gamma(x-y)\varphi(y)\mathrm{d}\sigma(y),\quad x\in\partial\Omega,$$

whose $L^2(\partial\Omega)$-adjoint $\mathcal{K}^*_{\partial\Omega}$ is also called the NP operator. Here we denote by $\frac{\partial}{\partial\nu_y}$ the outward normal derivative in y-variable on $\partial\Omega$. There holds the following jump relation:

$$\frac{\partial}{\partial\nu}\mathcal{S}_{\partial\Omega}[\varphi]\Big|_{\pm}(x)=\left(\pm\frac{1}{2}+\mathcal{K}^*_{\partial\Omega}\right)[\varphi](x),\quad x\in\partial\Omega,\tag{3}$$

where the subscript + (resp. −) indicate the limits to $\partial\Omega$ from outside (resp. inside) of Ω. Moreover, the Plemelj's symmetrization principle (also known as Calderón's identity) holds:

$$\mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega}=\mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}.\tag{4}$$

We denote $H^s(\partial\Omega)$, $s\in\mathbb{R}$, the L^2-Sobolev space on $\partial\Omega$, whose norm is written as $\|\cdot\|_s$. Define

$$\langle\varphi,\psi\rangle_{\mathcal{H}^*}:=-\langle\varphi,\mathcal{S}_{\partial\Omega}[\psi]\rangle_{L^2(\partial\Omega)}\tag{5}$$

for $\varphi, \psi \in H_0^{-1/2}(\partial\Omega) := \{\varphi \in H^{1/2}(\partial\Omega); \langle \varphi, 1\rangle_{L^2(\partial\Omega)} = 0\}$. Note that the right hand side of (5) is well-defined, since $\mathcal{S}_{\partial\Omega}$ maps $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. It is known that $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$ is an inner product on $H_0^{-1/2}(\partial\Omega)$, which induces the norm equivalent to the original norm of $H^{-1/2}(\partial\Omega)$:

$$\|\varphi\|_{\mathcal{H}^*} \approx \|\varphi\|_{-1/2} \tag{6}$$

for all $\varphi \in H_0^{-1/2}(\partial\Omega)$; see [8]. Put $\mathcal{H}_0^* := H_0^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$. Then the Plemelj's symmetrization principle (4) implies that $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $\mathcal{H}_0^*$.

Let us consider the symmetrization of $\mathcal{K}_{\partial\Omega}$. If $\mathcal{S}_{\partial\Omega}$ is invertible, multiplying $\mathcal{S}_{\partial\Omega}^{-1}$ from the right and left side in (4), we can obtain an analogue of the Plemelj's symmetrization for $\mathcal{K}_{\partial\Omega}$ and symmtrize it in the same way as in the case of $\mathcal{K}_{\partial\Omega}^*$. It is true in three dimensions; unfortunately, there exists a domain Ω in two dimensions such that $\mathcal{S}_{\partial\Omega}[\varphi_0] = 0$ in Ω for a nontrivial $\varphi_0 \in H^{-1/2}(\partial\Omega)$, see [14].

To overcome this difficulty, we define a variant of the single layer potential

$$\tilde{\mathcal{S}}_{\partial\Omega}[\varphi] = \begin{cases} \mathcal{S}_{\partial\Omega}[\varphi], & \text{if } \langle\varphi, 1\rangle_{L^2(\partial\Omega)} = 0, \\ -1, & \text{if } \varphi = \varphi_0, \end{cases}$$

where $\varphi_0 \in H^{-1/2}$ is an eigenfunction of $\mathcal{K}_{\partial\Omega}^*$ corresponding to the eigenvalue $1/2$ which is normalized as

$$\langle\varphi_0, 1\rangle_{L^2(\partial\Omega)} = 1. \tag{7}$$

Here we note that $\mathcal{K}_{\partial\Omega}^*$ is compact on $H^{-1/2}(\partial\Omega)$ and its spectrum $\sigma(\mathcal{K}_{\partial\Omega}^*) \subset (-1/2, 1/2]$; moreover, $1/2$ is simple, since $\sigma(\mathcal{K}_{\partial\Omega}^*|_{H_0^{-1/2}(\partial\Omega)}) \subset (-1/2, 1/2)$ and $\dim H^{1/2}(\partial\Omega)\backslash H_0^{-1/2}(\partial\Omega) = 1$ (see [3, 8, 10]). Then we have an extention of (4):

$$\tilde{\mathcal{S}}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^* = \mathcal{K}_{\partial\Omega}\tilde{\mathcal{S}}_{\partial\Omega},$$

and extend

$$\langle\varphi, \psi\rangle_{\mathcal{H}^*} = -\langle\varphi, \tilde{\mathcal{S}}_{\partial\Omega}[\psi]\rangle_{L^2(\partial\Omega)} \tag{8}$$

for $\varphi, \psi \in H^{-1/2}(\partial\Omega)$. Note that (6) is also extended to $H^{-1/2}(\partial\Omega)$. Then (8) is an inner product on $H^{-1/2}(\partial\Omega)$. We define the Hilbert space $\mathcal{H}^* = H^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$, on which $\mathcal{K}_{\partial\Omega}^*$ is symmetrized.

Let us symmetrize $\mathcal{K}_{\partial\Omega}$. $\tilde{\mathcal{S}}_{\partial\Omega}$ is a bijection from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$, so an analogue of the Plemelj's symmetrization holds:

$$\tilde{\mathcal{S}}_{\partial\Omega}^{-1}\mathcal{K}_{\partial\Omega} = \mathcal{K}_{\partial\Omega}^*\tilde{\mathcal{S}}_{\partial\Omega}^{-1}. \tag{9}$$

Define

$$\langle f, g\rangle_{\mathcal{H}} := -\langle f, \tilde{\mathcal{S}}_{\partial\Omega}^{-1}[g]\rangle_{L^2(\partial\Omega)} \tag{10}$$

for $f, g \in H^{1/2}(\partial\Omega)$. Then (10) is an inner product on $H^{1/2}(\partial\Omega)$ which induces the equivalence $\|\cdot\|_{\mathcal{H}} \approx \|\cdot\|_{1/2}$. Put $\mathcal{H} = H^{1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Then $\mathcal{K}_{\partial\Omega}$ is symmetrized on $\mathcal{H}$ by (9). We note that $\tilde{S}_{\partial\Omega}$ is unitaty and $\{\tilde{S}_{\partial\Omega}[\psi_j]\}_{j=1}^{\infty} \cup \{-1\}$ is an orthonormal bases of $\mathcal{H}$, where $\{\psi_j\}$ is an orthonormal basis of $\mathcal{H}_0^*$. In particular, we can choose $\{\psi_j\}_{j=1}^{\infty}$ as the normalized eigenvectors of $\mathcal{K}_{\partial\Omega}^*$ on $\mathcal{H}_0^*$.

For the details of the symmetrization of the NP operator, see [5].

3 Representation of the solution

If $\partial\Omega$ is smooth, at least $C^{1,\alpha}$ for some $\alpha > 0$, then it is known that $\mathcal{K}_{\partial\Omega}^*$ is compact on $\mathcal{H}^*$ (see [9], also [10]). Since $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $\mathcal{H}_0^*$, its eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ accumulate to 0. We remark that $|\lambda_j| < 1/2$ (see [6, 14]). Let ψ_j be the normalized eigenfunction corresponding to the eigenvalue λ_j in $\mathcal{H}_0^*$. Then we have an orthonormal basis $\tilde{S}_{\partial\Omega}[\{\psi_j\}_{j=0}^{\infty} \cup \{\varphi_0\}] = \{\mathcal{S}_{\partial\Omega}[\psi_j]\}_{j=1}^{\infty} \cup \{-1\}$ in $\mathcal{H}$.

Fix $z \in \mathbb{R}^d \setminus \overline{\Omega}$. Then $\Gamma(\cdot - z)$ belongs to $H^{1/2}(\partial\Omega)$, and so admits the following expansion:

$$\Gamma(x-z) = \sum_{j=1}^{\infty} c_j(z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + c_0(z), \quad x \in \partial\Omega, \tag{11}$$

for some constants $c_j(z)$ (depending on z) which satisfy

$$\sum_{j=1}^{\infty} |c_j(z)|^2 < \infty.$$

Since $-\langle \mathcal{S}_{\partial\Omega}[\psi_i], \psi_j\rangle_{L^2(\partial\Omega)} = \delta_{ij}$, where δ_{ij} is the Kronecker's delta, we see that

$$c_j(z) = -\int_{\partial\Omega} \Gamma(x-z)\psi_j(x)\mathrm{d}\sigma(x) = -\mathcal{S}_{\partial\Omega}[\psi_j](z), \quad j = 1, 2, 3, \dots.$$

We also see from (7) that

$$c_0(z) = \mathcal{S}_{\partial\Omega}[\varphi_0](z).$$

So, we obtain the following formula:

$$\Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad x \in \partial\Omega. \tag{12}$$

Observe that

$$\left\| \sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j] \right\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2 < \infty. \tag{13}$$

Since $\|\cdot\|_{\mathcal{H}} \approx \|\cdot\|_{1/2}$, we find from the trace theorem that $\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j]$ converges in $H^1(\Omega)$ and harmonic in Ω. So, we obtain the following expansion formula of the fundamental solution to the Laplacian.

Theorem 3.1. *It holds that*

$$\Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad x \in \overline{\Omega}, z \in \mathbb{R}^d \setminus \overline{\Omega}. \tag{14}$$

We now derive a representation of the solution to (1). By the jump relation (3), the equation (1) is equivalent to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \Delta u = a \cdot \nabla \delta_z & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u|_- = u|_+, \quad (\epsilon_c + i\delta) \left.\frac{\partial u}{\partial \nu}\right|_- = \epsilon_m \left.\frac{\partial u}{\partial \nu}\right|_+ & \text{on } \partial\Omega, \\ u(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty. \end{cases} \tag{15}$$

We seek the solution of (15) in the following form:

$$u_\delta(x) = F_z(x) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d. \tag{16}$$

where the potential $\varphi_\delta \in \mathcal{H}_0^*$ and

$$F_z(x) := a \cdot \nabla_x \Gamma(x-z), \quad x \neq z. \tag{17}$$

Note that $\Delta F_z(x) = a \cdot \nabla \delta_z(x)$. The solution $u_\delta(x)$ satisfies the equation of (15) on Ω and $\mathbb{R}^d \setminus \overline{\Omega}$; moreover, $u_\delta(x)$ decays as $O(|x|^{1-d})$, since $\varphi_0 \in \mathcal{H}_0^*$. Then, from the transmission condision in (15), we should solve the following integral equation

$$(\lambda_\delta I - \mathcal{K}_{\partial\Omega}^*)[\varphi_\delta] = \partial_\nu F_z \quad \text{on } \partial\Omega \tag{18}$$

($\partial_\nu F_z$ denotes the outward normal derivative of F_z on $\partial\Omega$). Here

$$\lambda_\delta := \frac{\epsilon_c + \epsilon_m + i\delta}{2(\epsilon_c - \epsilon_m) + 2i\delta} \to \frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} \quad \text{as } \delta \to 0.$$

From the spectral resolution of $\mathcal{K}_{\partial\Omega}^*$ on $\mathcal{H}_0^*$

$$\mathcal{K}_{\partial\Omega}^* = \sum_{j=1}^{\infty} \lambda_j \psi_j \otimes \psi_j, \tag{19}$$

the solution φ_δ to the integral equation (18) can be representad as

$$\varphi_\delta = \sum_{j=1}^{\infty} \frac{\alpha_j(z)}{\lambda_\delta - \lambda_j} \psi_j, \tag{20}$$

where

$$\alpha_j(z) := \langle \partial_\nu F_z, \psi_j \rangle_{\mathcal{H}^*}.$$

We can see from (17) that

$$\alpha_j(z) = -a \cdot \nabla \int_{\partial\Omega} \frac{\partial}{\partial \nu_x} \Gamma(x-z) \mathcal{S}_{\partial\Omega}[\psi_j](x) \mathrm{d}\sigma(x).$$

From (3) and (14), we have

$$\frac{\partial}{\partial \nu_x} \Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z) \frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\psi_j](x) = \sum_{j=1}^{\infty} \left(\frac{1}{2} - \lambda_j\right) \mathcal{S}_{\partial\Omega}[\psi_j](z) \psi_j(x).$$

It then follows that

$$\alpha_j(z) = \left(\frac{1}{2} - \lambda_j\right) a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z). \tag{21}$$

4 Anomalous localized resonance

The resonance occurs if and only if $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)} \in \sigma(\mathcal{K}^*_{\partial\Omega})$; if $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ is a non-zero eigenvalue of the NP operator, resonance occurs in the sense of (2) and have asymptotics $\|\nabla u_\delta\|_{L^2(\Omega)} \sim \delta$ as $\delta \to 0$; however, it is not localized, see [5].

Let us consider the resonance at the accumulation point of $\sigma(\mathcal{K}^*_{\partial\Omega})$, *i.e.*, $\epsilon_c + \epsilon_m = 0$. We assume that 0 is not an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$. Since $\partial\Omega$ is smooth, $\mathcal{K}^*_{\partial\Omega}$ is compact, hence 0 is an essential spectrum. It is worth mentioning that we are not aware of any domain other than disks on which the NP operator has 0 as an eigenvalue. If Ω is a disk, then $\mathcal{K}^*_{\partial\Omega} \equiv 0$ on $\mathcal{H}^*_0$.

When $\epsilon_c + \epsilon_m = 0$, we have

$$\lambda_\delta \approx \delta.$$

We first show that

$$\|\mathcal{S}_{\partial\Omega}[\varphi]\|^2_{L^2(\Omega)} \approx \|\varphi\|^2_{\mathcal{H}^*} \tag{22}$$

for all $\varphi \in \mathcal{H}^*_0$. In fact, we see from (3) and (19) that

$$\begin{aligned} \|\mathcal{S}_{\partial\Omega}[\varphi]\|^2_{L^2(\Omega)} &= \int_{\partial\Omega} \mathcal{S}_{\partial\Omega}[\varphi] \overline{\frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\varphi]\Big|_-} \mathrm{d}\sigma \\ &= \left\langle \varphi, \left(-\frac{1}{2} I + \mathcal{K}^*_{\partial\Omega}\right)[\varphi] \right\rangle_{\mathcal{H}^*} \\ &= \sum_{j=1}^{\infty} \left(\frac{1}{2} - \lambda_j\right) |\langle \varphi, \psi_j \rangle_{\mathcal{H}^*}|^2. \end{aligned}$$

Since $|\lambda_j| < 1/2$ and accumulates to 0, we have (22). Then we see from (20)

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)} \approx \|\varphi_\delta\|^2_{\mathcal{H}^*} \equiv \sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2}.$$

4.1 Anomalous localized resonance on ellipse and cloaking on ellipses

Assume that Ω is an ellipse in $\mathbb{R}^2$. We use the elliptic coordinates

$$x = (x_1, x_2) = (x_1(\rho,\theta), x_2(\rho,\theta)) \in \mathbb{R}^2,$$

where

$$x_1(\rho,\theta) = R\cos\theta\cosh\rho, \quad x_2(\rho,\theta) = R\sin\theta\sinh\rho, \quad \rho > 0, 0 \le \theta < 2\pi.$$

For $\rho_0 > 0$, let

$$\partial\Omega = \{(x_1(\rho_0,\omega), x_2(\rho_0,\omega)) \in \mathbb{R}^2;\ 0 \le \omega < 2\pi\}. \tag{23}$$

Then $\partial\Omega$ is an ellipse whose foci are $(\pm R, 0)$. The length element $d\sigma$ and the outward normal derivative $\frac{\partial}{\partial\nu}$ are given in terms of the elliptic coordinates by

$$d\sigma = \Xi d\omega, \quad \frac{\partial}{\partial\nu} = \Xi^{-1}\frac{\partial}{\partial\rho},$$

where

$$\Xi = \Xi(\rho_0,\omega) := R\sqrt{\sinh^2\rho_0 + \sin^2\omega}.$$

Let us define

$$\phi_n^c(\omega) := \Xi(\rho_0,\omega)^{-1}\cos n\omega, \quad \phi_n^s(\omega) := \Xi(\rho_0,\omega)^{-1}\sin n\omega, \quad n = 1, 2, \cdots.$$

Then we have

$$\mathcal{K}_{\partial\Omega}^*[\phi_n^c](\omega) = \alpha_n\phi_n^c(\omega), \quad \mathcal{K}_{\partial\Omega}^*[\phi_n^s](\omega) = -\alpha_n\phi_n^s(\omega),$$

where

$$\alpha_n = \frac{1}{2e^{2n\rho_0}}, \quad n = 1, 2, \cdots.$$

$\{\cos n\omega, \sin n\omega;\ n = 1, 2, \cdots\}$ is complete in $L_0^2(\partial\Omega)$, hence in $\mathcal{H}_0^*$ (see [10]), which means that $\mathcal{K}_{\partial\Omega}^*$ has the following eigenfunction expansion in $\mathcal{H}_0^*$:

$$\mathcal{K}_{\partial\Omega}^* = \sum_{n=1}^{\infty}\alpha_n\psi_n^c \otimes \psi_n^c - \sum_{n=1}^{\infty}\alpha_n\psi_n^s \otimes \psi_n^s,$$

where

$$\psi_n^c := \sqrt{\frac{ne^{n\rho_0}}{\pi\cosh n\rho_0}}\phi_n^c, \quad \psi_n^s := \sqrt{\frac{ne^{n\rho_0}}{\pi\sinh n\rho_0}}\phi_n^s. \tag{24}$$

Note that $\{\psi_n^c, \psi_n^s;\ n = 1, 2, \cdots\}$ is an orthonormal basis in $\mathcal{H}_0^*$. We also have

$$\mathcal{S}_{\partial\Omega}[\phi_n^c](x) = \begin{cases} -\dfrac{\cosh n\rho}{ne^{n\rho_0}}\cos n\theta, & \rho \le \rho_0, \\ -\dfrac{\cosh n\rho_0}{ne^{n\rho}}\cos n\theta, & \rho > \rho_0, \end{cases} \tag{25}$$

$$\mathcal{S}_{\partial\Omega}[\phi_n^s](x) = \begin{cases} -\dfrac{\sinh n\rho}{ne^{n\rho_0}} \sin n\theta, & \rho \le \rho_0, \\ -\dfrac{\sinh n\rho_0}{ne^{n\rho}} \sin n\theta, & \rho > \rho_0, \end{cases} \tag{26}$$

where $\mathcal{S}_{\partial\Omega}[\varphi]$ is the single layer potential of φ. See [4, 7].

Furthermore, by the change of variables,

$$\frac{\partial}{\partial x_1} = \frac{1}{R(\sinh^2\rho + \sin^2\theta)} \left(\cos\theta \sinh\rho \frac{\partial}{\partial\rho} - \sin\theta\cosh\rho\frac{\partial}{\partial\theta} \right), \tag{27}$$

$$\frac{\partial}{\partial x_2} = \frac{1}{R(\sinh^2\rho + \sin^2\theta)} \left(\sin\theta \cosh\rho \frac{\partial}{\partial\rho} + \cos\theta\sinh\rho\frac{\partial}{\partial\theta} \right). \tag{28}$$

Theorem 4.1. *Suppose that Ω is an ellipse given by (23). Then we have*

$$\|\nabla u_\delta\|^2_{L^2(\Omega)} \sim \begin{cases} \delta^{-3+\rho_z/\rho_0} |\log\delta| & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log\delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0, \end{cases} \tag{29}$$

as $\delta \to 0$.

Therefore, the quantity $E_\delta = \delta\|u_\delta\|^2_{L^2(\Omega)}$ blows up if $\rho_0 < \rho_z \le 2\rho_o$ while it tends to 0 as $\delta \to 0$ if $\rho_z > 2\rho_0$.

Proof. We only have to study the asymptotics of the following summation:

$$\begin{aligned} &\sum_{j=1}^\infty \frac{|a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2+\lambda_j^2} \\ &= \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2} |a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_n^c](z)|^2 + \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2} |a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_n^s](z)|^2 \\ &= \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2}\cdot\frac{e^{n\rho_0}\cosh n\rho_0}{n\pi} \left|a\cdot\nabla\left(e^{-n\rho}\cos n\theta\right)\right|^2 \\ &\quad + \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2}\cdot\frac{e^{n\rho_0}\sinh n\rho_0}{n\pi} \left|a\cdot\nabla\left(e^{-n\rho}\sin n\theta\right)\right|^2. \end{aligned} \tag{30}$$

Since $\cosh n\rho_0 \approx \sinh n\rho_0 \approx e^{n\rho_0}$,

$$\sum_{j=1}^\infty \frac{|a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2+\lambda_j^2} \sim \sum_{n=1}^\infty \frac{1}{\delta^2+\lambda_n^2}\cdot\frac{e^{2n\rho_0}}{n} \left[\left|a\cdot\nabla_z\left(e^{-n\rho_z}\cos n\omega_z\right)\right|^2 + \left|a\cdot\nabla_z\left(e^{-n\rho_z}\sin n\omega_z\right)\right|^2 \right]. \tag{31}$$

Let $U(\omega)$ be the rotation by the angle ω, namely,

$$U(\omega) = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}.$$

Using the change of variable formula (27) and (28), we have

$$a \cdot \nabla \left(e^{-n\rho} \cos n\theta\right) = \frac{-ne^{n\rho}}{R\left(\sinh^2 \rho + \sin^2 \theta\right)} a \cdot U(n\theta) b(\rho, \theta), \tag{32}$$

$$a \cdot \nabla \left(e^{-n\rho} \sin n\theta\right) = \frac{-ne^{-n\rho}}{R\left(\sinh^2 \rho + \sin^2 \theta\right)} a \cdot U(n\theta - \pi/2) b(\rho, \theta), \tag{33}$$

where

$$b(\rho, \theta) = (\cos\theta \sinh\rho, \sin\theta \cosh\rho) \in \mathbb{R}^2,$$

which implies

$$\left|a \cdot \nabla \left(e^{-n\rho} \cos n\theta\right)\right|^2 + \left|a \cdot \nabla \left(e^{-n\rho} \sin n\theta\right)\right|^2 = \frac{n^2 e^{-2n\rho} |a|^2 |b(\rho,\theta)|^2}{R^2 \left(\sinh^2 \rho + \sin^2 \theta\right)^2}. \tag{34}$$

From (31) and (34), we have

$$\sum_{j=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2} \sim \sum_{n=1}^{\infty} \frac{n e^{2n\rho_0} e^{-2n\rho}}{\delta^2 + \frac{1}{4} e^{-4n\rho_0}}. \tag{35}$$

Let

$$N = \left[-\frac{1}{2\rho_0} \log 2\delta\right],$$

which is the first integer such that $\delta > \frac{1}{2} e^{-2N\rho_0}$. Then one can easily see that

$$\sum_{n=1}^{\infty} \frac{n e^{2n\rho_0} e^{-2n\rho}}{\delta^2 + \frac{1}{4} e^{-4n\rho_0}} = \sum_{n \le N} + \sum_{n > N} \sim \sum_{n \le N} \frac{n e^{2n\rho_0} e^{-2n\rho}}{e^{-4n\rho_0}} + \frac{1}{\delta^2} \sum_{n > N} n e^{-2n(\rho_z - \rho_0)}.$$

Observe that

$$\sum_{n \le N} \frac{n e^{2n\rho_0} e^{-2n\rho}}{e^{-4n\rho_0}} \sim \sum_{n \le N} n e^{2n(3\rho_0 - \rho_z)} \sim \begin{cases} |\log \delta| \delta^{-3+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log \delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0. \end{cases}$$

On the other hand, we have

$$\frac{1}{\delta^2} \sum_{n > N} n e^{-2n(\rho_z - \rho_0)} \sim |\log \delta| \delta^{-3+\rho_z/\rho_0}.$$

So we infer that

$$\sum_{j=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2} \sim \begin{cases} |\log \delta| \delta^{-3+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log \delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0. \end{cases}$$

Since $\|\nabla F_z\|^2_{L^2(\Omega)}$ is bounded, we obtain (29) $\square$

To show ALR, we prove the following theorem.

Theorem 4.2. *Let Ω be an ellipse given by* (23)*. It holds for all x satisfying $\rho_x+\rho_z-4\rho_0 > 0$ that*

$$|u_\delta(x) - F_z(x)| \lesssim \sum_{n=1}^{\infty} e^{-n(\rho_x+\rho_z-4\rho_0)}. \tag{36}$$

In particular, let $\overline{\rho} > 0$ be such that $\overline{\rho} > 4\rho_0 - \rho_z$, then there exists some $C = C_{\overline{\rho}} > 0$ such that

$$\sup_{\rho_x \geq \overline{\rho}} |u_\delta(x) - F_z(x)| < C. \tag{37}$$

Proof. One can see from (16), (20) and (21) that

$$u_\delta(x) - F_z(x) = \sum_{n=1}^{\infty} \frac{1}{i\delta - \lambda_n}\left(\frac{1}{2} - \lambda_n\right) \{(a \cdot \nabla_z \mathcal{S}_{\partial\Omega}[\psi_n^c](z))\mathcal{S}_{\partial\Omega}[\psi_n^c](x) + (a \cdot \nabla_z \mathcal{S}_{\partial\Omega}[\psi_n^s](z))\mathcal{S}_{\partial\Omega}[\psi_n^s](x)\}.$$

It then follows from (24), (25) and (26) that

$$\begin{aligned} u_\delta(x) - F_z(x) =& \sum_{n=1}^{\infty} \frac{1}{i\delta - \lambda_n}\left(\frac{1}{2} - \lambda_n\right) \\ &\cdot \left\{\frac{e^{n\rho_0}\cosh n\rho_0}{n\pi}\left(a \cdot \nabla_z (e^{-n\rho_z}\cos n\omega_z)\right) e^{-n\rho_x}\cos n\omega_x \right. \\ &\left. + \frac{e^{n\rho_0}\sinh n\rho_0}{n\pi}\left(a \cdot \nabla_z (e^{-n\rho_z}\sin n\omega_z)\right) e^{-n\rho_x}\sin n\omega_x\right\}, \end{aligned}$$

where (ρ_z, ω_z) is the elliptic coordinates of z. Therefore, we have

$$|u_\delta(x) - F_z(x)| \lesssim \sum_{n=1}^{\infty} \frac{e^{4n\rho_0}}{n} \left\{|a \cdot \nabla_z (e^{-n\rho_z}\cos n\omega_z)| + |a \cdot \nabla_z (e^{-n\rho_z}\sin n\omega_z)|\right\} e^{-n\rho_x}.$$

We then see from (34) that

$$|u_\delta(x) - F_z(x)| \lesssim \sum_{n=1}^{\infty} \frac{e^{4n\rho_0}}{n} n e^{-n\rho_z} e^{-n\rho_x} = \sum_{n=1}^{\infty} e^{-n(\rho_x+\rho_z-4\rho_0)},$$

which proves (36). (37) is an immediate consequence of (36). □

Therefore, Theorems 4.1 and 4.2 imply that ALR occurs on ellipses in two dimensions.

4.2 Anomalous localized reaonance on balls

Assume that Ω is a ball in $\mathbb{R}^3$. We use the spherical coordinates

$$x = (x_1, x_2, x_3) = (x_1(r,\theta,\varphi), x_2(r,\theta,\varphi), x_3(r,\theta,\varphi)) \in \mathbb{R}^3$$

where

$$x_1(r,\theta,\varphi) = r\cos\theta\sin\varphi, \quad x_2(r,\theta,\varphi) = r\sin\theta\sin\varphi, \quad x_3(r,\theta,\varphi) = r\cos\varphi,$$
$$r \geq 0, 0 \leq \theta < 2\pi, 0 \leq \varphi \leq \pi.$$

Then we have $\partial\Omega = \{x \in \mathbb{R}^3;\ |x| = r_0\}$. The surface element $d\sigma$ and the outward normal derivative $\frac{\partial}{\partial\nu}$ are given in terms of the spherical coordinates by

$$d\sigma = r_0^2 \sin\varphi d\theta d\varphi, \quad \frac{\partial}{\partial\nu} = \frac{\partial}{\partial r}.$$

The Cartesian partial derivatives in the spherical coordinates are given by

$$\frac{\partial}{\partial x_1} = \cos\theta\sin\varphi\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial x_2} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\varphi}{r}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial x_3} = \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\varphi}.$$

Let $Y_n^m(\hat{x})$ be the orthonormal spherical harmonics of degree n, where $\hat{x} = \hat{x}(\theta,\varphi) = x/|x|$:

$$Y_n^m(\theta,\varphi) = (-1)^{(m+|m|)/2}\sqrt{\frac{2n+1}{4\pi}\cdot\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{im\varphi}.$$

Here, $P_n^{|m|}(x)$ is the associated Legendre polynomial with indices $n = 0, 1, \cdots$ and $|m| \leq n$. Then we have

$$\mathcal{K}^*_{\partial\Omega}[Y_n^m](x) = \frac{1}{2(2n+1)}Y_n^m(\hat{x}),$$
$$|x| = r_0, n = 0, 1, \cdots, m = -n, -n+1, \cdots, n-1, n.$$

$\{Y_n^m(\hat{x});\ n = 1, 2, \cdots, m = -n, -n+1, \cdots, n-1, n\}$ is complete in $L^2_0(\Omega)$, hence in $\mathcal{H}^*_0$, which means that $\mathcal{K}^*_{\partial\Omega}$ has the following eigenfunction expansion:

$$\mathcal{K}^*_{\partial\Omega} = \sum_{n=1}^{\infty}\frac{1}{2(2n+1)}\sum_{m=-n}^{n}\psi_n^m \otimes \psi_n^m,$$

where

$$\psi_n^m(x) = \sqrt{\frac{r_0^3}{2n+1}}Y_n^m(\hat{x}), \quad |x| = r_0.$$

Note that $\{\psi_n^m(x);\ |x| = r_0, n = 1, 2, \cdots, m = -n-n+1, \cdots, n-1, n\}$ is an orthonormal basis in $\mathcal{H}^*_0$. We also have

$$\mathcal{S}_{\partial\Omega}[Y_n^m](x) = \begin{cases} -\dfrac{1}{2n+1}\dfrac{r^n}{r_0^{n-1}}Y_n^m(\hat{x}), & \text{for } |x| = r \leq r_0, \\ -\dfrac{1}{2n+1}\dfrac{r_0^{n+2}}{r^{n+1}}Y_n^m(\hat{x}), & \text{for } |x| = r > r_0, \end{cases}$$

for $n = 0, 1, \cdots$, $m = -n, -n+1, \cdots, n-1, n$. See [2].

Theorem 4.3. *Suppose that Ω is a three dimensional ball. Then there is a constant $C > 0$ such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C. \tag{38}$$

Proof. By the symmetry, we can assume that $a = (0, 0, a_3)$. Therefore, we only have to study the asymptotics of the following summation:

$$\sum_{j=1}^{\infty} \frac{|a \cdot \nabla S_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2} = \sum_{n=1}^{\infty} \frac{a_3^2}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \frac{r_0^3}{2n+1} \cdot \left| a_3 \cdot \left(-\frac{1}{2n+1} \frac{\partial}{\partial x_3} \left(\frac{r_0^{n+2}}{r^{n+1}} Y_n^m(\hat{z}) \right) \right) \right|^2,$$

which turns out that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{a_3^2 r_0^3}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{1}{2n+1}\right)^3 \left(\frac{r_0}{r}\right)^{n+2} \sum_{m=-n}^{n} |-(n+1)\cos\varphi - im\sin\theta|^2 \, |Y_n^m(\hat{z})|^2 \\
\leq &\sum_{n=1}^{\infty} \frac{a_3^2 r_0^3}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{1}{2n+1}\right)^3 \cdot \left(\frac{r_0}{r}\right)^{n+2} \cdot 2(n+1)^2 \sum_{m=-n}^{n} |Y_n^m(\hat{z})|^2.
\end{aligned} \tag{39}$$

By the Unsöld's theorem

$$\sum_{m=-n}^{n} |Y_n^m(\hat{x})|^2 = \frac{2n+1}{4\pi}, \quad n = 0, 1, 2, \cdots,$$

the right hand side of (39) equals

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{a_3^2 r_0^3}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{1}{2n+1}\right)^3 \cdot \left(\frac{r_0}{r}\right)^{n+2} \cdot 2(n+1)^2 \cdot \frac{2n+1}{4\pi} \\
= &\frac{2a_3^2 r_0^3}{4\pi} \sum_{n=1}^{\infty} \frac{1}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{n+1}{2n+1}\right)^2 \cdot \left(\frac{r_0}{r}\right)^{n+2} \\
\leq &\frac{2a_3^2 r_0^3}{\pi} \sum_{n=1}^{\infty} (n+1)^2 \left(\frac{r_0}{r}\right)^{n+2} \\
= &\frac{2a_3^2 r_0^3}{\pi} \cdot \frac{r_0 + r}{(r - r_0)^3},
\end{aligned}$$

which proves the theorem. □

Theorem 4.3 implies that ALR does not occur on ball in three dimensions.

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